Commun. Math. Phys.
Digital Object Identifier (DOI) 10.1007/s00220-008-0494-5

Communications in
Mathematical Physics

Deligne Products of Line Bundles over Moduli Spaces of Curves

L. Weng[1,⋆], **D. Zagier**[2,3]

[1] Graduate School of Mathematics, Kyushu University, Fukuoka, Japan.
E-mail: weng@math.kyushu-u.ac.jp
[2] Max-Planck-Institut für Mathematik, Bonn, Germany
[3] Collège de France, Paris, France

Received: 22 June 2007 / Accepted: 9 October 2007

Abstract: We study Deligne products for forgetful maps between moduli spaces of marked curves by offering a closed formula for tautological line bundles associated to marked points. In particular, we show that the Deligne products for line bundles on the total spaces corresponding to "forgotten" marked points are positive integral multiples of the Weil-Petersson bundles on the base moduli spaces.

1. Introduction

Let $\mathcal{M}_{g,N}$ denote the moduli space of curves C of genus g with N ordered marked points $P_1, \ldots, P_N$, and $\pi = \pi_N : \mathcal{C}_{g,N} \to \mathcal{M}_{g,N}$ the universal curve over $\mathcal{M}_{g,N}$. (We are using the language of stacks here [3].) The marked points give sections $\mathbf{P}_i : \mathcal{M}_{g,N} \to \mathcal{C}_{g,N}$, $i = 1, \ldots, N$ of π.

The Picard group of $\mathcal{M}_{g,N}$ is known to be free of rank $N + 1$ [4] and has a $\mathbb{Z}$-basis given by the Mumford class λ (the line bundle whose fiber at C is $\det H^0(C, K_C) \otimes \det H^1(C, K_C)^{-1}$) and the "tautological line bundles" $\ell_i := \mathbf{P}_i^*(K_N)$, where K_N is the relative canonical line bundle (relative dualizing sheaf) of π [1,5]. The ℓ_i carry metrics in such a way that their first Chern forms give the Kähler metrics on $\mathcal{M}_{g,N}$ defined by Takhtajan-Zograf [9,10] in terms of Eisenstein series associated to punctured Riemann surfaces [11,12]. There is a further interesting element $\Delta \in \mathrm{Pic}(\mathcal{M}_{g,N})$, whose associated first Chern form (for a certain natural metric) gives the Kähler form for the Weil-Petersson metric on $\mathcal{M}_{g,N}$ [11,12]; it is given in terms of λ and the ℓ_i by the Riemann-Roch formula $\Delta = 12\lambda + \sum_{i=1}^N \ell_i$. We can also define Δ by the formula $\Delta := \big\langle K_N(\mathbf{P}_1 + \cdots + \mathbf{P}_N), K_N(\mathbf{P}_1 + \cdots + \mathbf{P}_N)\big\rangle_\pi$, where $\langle\cdot,\cdot\rangle_\pi : \mathrm{Pic}(\mathcal{C}_{g,N})^2 \to \mathrm{Pic}(\mathcal{M}_{g,N})$ denotes the Deligne pairing. We recall that the Deligne pairing is a bilinear map $\langle\, \cdot, \cdot \,\rangle_p : \mathrm{Pic}(Y)^2 \to \mathrm{Pic}(X)$ which is defined for any flat morphism $p : Y \to X$

⋆ Partially supported by the Japan Society for the Promotion of Science.

of relative dimension 1 and that it can be generalized to a multilinear map $\langle \cdot, \dots, \cdot \rangle_p$: $\mathrm{Pic}(Y)^{n+1} \to \mathrm{Pic}(X)$, the Deligne product, defined for any flat morphism $p : Y \to X$ of relative dimension n. (The precise definitions will be given in §3.)

