Distributions of Zeros for Non-Abelian Zeta Functions

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Abstract

Two levels of fine structures on distributions of zeros for non-abelian zeta functions are exposed. For one, we show that the classical delta type distributions for pair correlations of these zeros are of Dirac types. For the other, we introduce a new type of big Delta distributions for our zeros and conjecture that these big Delta distributions are closely related with GUE. Supportive evidences from numerical calculations are provided. In fact, treated are much more general zeta functions associated to reductive groups and their maximal parabolic subgroups.

Introduction

A well-known conjecture on distributions of Riemann zeros claims that they resemble that of Gaussian Unitary Ensembles. We in this paper study distributions of zeros for non-abelian zeta functions. By definition ([W0]), the rank n non-abelian zeta function is given by

$$\widehat{\zeta}_{\mathbb{Q},n}(s) := \int_{\mathcal{M}_{\mathbb{Q},n}} \left(e^{h^0(\mathbb{Q},\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg_{\rm ar}(\Lambda)} d\mu, \qquad \mathrm{Re}(s) > 1.$$

Here $\mathcal{M}_{\mathbb{Q},n}$ denotes moduli space of semi-stable lattices of rank n. It is known that $\widehat{\zeta}_{\mathbb{Q},1}(s) = \widehat{\zeta}(s)$ coincides with the complete Riemann zeta function and $\widehat{\zeta}_{\mathbb{Q},n}(s)$'s satisfy standard zeta properties. And for the Riemann hypothesis, when $n = 2, 3, 4, 5$, Ki, Lagarias, and Suzuki show that it does hold ([K, LS, S1, SW]); Moreover, based on extra symmetries, the author, using their techniques, shows that, for any fixed $n \geq 2$, all zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$ are on the line $\mathrm{Re}(s) = \frac{1}{2}$, except for (possibly) these lying in a bounded domain of s-plane. So it is natural to investigate distributions of these non-abelian zeta zeros. The initial works were done by Suzuki and myself independently on $n = 2$ many years ago. The outcome was that, instead of GUE, only Dirac type distribution appeared. It took quite long time for me to understand this result. The turning point was a joint work with Zagier

*This work is partially supported by JSPS.

([WZ]) on the Riemann Hypothesis of high rank zeta functions of elliptic curves over finite fields. From this work, we realize that there are two levels of structures for distributions of (arguments theta of) our zeta zeros: For one on theta in the classical sense, we simply get Dirac distributions; For the other, we successfully recover the Sato-Tate type distributions for high rank zeta zeros, using their infinitesimal structures, for non-CM elliptic curves defined over $\mathbb{Q}$. Indeed, for this second level, the key is a construction of the big Theta, obtained from original theta by blowing-up the infinitesimal structures around limit points ([W3]). In turn, this motivates our current works on parallel structures for zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$.

To explain this, let $\rho = \frac{1}{2} + \sqrt{-1}\,\gamma$'s be zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$. Arrange γ in an increasing order

$$0 \leq \gamma_{n,1} \leq \gamma_{n,2} \leq \cdots \leq \gamma_{n,3} \leq \ldots,$$

and, as usual, let

$$N_n(T) := \#\big\{k : 0 < \gamma_{n,k} < T\big\}$$

denote the number of zeta zeros with imaginary parts between 0 and T.

Theorem 1. *For the zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$, when $n \geq 2$,*[1] *we have*

(1) $N_n(T) = \dfrac{n}{2\,\pi}\,T\log T - \dfrac{n\log(2\pi e)}{2\,\pi}\,T + O(\log T)$;

(2) $\gamma_{n,k} = \dfrac{2\,\pi}{n}\,\dfrac{k}{\log k}\Big(1 + O\big(\dfrac{1}{\log k}\big)\Big)$;

(3) $\gamma_{n,k+1} - \gamma_{n,k} = \dfrac{2\,\pi}{n}\,\dfrac{1}{\log k} + O\Big(\dfrac{1}{\log^2 k}\Big)$.

Motivated by classical works on pair correlations of Riemann zeta zeros ([BH, H, M, O]), as an analogue of the classical pair correlation function, for $n \geq 2$, define the pair correlation function of high rank zeta zeros, by

$$\delta_{n,k} := \Big(\frac{n}{2\,\pi}\big(\gamma_{n,k+1} - \gamma_{n,k}\big)\Big) \cdot \log\Big(\frac{n}{2\,\pi}\gamma_{n,k}\Big).$$

Theorem 2. *For the zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$, when $n \geq 2$, we have*

$$\delta_{n,k} = 1 + O\Big(\frac{1}{\log k}\Big).$$

In particular, the distributions of non-abelian zeta zeros are very different from that of Riemann zeros, which conjecturally coincide with the GUE in the theory of random matrix. However, it turns out there is yet another level of structure for non-abelian zeta zeros. To explain this, also, motivated by our studies for function fields ([W3]) and classical works on pair correlations of Riemann zeros ([CGGGH-B, F1,2, G, M]), we introduce the big Δ functions for the pair correlations of our zeros.

[1] Here and in the sequel, when $n = 2$, stronger results hold. For details, please see Proposition 13 of §3.1.

Definition 3. *The big Delta pair correlation functions for the zeros of high rank non-abelian zeta function* $\widehat{\zeta}_{\mathbb{Q},n}(s)$ *are defined by*

$$\Delta_{n,k} := \big(\delta_{n,k} - 1\big) \cdot \log\big(\frac{n}{2\,\pi}\gamma_{n,k}\big). \tag{1}$$

The distributions of $\Delta_{n,k}$'s and δ_n's for the Riemann zeta function are expected to be closely related. For example, we, motivated by the conjectural connection between Riemann zeros and random matrix theory ([D, KS1,2, KeS, MS, M, O, Se, T]), have the following

Conjecture 4. *Denote by* $\mu(\Delta_n)$ *the measure introduced by* $\Delta_{n,k}$*, and* $\mu(\mathrm{GUE})$ *the corresponding one for the Gaussian unitary ensemble. Then*

$$\lim_{n\to\infty} \mathrm{Discrep}\Big(\mu(\Delta_{n,k}), \mu(\mathrm{GUE})\Big) = 0.$$

Here Discrep (μ, ν) *denotes the Kolomogorof-Smirnov distance of* μ *and* ν*, up to a normalization depending only on* n*.*

This is supported by some very impressive numerical calculations on zeros of low rank non-abelian zeta functions. For details, please refer to the figures at the end of this paper, or better, our web pages on Lab of Zeta Zeros at http://www2.math.kyushu-u.ac.jp/∼weng/zetas.