We are interested in computing the Deligne product explicitly for the forgetful map

$$\pi_{N,m} : \mathcal{M}_{g,N+m} \to \mathcal{M}_{g,N}, \qquad (C; P_1, \dots, P_{N+m}) \mapsto (C; P_1, \dots, P_N).$$

In other words, if we use $\tilde{\lambda}, \tilde{\Delta}, \tilde{\ell}_j$ to denote the Mumford, Weil-Petersson and tautological line bundles on $\mathcal{M}_{g,N+m}$, respectively, then we would like to compute $\langle L_1, \dots, L_{m+1} \rangle_{\pi_{N,m}}$ as a linear combination of $\ell_1, \dots, \ell_N$ and λ (or Δ) on $\mathcal{M}_{g,N}$, where each L_ν is one of $\tilde{\ell}_1, \dots, \tilde{\ell}_{N+m}$ and $\tilde{\lambda}$ (or $\tilde{\Delta}$). We have not solved this problem in general (though it is interesting and perhaps not intractable), but only in the case where each L_ν is one of the $\tilde{\ell}_i$, i.e., where $\tilde{\lambda}$ (or $\tilde{\Delta}$) does not appear. The formula we find expresses $\langle L_1, \dots, L_{m+1} \rangle_{\pi_{N,m}}$ in this case as a positive linear combination of Δ and those ℓ_i ($i = 1, \dots, N$) for which $\tilde{\ell}_i$ appear among the L_ν. In particular, if each of $L_1, \dots, L_{m+1}$ is one of the last m line bundles $\tilde{\ell}_{N+1}, \dots, \tilde{\ell}_{N+m}$, then $\langle L_1, \dots, L_{m+1} \rangle_{\pi_{N,m}}$ is simply a positive integer multiple of the Weil-Petersson class Δ, giving an interesting relation between the Weil-Petersson and the tautological line bundles.

2. Statement of the Theorem

As just explained, we want to compute the Deligne product $\langle L_1, \dots, L_{m+1} \rangle_{\pi_{N,m}}$, where each L_ν belongs to the set $\{\tilde{\ell}_1, \dots, \tilde{\ell}_{N+m}\}$. It turns out to be more convenient to use the multiplicities where the $\tilde{\ell}_i$ occur in $\{L_1, \dots, L_{m+1}\}$ as coordinates. We therefore introduce the notation

$$T_{N,m}(a_1, \dots, a_{N+m}) := \langle \underbrace{\tilde{\ell}_1, \dots, \tilde{\ell}_1}_{a_1}, \ \dots, \ \underbrace{\tilde{\ell}_{N+m}, \dots, \tilde{\ell}_{N+m}}_{a_{N+m}} \rangle_{\pi_{N,m}} \in \mathrm{Pic}(\mathcal{M}_{g,N}), \tag{1}$$

where $a_1, \dots, a_{N+m} \in \mathbb{Z}_{\geq 0}$ with $a_1 + \cdots + a_{N+m} = m + 1$. We will sometimes denote this element by $T_{N,m}(a_1, \dots, a_N; a_{N+1}, \dots, a_{N+m})$ or even, setting $a_{N+i} =: d_i$, by $T_{N,m}(a_1, \dots, a_N; d_1, \dots, d_m)$, to emphasize the different roles played by the indices corresponding to the points which are also marked in $\mathcal{M}_{g,N}$ and to those which are "forgotten" by the projection map $\pi_{N,m}$. Our main result is then:

Theorem. *Let $a_1, \dots, a_N, d_1, \dots, d_m \geq 0$ be non-negative integers with sum $m + 1$. Then the line bundle $T_{N,m}(a_1, \dots, a_N; d_1, \dots, d_m)$ defined in* (1) *is given in terms of the elements ℓ_i, $\Delta \in Pic(\mathcal{M}_{g,N})$ by the formula*

$$\left(\prod_{i=1}^{N} a_i!\right)\left(\prod_{j=1}^{m} d_j!\right) T_{N,m}(a_1, \dots, a_N; d_1, \dots, d_m) = C_1(\mathbf{d}) \sum_{i=1}^{N} a_i \left(\ell_i - \frac{\Delta}{\widetilde{N}}\right) + C_2(\mathbf{d})\, \Delta, \tag{2}$$

where $\widetilde{N} = N + 2g - 3$ *and the coefficients* $C_\nu(\mathbf{d}) = C_\nu(\widetilde{N}, d_1, \ldots, d_m)$ $(\nu = 1, 2)$ *depend only on the* d_i *and on* $\widetilde{N}$ *and are given explicitly by*

$$\begin{aligned} C_1(\mathbf{d}) &= \sum_{n=0}^{m} \frac{(m-n)!\,(\widetilde{N}+n-1)!}{(\widetilde{N}-1)!}\,\sigma_n, \\ C_2(\mathbf{d}) &= \sum_{n=0}^{m} \frac{(m-n+1)!\,(\widetilde{N}+n-2)!}{\widetilde{N}(\widetilde{N}-2)!}\,\sigma_n \end{aligned} \tag{3}$$

with $\sigma_n = \sigma_n(d_1, \ldots, d_m)$ *the* n^{th} *elementary symmetric polynomial in the* d_i*.*

Remark. 1. If $\widetilde{N} = 0$ or 1, then the factors $1/(\widetilde{N}-1)!$ and $1/\widetilde{N}(\widetilde{N}-2)!$ occurring in the formulas for $C_1(\mathbf{d})$ and $C_2(\mathbf{d})$ are to be interpreted as $\widetilde{N}/\widetilde{N}!$ and $(\widetilde{N}-1)/\widetilde{N}!$, respectively.

2. The proof (or rather, the recursive description of the $T_{N,m}$ on which it is based) will show that in the formula for $T_{N,m}(a_1, \ldots, a_N; d_1, \ldots, d_m)$ in terms of ℓ_i and Δ, all the coefficients are non-negative and integral (even though $\ell_1, \ldots, \ell_N, \Delta$ is not a $\mathbb{Z}$-basis of $\mathrm{Pic}(\mathcal{M}_{g,N})$). Neither property is obvious from the formulas (2) and (3), though one can see easily that both $C_1(\mathbf{d})$ and $C_2(\mathbf{d}) - (m+1-\sigma_1)C_1(\mathbf{d})/\widetilde{N}$, the coefficient of Δ on the right-hand side of (2), are polynomials in $\widetilde{N}$.
3. In Sect. 6 we will give an alternative explicit formula for the coefficients $C_1(\mathbf{d})$ and $C_2(\mathbf{d})$.
4. Notice that, as already mentioned in the Introduction, formula (2) in the special case when all the a_i are 0 says that $T_{N,m}(0, \ldots, 0; d_1, \ldots, d_m)$ is a multiple of Δ alone. In other words, all Deligne products of line bundles corresponding to points which are "forgotten" by $\pi_{N,m}$ are positive integral multiples of the Weil-Petersson bundle Δ.

3. The Deligne product

We start with some basic facts about Deligne products [2]. Let $\pi : X \to S$ be a flat family of algebraic varieties of relative dimension n. Then for any $n+1$ invertible sheaves $L_0, \ldots, L_n$ over X, following Deligne [2], we may introduce the *Deligne product*, denoted by $\langle L_0, \ldots, L_n\rangle(X/S)$ or $\langle L_0, \ldots, L_n\rangle_\pi$ or simply $\langle L_0, \ldots, L_n\rangle$, defined uniquely by the following axioms:

(**DP1**) $\langle L_0, \ldots, L_n\rangle(X/S)$ is an invertible sheaf on S, and is symmetric and multi-linear in the L_i's.

(**DP2**) $\langle L_0, \ldots, L_n\rangle(X/S)$ is locally generated by the symbols $\langle t_0, \ldots, t_n\rangle$, where the t_i are sections of L_i whose divisors have no common intersection, and these symbols satisfy the following property: if one multiplies one of the sections t_i by a rational function f on X, where $\bigcap_{j\neq i} \mathrm{div}(t_j) = \sum n_k Y_k$ is finite over S and $\mathrm{div}(f)$ has no intersection with any Y_k, then

$$\langle t_0, \ldots, f t_i, \ldots, t_n\rangle = \Big(\prod_k \mathrm{Norm}_{Y_k/S}(f)^{n_k}\Big) \cdot \langle t_0, \ldots, t_n\rangle.$$

(Here $\mathrm{Norm}_{Y_k/S}(f)$ is defined as follows: Since Y_k is finite over S, the function field of Y_k is a finite extension of that of S, and hence can be viewed as a finite dimensional vector space. Since f is in the function field of X, via restriction we may view f as an element of the function field of Y_k. Then multiplication by f defines a linear map of the vector space of the function field of Y_k over the function field of S. By definition, the determinant of this linear map is called the norm of f with respect to Y_k/S.)