Our method works for much more general zeta functions $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ associated to Chevalley groups G and their maximal parabolic subgroups P. Indeed, based on a beautiful work [KKS], we have

Theorem 5. *Assuming the volume conjecture, for Chevelley groups* G *of rank* ≥ 2 *and their maximal parabolic subgroup* P *defined over* $\mathbb{Q}$*, we have*

$$\delta_k^{G/P} = 1 + O\Big(\frac{1}{\log k}\Big).$$

Here $\delta_k^{G/P} := \frac{d_P}{\pi}\big(\gamma_{k+1}^{G/P} - \gamma_k^{G/P}\big) \cdot \log\big(\frac{d_P}{\pi}\gamma_k^{G/P}\big).$

Similarly, we have the corresponding big Delta pair correlation functions for the zeros of $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$:

$$\Delta_k^{G/P} := \big(\delta_k^{G/P} - 1\big) \cdot \log\big(\frac{d_P}{\pi}\gamma_k^{G/P}\big). \tag{2}$$

At the moment, these general Δ's still prove to be very mysterious, even the strongest form of our conjectures predicts that $\Delta_k^{G/P}$'s obey GUE.

The contents of this paper are as follows. In §1, we recall some basic constructions and properties for non-abelian zeta functions and zeta functions associated to (G, P). In §2, we state our main results, and in §3, we prove them.

1 Non-Abelian Zetas and Their Generalizations

1.1 Non-Abelian Zeta Functions for Number Fields

Let F be a number field with $\mathcal{O}_F$ the integer ring and Δ_F the absolute value of discriminant. Then a rank n projective $\mathcal{O}_F$-module M is isomorphic to $\mathcal{O}_F^{n-1} \oplus \mathfrak{a}$ with $\mathfrak{a}$ a fractional ideal of F. And, by the Minkowski embedding $F \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, we may view a rank n projective $\mathcal{O}_F$-module naturally as a sub-$\mathcal{O}_F$-module of $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r$. By an $\mathcal{O}_F$-lattice of rank n, we mean a pair (M,h) consisting of a projective $\mathcal{O}_F$-module M of rank n, a metric h on $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^n$ and a Minkowski embedding $M \hookrightarrow F^r \hookrightarrow \big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r$. An $\mathcal{O}_F$-lattice $L=(M,h)$ is called semi-stable if $\mu(L_1) \leq \mu(L)$ for all $\mathcal{O}_F$-sublattices L_1 of L. Here, as usual, $\mu(L) := \mathrm{deg}_{\mathrm{ar}}(L)\big/\mathrm{rank}(L)$, with $\mathrm{deg}_{\mathrm{ar}}$ the Arakelov degree ([L]).

Denote by $\mathcal{M}_{F,n}$, resp. $\mathcal{M}_{F,n}[\Delta_F^{1/2}]$, resp. $\mathcal{M}_{F,n}[\geq \Delta_F^{1/2}]$, the moduli space of semi-stable $\mathcal{O}_F$-lattices of rank n, resp. of rank n and co-voloume $\Delta_F^{1/2}$, or the same, of the Arakelov degree 0, resp. of rank n and co-voloume $\geq \Delta_F^{1/2}$. It is well-known that, as sub-spaces of all $\mathcal{O}_F$-lattices of rank n, there exist natural measures $d\mu$ on $\mathcal{M}_{F,n}$, say, induced from the natural Tamagawa measure on the associated adelic space $SL_n(\mathbb{A}_F)$. By definition ([W0]), the *rank n non-abelian zeta function $\widehat{\zeta}_{F,n}(s)$ of F* is the integration

$$\widehat{\zeta}_{F,r}(s) := |\Delta_F|^{\frac{r}{2}s} \int_{\mathcal{M}_{F,r}} \Big(e^{h^0(F,L)} - 1\Big) \left(e^{-s}\right)^{\mathrm{deg}_{\mathrm{ar}}(L)} d\mu(L), \qquad \mathrm{Re}(s) > 1.$$

Here $h^0(F,L)$ denotes the 0-th arithmetic cohomology of the lattice L. These zeta functions satisfy standard properties of zeta functions:

Theorem 6. (Zeta Facts)
(0) $\widehat{\zeta}_{F,1}(s) \doteq \widehat{\zeta}_F(s)$ is the completed Dedekind zeta function of F;
(1) **(Meromorphic continuation)** *$\widehat{\zeta}_{F,n}(s)$ is well-defined when* $\mathrm{Re}(s) > 1$ *and admits a unique meromorphic continuation, denoted also by $\widehat{\zeta}_{F,n}(s)$, to the whole complex s-plane;*
(2) **(Functional equation)** *$\widehat{\zeta}_{F,n}(1-s) = \widehat{\zeta}_{F,n}(s)$;*
(3) **(Singularities & Residues)** *$\widehat{\zeta}_{F,n}(s)$ has two singularities, all simple poles, at $s=0,1$, with residues given by* $\pm\mathrm{Vol}\big(\mathcal{M}_{F,n}[\Delta_F^{\frac{1}{2}}]\big)$.

This theorem is proved tautologically in [W0], using an arithmetic cohomology theory for number fields. Indeed, the functional equation and the singularity and residues statements are direct consequences of the arithmetic duality with respect to the Arakelov dualizing lattice ω_F of F:

$$h^1_{\mathrm{ar}}(F, \omega_F \otimes L^\vee) = h^0_{\mathrm{ar}}(F,L),$$

and the arithmetic Riemann-Roch theorem:

$$h^0_{\mathrm{ar}}(F,L) - h^1_{\mathrm{ar}}(F,L) = \mathrm{deg}_{\mathrm{ar}}(L) - \frac{n}{2}\log|\Delta_F|.$$

Moreover, with them, a formal calculation leads to the expression

$$\widehat{\zeta}_{F,n}(s) = I_{F,n}(s) + I_{F,n}(1-s) + \mathrm{Vol}\Big(\mathcal{M}_{F,n}[\Delta_F^{\frac{1}{2}}]\Big) \cdot \Big(\frac{1}{s-1} - \frac{1}{s}\Big)$$

where

$$I_{F,n}(s) = \int_{L \in \mathcal{M}_{F,n}[\geq \Delta_F^{\frac{1}{2}}]} \Big(e^{h^0(F,L)} - 1\Big) \cdot \mathrm{Vol}(L)^s \cdot d\mu(L).$$

Finally, the convergence is given by the equivalence of the follows:

(1) Rank one $\mathcal{O}_F$-lattice A is arithmetic positive;

(2) Rank one $\mathcal{O}_F$-lattice A is arithmetic ample; and

(3) For rank one $\mathcal{O}_F$-lattice A and any $\mathcal{O}_F$-lattice L,

$$\lim_{n \to \infty} h^1(F, A^n \otimes L) = 0.$$

Or better, we can get the convergence from an effective arithmetic vanishing theorem for semi-stable lattices: For semi-stable $\mathcal{O}_F$-lattice L of rank n satisfying $\deg_{\mathrm{ar}}(L) \leq -[F:\mathbb{Q}] \cdot (n \log n)/2$, we have

$$h^0(F,L) \leq \frac{3^{n[F:\mathbb{Q}]}}{1 - \log 3/\pi} \cdot \exp\Big(-\pi[F:\mathbb{Q}] \cdot e^{-\mu(L)}\Big).$$

For more details, please refer to [W0].