(**DP3**) If t_n is a section of L_n such that all components D_α of the divisor $\mathrm{div}(t_n) = \sum_\alpha n_\alpha D_\alpha$ are flat (of relative dimension $n-1$) over S, then we have a canonical isomorphism

$$\langle L_0, \ldots, L_n\rangle(X/S) \simeq \langle L_0, \ldots, L_{n-1}\rangle(\mathrm{div}(t_n)/S) := \otimes_\alpha \langle L_0, \ldots, L_{n-1}\rangle(D_\alpha/S)^{\otimes n_\alpha}.$$

Roughly speaking, the Deligne product may be built up as follows using the above axioms: one first uses (DP3) to make an induction on the relative dimension so as to reduce to special cases, say, $n = 1$, by using a certain choice of sections, and then shows with the help of axiom (DP2) that this construction does not depend on the choice of sections of line bundles and is symmetric by virtue of the Weil reciprocity law.

As a consequence of these axioms and the uniqueness, we know that the Deligne product formalism is compatible with any base change, and that the products satisfy the following compatibility relations with respect to compositions of flat morphisms:

Proposition 1. ([2]). *Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimension n and m, respectively. Then:*

(a) For invertible sheaves $L_0, \ldots, L_n$ on X and $H_1, \ldots, H_m$ on Y, we have

$$\langle\langle L_0, \ldots, L_n\rangle_f,\ H_1, \ldots, H_m\rangle_g \simeq \langle L_0, \ldots, L_n, f^*H_1, \ldots, f^*H_m\rangle_{g\circ f}. \quad (4a)$$

(b) For invertible sheaves $L_1, \ldots, L_n$ on X and $H_0, \ldots, H_m$ on Y, we have

$$\langle\langle f^*H_0, L_1 \ldots, L_n\rangle_f,\ H_1, \ldots, H_m\rangle_g \simeq \langle H_0, H_1, \ldots, H_m\rangle_g^{f_*(c_1(L_1)\cdots c_1(L_n))}. \quad (4b)$$

A special case of Proposition 1 which will be needed later is the formula

$$\langle L, f^*H_0, \ldots, f^*H_m\rangle_{g\circ f} \simeq \deg_f(L)\,\langle H_0, \ldots, H_m\rangle_g \quad (5)$$

for f, g as in the proposition with $n = 1$ and for any bundles L on X and $H_0, \ldots, H_m$ on Y. To get this, we use part (a) of the proposition to write the left-hand side as $\langle\langle L, f^*H_0\rangle_f, H_1, \ldots, H_m\rangle_g$ and then part (b) to write $\langle L, f^*H_0\rangle_f$ as $\deg_f(L)H_0$.

Remark. Recall that there is a map c_1 from $\mathrm{Pic}(X)$ to the codimension 1 part $\mathrm{CH}^1(X)$ of the Chow group of X. If $f : X \to Y$ is flat of relative dimension n and $L_0, \ldots, L_n$ belong to $\mathrm{Pic}(X)$, then the image of $\langle L_0, \ldots, L_n\rangle$ under c_1 is equal to the image of the product $c_1(L_0)\cdots c_1(L_n)$ under the push-forward map $f_* : \mathrm{CH}^{n+1}(X) \to \mathrm{CH}^1(Y)$. At this level, formulas (4a), (4b) and (5) are just specializations of the general projection formula $g_*(f_*(A)\cdot B) = (gf)_*(A \cdot f^*(B))$, valid for any flat morphisms $f : X \to Y$ and $g : Y \to Z$ and elements $A \in \mathrm{CH}(X)$, $B \in \mathrm{CH}(Y)$.

4. Geometric Preparations

We now apply the Deligne product to universal curves over moduli spaces. As in §1, we denote by K_N the relative canonical line bundle of $\pi = \pi_N : \mathcal{C}_{g,N} \to \mathcal{M}_{g,N}$, by $\mathbf{P}_i$ $(1 \le i \le N)$ the N sections of π and by $\ell_i = \mathbf{P}_i^*(K_N)$ the i^{th} tautological line bundle on $\mathcal{M}_{g,N}$. We write L_i $(1 \le i \le N+1)$ for the line bundles on $\mathcal{C}_{g,N}$ defined in the same way, where $\mathcal{C}_{g,N}$ is identified with $\mathcal{M}_{g,N+1}$. Also for convenience, we denote the bundle $\mathcal{O}_{\mathcal{C}_{g,N}}(\mathbf{P}_i)$ $(1 \le i \le N)$ simply by $\mathbf{P}_i$. The following properties can be found in [6,7]. (Deligne products are not used in Knudsen's original papers, but the verbatim change is rather trivial. See e.g., [11,12].)