Concerning the Riemann Hypothesis, we have is the following

Theorem 7. *(1)* **(Weak RH)** *For $n \geq 2$, outside a bounded domain of the complex s-plane,*

$$\widehat{\zeta}_{\mathbb{Q},n}(s) = 0 \qquad \textit{implies} \qquad \mathrm{Re}(s) = \frac{1}{2}.$$

(2) **(RH for low ranks)** *([K, LS, S1]) When $n = 2, 3, 4, 5$,*

$$\widehat{\zeta}_{\mathbb{Q},n}(s) = 0 \qquad \textit{implies} \qquad \mathrm{Re}(s) = \frac{1}{2}.$$

The weak Riemann Hypothesis above is due to myself based on extra symmetries and a method of Ki. See e.g., [K], [KKS, §4]. In fact, by the special uniformity of zeta functions, high rank non-abelian zeta functions coincide with zeta functions for $(SL_n, P_{n-1,1})$, where $P_{n-1,1}$ denotes the standard maximal parabolic subgroup of SL_n corresponding to the partition $n = (n-1)+1$. These latest zeta functions are special cases of the so-called Weng zeta functions for reductive algebraic groups G and their maximal parabolic subgroups P. Thanks to the beautiful work of Ki-Komori-Suzuki ([KKS]), we now have the weak Riemann Hypothesis for zeta functions of (G,P) assuming the volume conjecture. On the other hand, the volume conjecture is proved for the group SL_n in [W1], as a special case of the conjecture on Parabolic Reduction, Stability and the Masses for general reductive groups, based on a result of Lafforgue on Arthur's analytic truncation and an advanced version of Rankin-Selberg & Zagier method.

1.2 Zeta Functions for $(G,P)/\mathbb{Q}$

Let G be a split reductive algebraic group defined over F with associated Borel subgroup B and its maximal split sub-torus T. Denote the corresponding root system by

$$\Big(\Delta,\,\Lambda,\,\Phi=\Phi^+\cup\Phi^-,\,\Phi^\vee,\,W,\,\widehat{\Delta},\,\widehat{\Lambda},\,\rho\Big),$$

where, $\Delta=\{\alpha_1,\dots,\alpha_r\}$ is the set of simple roots, $\Lambda=\{\lambda_1,\dots,\lambda_r\}$ the set of fundamental weights, Φ the set of roots with Φ^+, resp. Φ^- of positive roots, resp. negative roots, $\Phi^\vee=\{\alpha^\vee:\alpha\in\Phi\}$ the set of coroots, W the Weyl group, $\widehat{\Delta}\subset\Phi^\vee$ the set of simple co-roots, $\widehat{\Lambda}=\{\varpi_1,\dots,\varpi_r\}$ the set of fundamental co-weights, and $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$ the Weyl vector. For each $w\in W$, set also $\Phi_w:=\Phi^+\cap w^{-1}\Phi^-$.

Denote by $X(G)_\mathbb{R}$ the $\mathbb{R}$-span of fundamental weights and $X(G)^*_\mathbb{R}$ the $\mathbb{R}$-span of simple roots. There is a natural W-invariant bi-linear pairing $\langle\cdot,\cdot\rangle:X(G)_\mathbb{R}\times X(G)^*_\mathbb{R}\to\mathbb{R}$ such that $\langle\lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$. Introduce a particular coordinate system on $X(G)_\mathbb{R}$ by

$$\lambda=\sum_{i=1}^r(1+s_i)\,\lambda_i=\rho+\sum_{i=1}^r s_i\,\lambda_i.$$

Following [W1], define the *period of G over F* by

$$\omega_F^G(\lambda):=\sum_{w\in W}\frac{1}{\prod_{\alpha\in\Delta}\langle\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,\,w\alpha<0}\frac{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_F(\langle\lambda,\alpha^\vee\rangle+1)}.$$

Here $\widehat{\zeta}_F(s)$ denotes the complete Dedekind zeta function of F. These periods can be obtained from regularized integrations over cones for (constant terms of) certain Siegel type Eisenstein series.

In general, $\omega_F^G(\lambda)$ is a several variables function. To get a genuine one variable zeta function, fix a maximal standard parabolic subgroup P of G. Then, by Lie theory ([Hu]), P corresponds to a unique simple root, which we denote by α_P, or α_p with $p\in\{1,2,\dots,r\}$. Following [W2], we define the *period of* $(G,P)/F$ by

$$\omega_F^{G/P}(s):=\mathrm{Res}_{\substack{\langle\lambda,\alpha^\vee\rangle=1\\ \alpha\in\Delta_P}}\,\omega_F^G(\lambda)=\mathrm{Res}_{\substack{\langle\lambda,\alpha_i^\vee\rangle=1\\ 1\le i\le r,\,i\ne p}}\,\omega_F^G(\lambda),$$

where $s=s_P$ and $\Delta_P=\Delta\backslash\{\alpha_P\}$. This latest period is essentially the zeta function $\widehat{\zeta}_F^{G/P}(s)$ associated to $(G,P)/F$: What is left is merely a normalization of clearing out the factors involving Dedekind zeta functions appeared in the denominators after taking residues. For details, please refer to [W2]. Indeed, as proved in [Ko], our zeta function $\widehat{\zeta}_F^{G/P}(s)$ is given by

$$\widehat{\zeta}_F^{G/P}(s)=\omega_F^{G/P}(s)\cdot\prod_{k=0}^\infty\prod_{h=2}^\infty\widehat{\zeta}_F(ks+h)^{M_P(k,h)},\qquad(*)$$

where, for $w \in \mathfrak{W}_P$[2] and $(k,h) \in \mathbb{Z}^2$,

$$\begin{aligned} N_{p,w}(k,h) :=& \#\big\{\alpha \in w^{-1}\Phi^- : \langle \lambda_p, \alpha^\vee \rangle = k, \langle \rho, \alpha^\vee \rangle = h\big\}, \\ M_p(k,h) :=& \max\nolimits_{w \in \mathfrak{W}_p} \big(N_{p,w}(k,h-1) - N_{p,w}(k,h)\big). \end{aligned}$$

Main structures exposed for $\widehat{\zeta}_F^{G/P}(s)$ can be summarized in the following:

Theorem 8. *(i)* **(Special uniformity)** *([W1,2]) Up to a certain constant factor depending on F and n,*

$$\widehat{\zeta}_{F,n}(s) \;=\; \widehat{\zeta}_F^{SL_n/P_{n-1,1}}(-ns);$$

(ii) **(Functional equation)** *([W2]‖[Ko]) Let $c_P = 2\langle \lambda_P - \rho_P, \alpha_P^\vee \rangle$*

$$\widehat{\zeta}_F^{G/P}(-c_P - s) \;=\; \widehat{\zeta}_F^{G/P}(s);$$

(iii) **(Weak Riemann hypothesis)** *([W2]‖[KKS, also K, LS, S1,S2, SW]) Outside a bounded domain in the complex s-plane,*

$$\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s) = 0 \qquad \textit{implies} \qquad \mathrm{Re}\,(s) = -c_P/2,$$

provided that the residue of $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ at $s = 1$ coincides the volume of semi-stable principal G-lattices over $\mathbb{Q}$ of degree 0.