Proposition 2. ([6]) *With the above notations, we have*

(a) $\langle \mathbf{P}_i, \mathbf{P}_j \rangle_\pi \simeq \mathcal{O} \quad (i, j = 1, \ldots, N, \quad i \neq j);$
(b) $\langle K_N(\mathbf{P}_i), \mathbf{P}_i \rangle_\pi \simeq \mathcal{O} \quad (i = 1, \ldots, N);$
(c) $L_i \simeq \pi^*\ell_i + \mathbf{P}_i \quad (i = 1, \ldots, N);$
(d) $L_{N+1} \simeq K_N(\mathbf{P}_1 + \cdots + \mathbf{P}_N).$

The next proposition, which is a slight extension of Proposition 2, contains all the geometric information which we will need to compute the Deligne products in (1). We use the same notations as above, but also denote by $\pi' = \pi_{N-m,m}$ the forgetful map from $\mathcal{M}_{g,N}$ to $\mathcal{M}_{g,N-m}$ for some $m \ge 0$ and use $\xi_1, \ldots, \xi_m$ to denote m general elements of $\mathrm{Pic}(\mathcal{C}_{g,N})$.

Proposition 3. *With the above notations, we have*

(a) $\langle \mathbf{P}_i, \mathbf{P}_j, \xi_1, \ldots, \xi_m \rangle_{\pi' \circ \pi} \simeq \mathcal{O} \quad (i, j = 1, \ldots, N, \quad i \neq j);$
(b) $\langle \mathbf{P}_i, L_i, \xi_1, \ldots, \xi_m \rangle_{\pi' \circ \pi} \simeq \mathcal{O} \quad (i = 1, \ldots, N);$
(c) $\langle \mathbf{P}_i, L_{N+1} \rangle_\pi \simeq \mathcal{O} \quad (i = 1, \ldots, N);$
(d) $\deg_\pi (L_{N+1}) = 2g - 2 + N.$

Proof. Since the sections $\mathbf{P}_i$ and $\mathbf{P}_j$ are disjoint for $i \neq j$, the pull-back of $\mathcal{O}(\mathbf{P}_i)$ to $\mathbf{P}_j$ is trivial (Prop. 2(a)). Therefore axiom (DP3) from §3 implies (a). Next, we use Prop. 2(c) to write

$$\langle L_i, \mathbf{P}_i, \xi_1, \ldots, \xi_m \rangle_{\pi' \circ \pi} \simeq \langle \pi^*\ell_i, \mathbf{P}_i, \xi_1, \ldots, \xi_m \rangle_{\pi' \circ \pi} + \langle \mathbf{P}_i, \mathbf{P}_i, \xi_1, \ldots, \xi_m \rangle_{\pi' \circ \pi}.$$

By (DP3), the first term equals $\langle \ell_i, \xi_1|_{\mathbf{P}_i}, \ldots, \xi_m|_{\mathbf{P}_i} \rangle_{\pi'}$ (actually multiplied by $\deg_\pi(\mathbf{P}_i)$, but the relative degree of a section is 1), while the second term is $-\langle \ell_i, \xi_1|_{\mathbf{P}_i}, \ldots, \xi_m|_{\mathbf{P}_i} \rangle_{\pi'}$ by Prop. 2(b) (adjunction formula). This proves (b). Part (c) follows from Prop. 2(d), since $\langle \mathbf{P}_i, \ K_N(\mathbf{P}_i) \rangle_\pi$ vanishes by the adjunction formula and all $\langle \mathbf{P}_i, \mathbf{P}_j \rangle_\pi$ with $j \neq i$ vanish by (a). Part (d) also follows from Prop. 2(d) by taking the relative degree of both sides. We also mention the stronger statement that $\langle L_{N+1}, L_i \rangle_\pi \simeq (2g - 2 + N)\, \ell_i$ for $i = 1, \ldots, N$. The proof of this is similar to the other parts of the proposition, but we omit it since this result will not be used in the sequel. □

5. The Recursion Formula for $T_{N,m}$

In this section, we use the results of §4 to give a recursion formula and initial data for the line bundles (1) which determine them completely in $\mathrm{Pic}(\mathcal{M}_{g,N})$. These recursions will be solved in §6.

The recursion formula which we will prove for the $T_{N,m}$ is as follows.