2 Main Theorems

Now assuming the Riemann hypothesis for $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ and consider the zeros $\rho = -c_P/2 + \sqrt{-1}\,\gamma$ of $\widehat{\zeta}_F^{G/P}(s)$ on the central line $\mathrm{Re}(s) = -c_P/2$. Arrange γ in an increasing order

$$0 \le \gamma_1^{G/P} \le \gamma_2^{G/P} \le \cdots \le \gamma_n^{G/P} \le \ldots,$$

and, as usual, let

$$N^{G/P}(T) := \#\big\{n : 0 < \gamma_n^{G/P} < T\big\}$$

denote the number of our zeta zeros with imaginary parts between 0 and T. Also introduce

$$\begin{aligned} d_P :=& \frac{1}{2}\sum_{k=1}^{\infty} k \cdot N_P(k, [(kc_P - 1)/2]) \\ e_P :=& \frac{1}{2}\sum_{k=1}^{\infty} k \log k \cdot N_P(k, [(kc_P - 1)/2]), \end{aligned}$$

where $N_P(k,h) := \#\big\{\alpha \in \Phi : \langle \lambda_p, \alpha^\vee \rangle = k, \langle \rho, \alpha^\vee \rangle = h\big\}$.

[2]The definitions of $\mathfrak{W}_P$ and ρ_P below will be given in §3.

Theorem 9. *Assume the volume conjecture for G,[3] we have,*

(1) $N^{G/P}(T) = \frac{d_P}{\pi} T \log T + \frac{e_P - d_P \log(2\pi e)}{\pi} T + O(\log T)$;

(2) $\gamma_n^{G/P} = \frac{\pi}{d_P} \frac{n}{\log n} \Big(1 + O(\frac{1}{\log n})\Big)$;

(3) $\gamma_{n+1}^{G/P} - \gamma_n^{G/P} = \frac{\pi}{d_P} \frac{1}{\log n} + O\Big(\frac{1}{\log^2 n}\Big)$.

Based on this, as an analogue of the classical pair correlation function, introduce the pair correlation function small delta of these zeta zeros by

$$\delta_n^{G/P} := \Big[\frac{d_P}{\pi}\Big(\gamma_{n+1}^{G/P} - \gamma_n^{G/P}\Big)\Big] \cdot \log\Big(\frac{d_P}{\pi}\gamma_n^{G/P}\Big).$$

Theorem 10. *With the same conditions as in Theorem 9, we have,*

$$\delta_n^{G/P} = 1 + O\Big(\frac{1}{\log n}\Big).$$

Since, for special linear group SL_n, the volume conjecture is proved in [W1, §§3.1, 3.4, and 4.8] using Arthur's analytic truncation and Lafforgue's arithmetic truncation, we have the following unconditional

Theorem 11. *For the zeros of $\widehat{\zeta}_{\mathbb{Q},n}(s)$, when $n \geq 3$, we have*

(1) $N_n(T) = \frac{n}{2\pi} T \log T - \frac{n \log(2\pi e)}{2\pi} T + O(\log T)$;

(2) $\gamma_{n,k} = \frac{2\pi}{n} \frac{k}{\log k} \Big(1 + O(\frac{1}{\log k})\Big)$;

(3) $\gamma_{n,k+1} - \gamma_{n,k} = \frac{2\pi}{n} \frac{1}{\log k} + O\Big(\frac{1}{\log^2 k}\Big)$.

(4) $\delta_{n,k} = 1 + O\Big(\frac{1}{\log k}\Big)$.

Consequently, the distributions of our zeta zeros are very different from that of Riemann zeros, which conjecturally coincide with the Gaussian Unitary Ensemble in random matrix theory. However, it turns out there is yet another level of structure for these zeta zeros. To explain this, also, motivated by our study for function fields, we introduce the big Δ functions for the pair correlations of our zeta zeros.

Definition 12. *The big Delta pair correlation functions for the zeros of $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ are defined by*

$$\Delta_k^{G/P} := \big(\delta_k^{G/P} - 1\big) \cdot \log\big(\frac{d_P}{\pi}\gamma_k^{G/P}\big).$$

[3] Here and in the next theorem, in order to get the weak RH for $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$, we add the assumption on volume conjecture. For details, see [KKS].

The distributions of $\Delta_n^{G/P}$'s and δ_n's for the Riemann zeta function are supposed to be closely related. For example, we have the conjecture of the introduction, supported by some very impressive numerical calculations on zeros of low rank non-abelian zeta functions. For details, please refer to the figures at the end of this paper, or better, our web pages on Lab of Zeta Zeros at http://www2.math.kyushu-u.ac.jp/~weng/zetas.

3 Proof of Main Theorems

3.1 Rank two zeta function

To start with, we prove the following stronger result for rank 2 zeta function of rationals, due to Suzuki and myself independently, to indicate the analytic structure involved.

Proposition 13. *For the zeros $\gamma_{2,k}$'s of $\widehat{\zeta}_{\mathbb{Q},2}(s)$, we have*

(1) $N_2(T) = \frac{1}{\pi} T \log T - \frac{1}{\pi} T \log(\pi e) + O\big(\frac{\log T}{\log\log T}\big)$;

(2) $\gamma_{2,k} = \pi \frac{k}{\log k}\Big(1 + O\big(\frac{1}{\log k}\big)\Big)$;

(3) $\gamma_{2,k+1} - \gamma_{2,k} = \pi \frac{1}{\log k} + O\big(\frac{1}{\log k \log\log k}\big)$;

(4) $\delta_{2,k} := \frac{\gamma_{2,k+1} - \gamma_{2,k}}{\pi} \log \frac{\gamma_{2,k}}{\pi} = 1 + O\big(\frac{1}{\log\log k}\big)$.

Proof. For $s = \frac{1}{2} + it$, we have $\overline{\widehat{\zeta}(2s-1)} = \widehat{\zeta}(2s)$. Consequently, by [W2, §A.1.1],

$$\widehat{\zeta}_{\mathbb{Q},2}\big(\frac{1}{2} + it\big) = \frac{1}{2}\Big|\frac{\widehat{\zeta}(1+2it)}{(-\frac{1}{2}+it)}\Big| \cdot \cos\theta_2(t),$$

where $\theta_2(t)$ denotes the argument of $\widehat{\zeta}(1+2it)/(-1/2+it)$. So to understand the distributions of rank two zeta function, it suffices to know the asymptotics of $\theta_2(t) = \arg\Gamma(\frac{1}{2}+it) + \arg\zeta(\frac{1}{2}+it) - \arg(\pi^{\frac{1}{2}+it}) - \arg(-\frac{1}{2}+it)$ when $t \to \infty$. Now by Stirlings' formula, $\arg\Gamma(\sigma+it) = t\log t - t + \frac{\pi}{2}(\sigma - \frac{1}{2}) + O(\frac{1}{t})$. Moreover, from [T, Thm 5.16], $\arg\zeta(1+2it) = O\big(\log t/\log\log t\big)$. Consequently, $\theta_2(t) = t\log t - t(1+\log\pi) + O\big(\frac{\log t}{\log\log t}\big)$. But $\gamma_{2,n+1} - \gamma_{2,n} = \pi$. So, asymptotically, $N_2(T) = \theta_2(T)/\pi$. This proves (1). To prove (2), since $N_2(\gamma_{2,n} - 1) \leq n \leq N_2(\gamma_{2,n} + 1)$. Consequently, $\gamma_{2,n} \log\gamma_{2,n} \sim \pi n$. Hence $\gamma_{n,2} \sim \pi n/\log n$. In particular, $\pi N_2(\gamma_{2,n} \pm 1) = (\gamma_{2,n} \pm 1)\big(\log(\gamma_{2,n} \pm 1) - \log\pi - 1\big) + O\big(\log(\gamma_{2,n} \pm 1)/\log\log(\gamma_{2,n} \pm 1)\big)$. With a simple manipulation, we obtain (2). To go further, as in the proof of (1), using Stirings' formula and [T, Thm 5.16], we have, for small h, $\theta_2(t+h) - \theta_2(t) = h\log t\big(1 + O\big(\frac{1}{\log\log t}\big)\big) + O\big(\frac{1}{t}\big)$. To apply this, set $t = \gamma_{2,n}$ and $h = \gamma_{2,n+1} - \gamma_{2,n}$. Then, using $\gamma_{2,n+1} - \gamma_{2,n} = \pi$ again, we get (3) and hence prove the theorem, since (4) is a direct consequence of (2) and (3).

3.2 Proof of main theorems

Step 1. *Fine symmetric structures of* $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$.

Let P be a standard parabolic subgroup of G. Denote by $P = M_P N_P$ the Levi decomposition of P, $\mathfrak{n}_P$ the Lie algebra of N_P, and T_P the maximal central subgroup of M_P with $\mathfrak{a}_P$ its Lie algebra. Let Δ_P be the set of roots for (P, A_P), i.e., the finite subset of non-zero elements in $X(A_P)_{\mathbb{Q}}$ parametrizing the decomposition $\mathfrak{n}_P = \oplus_{\alpha\in\Phi_P}\mathfrak{n}_\alpha$ of the eigenspace under the adjoint action $\mathrm{Ad} : A_P \to GL(\mathfrak{n}_P)$ of A_P, where, as usual, $\mathfrak{n}_\alpha := \{X_\alpha \in \mathfrak{n}_P : \mathrm{Ad}(a)X_\alpha = a^\alpha \cdot X_\alpha,\ \forall a \in A_P\}$. Note that $\Phi_P \subset X(A_P)_{\mathbb{Q}} \subset X(A_P)_{\mathbb{Q}} \otimes \mathbb{R} \simeq \mathfrak{a}_P^*$. Similar to the Weyl vector, introduce its P-version by $\rho_P := \dfrac{1}{2}\sum_{\alpha\in\Phi_P} (\dim \mathfrak{n}_\alpha)\, \alpha$.

By Lie theory ([Hu]), there is a natural order reversing bijection

$$\big\{P : \text{standard parabolic subgroup of } G\big\} \longleftrightarrow \big\{\Delta^P \subset \Delta\big\}$$

such that $\mathfrak{a}_P = \big\{H \in \mathfrak{a} : \alpha(H) = 0,\ \forall\alpha \in \Delta^P\big\}$. Then Δ^P forms a basis of $\mathfrak{a}_P$. Let Δ_P be the set of linear forms on $\mathfrak{a}_P$ obtained by restrictions of elements of $\Delta_0\backslash\Delta_0^P$: $\Delta_P := \big\{\alpha|_{\mathfrak{a}_P} \in \mathfrak{a}_P^* :\ \exists\alpha \in \Delta_0\backslash\Delta_0^P\big\}$. It is well-known that for any $\alpha \in \Phi_P$, $\alpha = \sum_{\beta\in\Delta_P} n_\beta\beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Even Δ_P is not really a root system in the usual sense, with this proproty, it is still possible to introduce $\Phi_P^\pm$ such that $\Phi_P = \Phi_P^+ \sqcup \Phi_P^-$, $\Phi_P^- = -\Phi_P^+$. Indeed, we can and will identify Φ_P as a subset of Φ from the above construction, so that, simply, $\Phi_P^+ := \Phi^+ \cap \Phi_P$. In this language, then $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P}\alpha$. Following [Ko], introduce the constant

$$c_P := 2\langle \lambda_P - \rho_P, \alpha_p^\vee\rangle.$$

From now on, assume that P is maximal. Then $\Delta^P = \{\alpha_P\} = \{\alpha_p\}$ consisting of a single element $(1 \leq p \leq r)$. By definition, $\omega_{\mathbb{Q}}^{G/P}(s) = \sum_{w\in W} T_w$, where $T_w(s) := \lim_{\lambda\to\lambda_P} \frac{\prod_{\alpha\in\Delta_P}\langle\lambda-\rho,\alpha^\vee\rangle}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho,\alpha^\vee\rangle} \prod_{\alpha>0,\, w\alpha<0} \frac{\widehat{\zeta}\,(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}\,(\langle\lambda,\alpha^\vee\rangle+1)}$. Note that $\lim_{\lambda\to\lambda_p} \langle \lambda - \rho, \alpha^\vee\rangle \equiv 0,\ \forall\alpha \in \Delta_P$. So, to obtain a non-trivial $T_w(s)$ within the period $\omega_{\mathbb{Q}}^{G/P}(s)$, there should be a total cancellation for all factors $\langle\lambda - \rho, \alpha^\vee\rangle$, $\alpha \in \Delta_P$. In particular, $T_w(s) \not\equiv 0$ if and only if $\Delta_P \subset w^{-1}(\Delta \cup \Phi^-)$, since $\widehat{\zeta}(s)$ has poles only at $s = 0$, 1, which are also know to be simple. Accordingly, we conclude that

$$\omega_{\mathbb{Q}}^{G/P}(s) = \sum_{w\in\mathfrak{W}_P} T_w \qquad \text{with} \qquad \mathfrak{W}_P := \big\{w \in W|\Delta_P \subset w^{-1}(\Delta \cup \Phi^-)\big\}.$$

We will call elements w of $\mathfrak{W}_P$ *special* (with respect to P).