Proposition 4. (String Equation) *For $m \geq 0$ and any integers $a_1, \ldots, a_{N+m} \geq 0$, we have*

$$T_{N,m+1}(a_1, \ldots, a_{N+m}, 0) = \sum_{i=1}^{N+m} T_{N,m}(a_1, \ldots, a_i - 1, \ldots, a_{N+m}),$$

with the convention that $T_{N,m+1}(a_1, \ldots, a_{N+m}) = 0$ if any $a_i < 0$.

Recall that the indices a_i with $i > N$ in $T_{N,m}(a_1, \ldots, a_{N+m})$ play a different role than the a_i with $i \leq N$ and that we also use the notations d_j for a_{N+j} $(1 \leq j \leq m)$ and $T_{N,m}(a_1, \ldots, a_N; d_1, \ldots, d_m)$ for $T_{N,m}(a_1, \ldots, a_{N+m})$. Proposition 4 lets us reduce the calculation of these bundles by induction to the case when every d_i is strictly positive. (If any d_j is zero, we can put it in the last position, because $T_{N,m}$ is symmetric in the d's.) But since $\sum_{i=1}^{N} a_i + \sum_{j=1}^{m} d_j = m+1$, this can only happen if $(d_1, \ldots, d_m) = (1, \ldots, 1)$ or $(2, 1, \ldots, 1)$. There are therefore only two initial cases which have to be considered. The values of $T_{N,m}$ in these two cases are given by the following:

Proposition 5. *The line bundles $T_{N,m}(a_1, \ldots, a_N; d_1, \ldots, d_m)$ in the two cases when all the d_i are strictly positive are given by the formulas*

$$T_{N,m}(1, \underbrace{0, \ldots, 0}_{N-1}; \underbrace{1, \ldots, 1}_{m}) = \frac{(\widetilde{N} + m)!}{\widetilde{N}!}\, \ell_1 \qquad (m \geq 0)$$

and

$$T_{N,m}(\underbrace{0, \ldots, 0}_{N}; 2, \underbrace{1, \ldots, 1}_{m-1}) = \frac{(\widetilde{N} + m)!}{(\widetilde{N} + 1)!}\, \Delta \qquad (m \geq 1),$$

where $\widetilde{N} = N + 2g - 3$.

Proposition 5 in turn can be deduced by induction over m from the special cases

$$T_{N,0}(1, \underbrace{0, \ldots, 0}_{N-1};) = \ell_1, \qquad T_{N,1}(\underbrace{0, \ldots, 0}_{N}; 2) = \Delta$$

(the first of which is trivial because the Deligne product is simply the identity map, and the second by the very definition of Δ) and from the following companion result to Proposition 4.

Proposition 6. (Dilaton Equation) *For $m \geq 0$ and any integers $a_1, \ldots, a_{N+m} \geq 0$, we have*

$$T_{N,m+1}(a_1, \ldots, a_{N+m}, 1) = (N + m + 2g - 2)\, T_{N,m}(a_1, \ldots, a_{N+m}).$$

The proofs of Propositions 4 and 6 are similar to one another and will be given together. For convenience, we use the abbreviated notation $\langle S_1^{\circ k_1}, S_2^{\circ k_2}, \ldots, S_n^{\circ k_n} \rangle_f$ to denote the Deligne product

$$\langle \underbrace{S_1, \ldots, S_1}_{k_1}, \underbrace{S_2, \ldots, S_2}_{k_2}, \ldots, \underbrace{S_n, \ldots, S_n}_{k_n} \rangle_f$$

for any line bundles S_i and any integers $k_i \geq 0$. We also replace the "N" of §4 by "$N+m$" and use the same conventions as there, i.e., ℓ_i $(1 \leq i \leq N+m)$ denotes the i^{th} tautological line bundle on $\mathcal{M}_{g,N+m}$ and L_i $(1 \leq i \leq N+m+1)$ the i^{th} tautological line bundle on $\mathcal{M}_{g,N+m+1}$, while π and π' denote the projections from $\mathcal{M}_{g,N+m+1}$ to $\mathcal{M}_{g,N+m}$ and from $\mathcal{M}_{g,N+m}$ to $\mathcal{M}_{g,N}$, respectively. Finally, we set $M = N+m$. With these notations, the two formulas to be proved become

$$\big\langle L_1^{\circ a_1}, \ldots, L_M^{\circ a_M} \big\rangle_{\pi' \circ \pi} = \sum_{i=1}^{M} \big\langle \ell_1^{\circ a_1}, \ldots, \ell_i^{\circ (a_i - 1)}, \ldots, \ell_M^{\circ a_M} \big\rangle_{\pi'} \tag{6}$$