To facilitate our ensuing discussions, we make the following preparations following [KKS]. Let

$$X_P(s) := Q_P(s) \cdot \Big(F_P(s) \cdot \omega_{\mathbb{Q}}^{G/P}(s)\Big)$$

where $F_P(s) := \prod_{\alpha\in\Phi^-} \widehat{\zeta}\left(\langle \lambda_P s+\rho, \alpha^\vee\rangle\right)$ and $Q_P(s) := \prod_{w\in\mathfrak{W}_P} q_{P,w}(s)$ for

$$q_{P,w}(s) := \prod_{w\in\mathfrak{W}_P} \Big[2^{|\Delta_P\cap w^{-1}\Phi^+|} \prod_{\alpha\in(w^{-1}\Delta)\cap\Delta_P} \Big(\big(\langle\lambda_s+\rho,\alpha^\vee\rangle-1\Big) \times \prod_{\alpha\in\Phi^+\backslash\Delta_P} \Big(\langle\lambda_s+\rho,\alpha^\vee\rangle+\delta_{\alpha,w}\Big)\Big(\langle\lambda_s+\rho,\alpha^\vee\rangle+\delta_{\alpha,w}-1\Big)\Big],$$

with

$$\delta_{\alpha,w} := \begin{cases} 1 & \alpha\in w^{-1}\Phi^+, \\ 0 & \alpha\in w^{-1}\Phi^-. \end{cases}$$

Then, we may write down $X_P(s)$ as

$$X_P(s) = \sum_{w\in\mathfrak{W}_P} Q_{P,w}(s)\cdot X_{P,w}(s),$$

where

$$X_{p,w} := \prod_{\alpha\in\Phi^+\backslash\Phi_P^+} \widehat{\zeta}\left(\langle\lambda_s+\rho,\alpha^\vee\rangle+\delta_{\alpha,w}\right), \qquad Q_{p,w}(s) := C_{P,w}\cdot\widetilde{Q}_{P,w}(s),$$

with $C_{p,w} := \widehat{\zeta}\,(2)^{|\Delta_P\cap w^{-1}\Phi^+|}\prod_{\alpha\in\Phi_P^+\backslash\Delta_P}\widehat{\zeta}\left(\langle\rho,\alpha^\vee\rangle+\delta_{\alpha,w}\right)$, consisting of special zeta values, and $\widetilde{Q}_{P,w}(s) := \dfrac{Q_P(s)}{q_{P,w}(s)}$, consisting of rational functions.

Let now

$$l_p(w) := \sum_{\alpha\in\Phi^+\backslash\Phi_P^+}(1-\delta_{\alpha,w}).$$

Then, $l_P(w) = \#\big(\Phi_w\backslash\Phi_P^+\big)$, from which we get a natural decomposition of $\mathfrak{W}_P$ by

$$\begin{aligned} \mathfrak{W}_p^< &:= \{w\in\mathfrak{W}_P \mid l_p(w) < \#\big(\Phi^+\backslash\Phi_P^+\big)\}, \\ \mathfrak{W}_p^o &:= \{w\in\mathfrak{W}_P \mid l_p(w) = \#\big(\Phi^+\backslash\Phi_P^+\big)\}, \\ \mathfrak{W}_p^> &:= \{w\in\mathfrak{W}_P \mid l_p(w) > \#\big(\Phi^+\backslash\Phi_P^+\big)\}. \end{aligned}$$

Consequently, by Prop. 5.8 of [KKS], the up-shot of this discussion, we have

$$X_P(s) = E_P(s) \pm E_P(-c_P-s). \tag{$**$}$$

where

$$E_P(s) := \sum_{w\in\mathfrak{W}_P^<} Q_{P,w}(s)X_{P,w}(s) + \frac{1}{2}\sum_{w\in\mathfrak{W}_P^o} Q_{P,w}(s)X_{P,w}(s).$$

Here, if $\mathfrak{W}_P^o=\emptyset$, the second term is defined to be zero. In particular,

$$X_P(-c_P-s) = X_P(s).$$

Finally, introduce

$$\xi^{G/P}(s) := \frac{X_P(s)}{R_P(s)D_P(s)}$$

where

$$D_P(s) := \prod_{k=1}^{\infty}\prod_{h=2}^{\infty}\xi(ks+h)^{N_P(k,h-1)-M_P(k,h)},$$
$$R_P(s) := \text{g.c.d.}\big\{Q_{P,w} : w \in \mathfrak{W}_P\big\}.$$

Then, by (**), for $\varepsilon_P(s) := \dfrac{E_P(s)}{R_P(s)D_P(s)}$,

$$\xi^{G/P}(s) = \varepsilon_P(s) \pm \varepsilon_P(-c_P - s).$$

Moreover, by (*), or better by [KKS, §5.2], $\xi^{G/P}(s)$ is an entire function obtained from $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ by changing $\widehat{\zeta}(s)$ to $\xi(s) := s(s-1)\cdot\widehat{\zeta}(s)$ first and then multiplying the resulting function with the least common multiple of all polynomials appeared in the denominators of T_w for $w \in \mathfrak{W}_P$. In particular, $\xi^{G/P}(s)$ has the same non-trivial zeros as $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$ away from real axis. So to understand distributions of zeros of $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$, it suffices to treat $\xi^{G/P}(s)$.

Step 2. *Asymptotic behaviors of* $\arg\varepsilon_P\big(-c_P/2+\sqrt{-1}\,t\big)$.

Let $\theta_P(t)$ be the argument of $\varepsilon_P\big(-c_P/2+it\big)$. We have

$$\xi_P\Big(-\frac{c_P}{2}+it\Big) = \Big|\varepsilon_P\Big(-\frac{c_P}{2}+it\Big)\Big|\cdot\Big(e^{i\theta_P(t)} \pm e^{-i\theta_P(t)}\Big),$$

since $\overline{\varepsilon_P\big(-c_P/2+it\big)} = \varepsilon_P\big(-c_P/2-it\big)$. Hence, the zeros of $\xi_P(-c_P/2+it)$ correspond in one-to-one with the zeros of $\cos\theta_P(t)$ or $\sin\theta_P(t)$, or better, with the solutions of either $\theta_P(t) \in \dfrac{\pi}{2}+\pi\,\mathbb{Z}$ or $\theta_P(t) \in \pi\,\mathbb{Z}$. Therefore, to understand distributions of these zeros, it suffices to obtain asymptotic behaviors of $\theta_P(t)$ when $|t| \to +\infty$. For this purpose, let

$$Q_P^{\ddagger}(s) := \sum_{w\in\mathfrak{W}_P^{\ddagger}} Q_{p,w}(s) \qquad \text{with} \qquad \mathfrak{W}_P^{\ddagger} := \{w \in \mathfrak{W}_P \,|\, l_p(w) = 0\}.$$

Then by (6.2) of [KKS, p.16],

$$\varepsilon_P(s) = \frac{Q_P^{\ddagger}(s)}{R_P(s)}\cdot\frac{X_{P,\mathrm{id}}(s)}{D_P(s)}\cdot\Big(1+r_P(s)\Big) \qquad \text{and} \qquad \big|\,r_p(s)\,\big| < 1.$$

Since $\big|\,r_p(s)\,\big| < 1$, $\arg\big(1+r_P(s)\big) \le \frac{\pi}{2}$. Consequently,

$$\theta_P(t) = \arg\Big(\frac{Q_P^{\ddagger}(s)}{R_P(s)}\Big|_{s=-\frac{c_P}{2}+it}\Big) + \arg X_{P,\mathrm{id}}\Big(-\frac{c_P}{2}+it\Big) + O(1).$$

The first term is simply $O(1)$, since $Q^{\ddagger}_P(s)$, $R_P(s)$ are polynomials. To treat the second term, we use the formula (9.3) of [KKS]

$$\frac{X_{P,\mathrm{id}}(s)}{D_p(s)} = \prod_{k=1}^{\infty} \prod_{h>(kc_P+1)/2} \xi(ks+h)^{N_P(k,h-1)-N_P(k,h)}.$$

Note that, when $s = -\frac{c_P}{2} + it$, $\mathrm{Re}\,(ks+h) = -\frac{c_P}{2}k + h > \frac{1}{2}$. Thus, recall that the above products are of finite type, we have, by the Stirlings formula,

$$\begin{aligned}
\arg \frac{X_{P,\mathrm{id}}(s)}{D_P(s)}\Big|_{s=-\frac{c_P}{2}+it} =& \sum_{k=1}^{\infty} \sum_{h>(kc_P+1)/2} \Big(N_P(k,h-1) - N_P(k,h)\Big) \\
& \times \Big(\arg(ks+h)(ks+h-1)\big|_{s=-\frac{c_P}{2}+it} \\
& \qquad + \arg \pi^{\frac{c_P}{4}k - \frac{h}{2} - \frac{ikt}{2}} + \arg \Gamma\big(-\frac{c_P}{4}k + \frac{h}{2} + \frac{ikt}{2}\big) \\
& \qquad + \arg \zeta\big(-\frac{c_P}{2}k + h + ikt\big)\Big) \\
= \sum_{k=1}^{\infty} \sum_{h>(kc_P+1)/2} & \Big(N_P(k,h-1) - N_P(k,h)\Big) \\
& \times\Big(O(1) - \frac{k}{2}t\log\pi + \frac{k}{2}t\big(\log(\frac{k}{2}t) - 1\big) + O(\frac{1}{t}) + O(\log t)\Big) \\
= \sum_{k=1}^{\infty} N_P\Big(k, [\frac{kc_P-1}{2}]\Big) & \cdot \Big(\frac{k}{2}t\,\big(\log t - \log(2\pi e) + \log k\big) + O(\log t)\Big).
\end{aligned}$$

Here, to conclude that $\arg\zeta\Big(-\dfrac{c_P}{2}k + h + ikt\Big) = O(\log|t|)$, we have used the original Riemann Hypothesis when $\frac{1}{2} < -\frac{c_P}{2}k + h < 1$ and the following classical lemma when $-\frac{c_P}{2}k + h \geq 1$.

Lemma 14. *([Lem 9.4, T], [Lem 12.1, KKS]) Let $0 \leq \alpha < \beta < \sigma_0$, $T > 10$. Let $f(s)$ be an analytic function, real valued for real s, and regular for $\sigma \geq \alpha$ except at finitely many poles on the real line. If*

$$\big|\mathrm{Re}\big(f(\sigma+it)\big)\big| \geq m > 0$$

and

$$|f(\sigma_1 + it_1)| \leq M_{\sigma,t} \qquad \forall \sigma_1 \geq \sigma,\ 1 \leq t_1 \leq t.$$

Then, for any T different from ordinate of a zero of $f(s)$,

$$\Big|\arg f(\sigma+iT)\Big| \leq \frac{\pi}{\log\frac{\sigma_0-\alpha}{\sigma_0-\beta}}\Big(\log M_{\alpha,T+2} + \log\frac{1}{m}\Big) + \frac{3}{2}\pi.$$

All this then proves the following

Proposition 15. *We have*

$$\theta_P(T) = T\log T\cdot d_P + T\cdot\big(e_P - d_P\log(2\pi e)\big) + O(\log T)$$

Step 3. *Distributions of zeros for* $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$.

To complete our proof of Theorems 9 and 10, we use

Lemma 16. *Assume that*

$$\theta_P(\gamma_{n+1}^{G/P}) - \theta_P(\gamma_n^{G/P}) = C, \qquad N^{G/P}(\gamma_n^{G/P}) \sim \frac{1}{C'}\,\theta_P(\gamma_n^{G/P}) + O(1),$$

and that $N^{G/P}(T) = C_1 T\log T + C_2 T + O(\log T)$. *Then*

$$\gamma_n^{G/P} = \frac{1}{C_1}\frac{n}{\log n}\Big(1+O\big(\frac{1}{\log n}\big)\Big); \qquad \gamma_{n+1}^{G/P}-\gamma_n^{G/P} = \frac{1}{C_1}\frac{1}{\log n}+O\Big(\frac{1}{\log^2 n}\Big).$$

Proof. We start with the dominant term for $\gamma_n = \gamma_n^{G/P}$. From our assumption on $N = N^{G/P}$, $N(\gamma_n\pm 1) \sim C_1(\gamma_n\pm 1)\log(\gamma_n\pm 1) + C_2(\gamma_n\pm 1) \sim C_1\gamma_n\log\gamma_n$. But, by definition, $N(\gamma_n - 1)\leq n\leq N(\gamma_n+1)$. Hence $n\sim C_1\gamma_n\log\gamma_n$ and $\log n\sim\log\gamma_n$. Consequently, $\gamma_n\sim\frac{1}{C_1}\frac{n}{\log n}$. To get the precise asymptotic behaviors, we use

$$N(\gamma_n\pm 1) = C_1(\gamma_n\pm 1)\log(\gamma_n\pm 1) + C_2(\gamma_n\pm 1) + O\big(\log(\gamma_n\pm 1)\big).$$

As above, then, we get $n = C_1\gamma_n\log\gamma_n + O(\gamma_n)$, or better, since $\gamma_n\sim\frac{1}{C_1}\frac{n}{\log n}$, $C_1\gamma_n\log\gamma_n = n\cdot\Big(1+O\big(\frac{1}{\log n}\big)\Big)$. Therefore $\gamma_n = \frac{1}{C_1}\cdot\frac{n}{\log n}\Big(1+O\big(\frac{1}{\log n}\big)\Big)$.