(with the usual convention that $\big\langle \cdots, \ell_i^{\circ (a_i-1)}, \cdots \big\rangle = 0$ if $a_i = 0$) and

$$\big\langle L_1^{\circ a_1}, \ldots, L_M^{\circ a_M}, L_{M+1} \big\rangle_{\pi' \circ \pi} = (2g-2+M)\, \big\langle \ell_1^{\circ a_1}, \ldots, \ell_M^{\circ a_M} \big\rangle_{\pi'}. \tag{7}$$

To prove these equations, we proceed as follows. Using Prop. 2(c), Prop. 3(b) and then Prop. 2(c) again (all of them with N replaced by M), we obtain

$$\begin{aligned}
\big\langle L_1^{\circ a_1}, \xi_1, \ldots, \xi_r \big\rangle_{\pi' \circ \pi} &= \big\langle L_1, \big(\pi^* \ell_1 + \mathbf{P}_1\big)^{\circ (a_1 - 1)}, \xi_1, \ldots, \xi_r \big\rangle_{\pi' \circ \pi} \\
&= \big\langle L_1, \big(\pi^* \ell_1\big)^{\circ (a_1 - 1)}, \xi_1, \ldots, \xi_r \big\rangle_{\pi' \circ \pi} \\
&= \big\langle \big(\pi^* \ell_1\big)^{\circ a_1}, \xi_1, \ldots, \xi_r \big\rangle_{\pi' \circ \pi} \\
&\quad + \big\langle \mathbf{P}_1, \big(\pi^* \ell_1\big)^{\circ (a_1 - 1)}, \xi_1, \ldots, \xi_r \big\rangle_{\pi' \circ \pi}
\end{aligned}$$

for any line bundles $\xi_1, \ldots, \xi_r$ in $\mathrm{Pic}(\mathcal{C}_{g,M})$, where $a_1 + r = m+2$. Now if there are a_2 indices i with $\xi_i = L_2$, then we can do the same with L_2 as we did with L_1. This gives four terms a priori, but one of them is $\big\langle \mathbf{P}_1, \big(\pi^* \ell_1\big)^{\circ (a_1-1)}, \mathbf{P}_2, \big(\pi^* \ell_2\big)^{\circ (a_2 - 1)}, \ldots \big\rangle_{\pi' \circ \pi}$ and this vanishes by Prop. 3(a). Continuing, we find

$$\begin{aligned}
&\big\langle L_1^{\circ a_1}, \ldots, L_M^{\circ a_M}, L_{M+1}^{\circ a} \big\rangle_{\pi' \circ \pi} \\
&\quad = \big\langle \big(\pi^* \ell_1\big)^{\circ a_1}, \ldots, \big(\pi^* \ell_M\big)^{\circ a_M}, L_{M+1}^{\circ a} \big\rangle_{\pi' \circ \pi} \\
&\qquad + \sum_{i=1}^{M} \big\langle \mathbf{P}_i, \big(\pi^* \ell_1\big)^{\circ a_1}, \ldots, \big(\pi^* \ell_i\big)^{\circ (a_i - 1)}, \ldots, \big(\pi^* \ell_M\big)^{\circ a_M}, L_{M+1}^{\circ a} \big\rangle_{\pi' \circ \pi}
\end{aligned} \tag{8}$$

for any $a_i,\ a \geq 0$. We have to evaluate this in two cases, when $a = 0$ and when $a = 1$.

If $a = 0$, then the first term in (8) vanishes, because each of the arguments of the Deligne product is a pull-back under π. For the second term, we note that

$$\big\langle \mathbf{P}_i, \pi^* \xi_1, \ldots, \pi^* \xi_{m+1} \big\rangle_{\pi' \circ \pi} = \big\langle \xi_1, \ldots, \xi_{m+1} \big\rangle_{\pi'}$$

for any $\xi_1, \ldots, \xi_{m+1} \in \mathrm{Pic}(\mathcal{M}_{g,M})$. (This follows from (DP3), because P_i is a section of π.) This gives Eq. (6) and hence Proposition 4 (string equation).