To prove the second statement, we shift our attention to $\theta = \theta_P$. Then, for $T\gg 0$, $\Delta T > 0$, we have

$$\begin{aligned}\theta(T+\Delta T) - \theta(T) &= C' N^{G/P}(T+\Delta T) - C' N^{G/P}(T)\\ &= C'C_1 T\log\frac{T+\Delta T}{T} + C'C_1\Delta T\log(T+\Delta T) + C'C_2\Delta T + O\big(\log\frac{T+\Delta T}{T}\big)\\ &= C'C_1\Delta T\big(\log T + 1\big) + O\big(\frac{1}{T}\big),\end{aligned}$$

since $T\log\dfrac{T+\Delta T}{T} = \log\big(1+\dfrac{\Delta T}{T}\big)^T = O(1)$. In particular, by taking $T = \gamma_n$ and $\Delta T = \gamma_{n+1}-\gamma_n$, we get

$$C = \theta(\gamma_{n+1}) - \theta(\gamma_n) = C'C_1(\gamma_{n+1}-\gamma_n)\big(\log\gamma_n + 1\big) + O\big(1/\gamma_n\big).$$

Hence $\gamma_{n+1}-\gamma_n\sim\frac{C}{C'C_1}\frac{1}{\log\gamma_n}$. So $C = C'C_1(\gamma_{n+1}-\gamma_n)\log\gamma_n + O(\frac{1}{\log\gamma_n})$. Therefore,

$$\gamma_{n+1}-\gamma_n = \frac{C}{C'C_1}\frac{1}{\log n} + O\big(\frac{1}{\log^2 n}\big).$$

This then completes the proof of the lemma and hence also Theorems 9, 10, since, for $\widehat{\zeta}_{\mathbb{Q}}^{G/P}(s)$, we have $C = C' = \pi$.

Proof of Theorems 1 and 2. Theorem 2 is a direct consequence of Theorem 1. As for Theorem 1, note that, when $P = P_{n-1,1}$, $\Phi^+ = \{\boldsymbol{e}_i - \boldsymbol{e}_j \,|\, 1 \leq i < j \leq n\}$, $\rho = \frac{1}{2}\sum_{i=1}^{n}(n+1-2i)\,\boldsymbol{e}_i$, and $\lambda_P = \frac{1}{n}\big(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{n-1} - (n-1)\,\boldsymbol{e}_n\big)$. So, for $i < j$, $\langle \lambda_P, \boldsymbol{e}_i - \boldsymbol{e}_j \rangle = \delta_{jn}$. Consequently, $N_P(k,h) = 0$ unless $k \leq 1$. This implies that $e_P = 0$ and $2d_P = N_P\big(1, [n - 1/2]\big)$ when $n \geq 3$. By a direct calculation, we know that then $d_P = 1/2$. Consequently, we have, for the k, h involved, $-c_P/2\,k + h \geq 1$. So, for zeros of non-abelian zeta functions, we do not really need to assume the Riemann Hypothesis as in Step 2 above. This, together with Theorems 8(1) and 9, completes the proof of Theorem 1 (and hence also Theorem 11).

To end this paper, from Proposition 13, we define the big Delta pair correlation function for rank 2 zeta $\widehat{\zeta}_{\mathbb{Q},2}(s)$ by

$$\Delta_{2,k} = \big(\delta_{2,k} - 1\big) \cdot \log\big(\log \frac{\gamma_{2,k}}{\pi}\big). \tag{3}$$

Motivated by GUE, we calculate the first 138,068 zeros of $\widehat{\zeta}_{\mathbb{Q},2}(\sigma + it)$ (with $0 \leq t \leq 50,000$), using a Mathematica program of Katayama-Suzuki-Weng. As a result, we are able to obtain the following very impressive figure for the distributions of zeros of rank 2 zeta function $\widehat{\zeta}_{\mathbb{Q},2}(s)$ using $\Delta_{2,k}$.

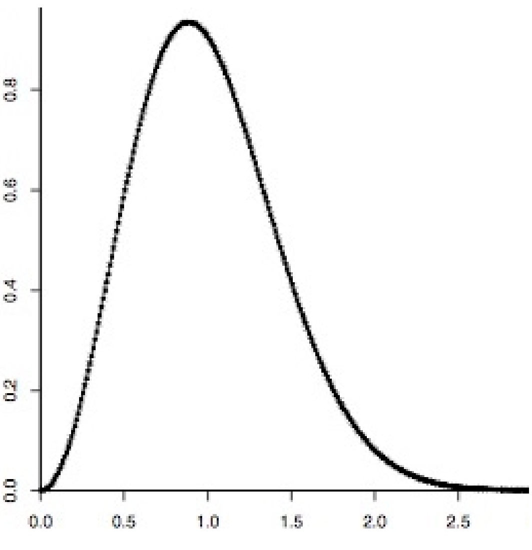

Figure 1: Conjectural distributions of Riemann zeta zeros using classical δ_n

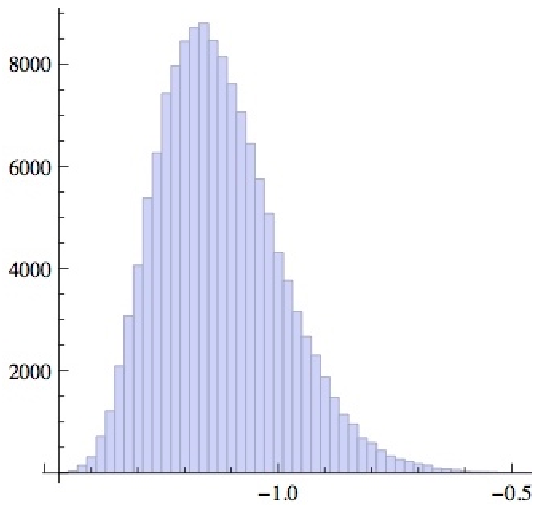

Figure 2: Distributions of rank two zeta zeros using secondary $\Delta_{2,n}$

For details of the first 138,068 zeros of rank 2 zeta function $\widehat{\zeta}_{\mathbb{Q},2}(s)$, please refer to our web page http://www2.math.kyushu-u.ac.jp/∼weng/zetas.html

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