If $a = 1$, then the second term of (8) vanishes, because

$$\big\langle \mathbf{P}_i, \pi^* \xi_1, \ldots, \pi^* \xi_m, L_{M+1} \big\rangle_{\pi' \circ \pi} = \big\langle \big\langle \mathbf{P}_i, L_{M+1} \big\rangle_{\pi}, \xi_1, \ldots, \xi_m \big\rangle_{\pi'} = 0$$

for any $\xi_1, \ldots, \xi_m \in \mathrm{Pic}(\mathcal{M}_{g,M})$, by Prop. 1 (a) and Prop. 3 (c). The first term in (8) is equal to the right-hand side of (7) by Eq. (5) and Prop. 3 (d). This proves Eq. (7) and hence Proposition 6 (dilaton equation).

6. Proof of the Main Theorem

Since the recursion formula and initial values given in Propositions 4 and 5 determine the elements $T_N(a_1, \ldots, a_{N+m}) \in \mathrm{Pic}(\mathcal{M}_{g,N})$ uniquely, we can prove Theorem 1 by showing that the elements $T_N(a_1, \ldots, a_{N+m})$ defined by (2) and (3) satisfy these three equations. A direct proof of this is possible, but rather complicated, involving a series of lemmas about elementary symmetric polynomials and multinomial coefficients. This proof can be simplified considerably by a judicious use of generating functions, but remains quite complicated. A much simpler proof is obtained using the following trick. By the substitution $t = x/(1+x)$ and Euler's formula for the beta integral (or simply by integration by parts and induction on α and β) we see that

$$\int_0^\infty \frac{x^\alpha\, dx}{(x+1)^{\alpha+\beta+2}} = \int_0^1 t^\alpha (1-t)^\beta\, dt = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}$$

for any integers $\alpha,\ \beta \geq 0$. Hence, if we define a polynomial $F(x) = F_{d_1,\ldots,d_m}(x)$ by

$$F(x) = \prod_{j=1}^{m} \bigl(x + d_j\bigr) = \sum_{n=0}^{m} \sigma_n\, x^{m-n}, \qquad \sigma_n = \sigma_n(d_1, \ldots, d_m),$$

then we have

$$\int_0^\infty \frac{F(x)\, dx}{(x+1)^{\widetilde{N}+m+1}} = \sum_{n=0}^{m} \sigma_n\, \frac{(m-n)!\,(\widetilde{N}+n-1)!}{(\widetilde{N}+m)!} = \frac{(\widetilde{N}-1)!}{(\widetilde{N}+m)!}\, C_1(\mathbf{d}) \tag{9a}$$

and

$$\int_0^\infty \frac{x\, F(x)\, dx}{(x+1)^{\widetilde{N}+m+1}} = \sum_{n=0}^{m} \sigma_n\, \frac{(m-n+1)!\,(\widetilde{N}+n-2)!}{(\widetilde{N}+m)!} = \frac{\widetilde{N}\,(\widetilde{N}-2)!}{(\widetilde{N}+m)!}\, C_2(\mathbf{d}) \tag{9b}$$

with $C_\nu(\mathbf{d}) = C_\nu(\widetilde{N}; d_1, \ldots, d_m)$ as in Eq. (3) (or the first remark after Theorem 1 if $\widetilde{N}$ is 0 or 1). In particular, we have

$$C_1(\widetilde{N}; \underbrace{1, \ldots, 1}_{m}) = \frac{(\widetilde{N}+m)!}{(\widetilde{N}-1)!} \int_0^\infty \frac{dx}{(x+1)^{\widetilde{N}+1}} = \frac{(\widetilde{N}+m)!}{\widetilde{N}!},$$

$$C_2(\widetilde{N}; \underbrace{1, \ldots, 1}_{m}) = \frac{(\widetilde{N}+m)!}{\widetilde{N}(\widetilde{N}-2)!} \int_0^\infty \frac{x\, dx}{(x+1)^{\widetilde{N}+1}} = \frac{(\widetilde{N}+m)!}{\widetilde{N}\cdot\widetilde{N}!},$$

$$C_2(\widetilde{N}; 2, \underbrace{1, \ldots, 1}_{m-1}) = \frac{(\widetilde{N}+m)!}{\widetilde{N}(\widetilde{N}-2)!} \int_0^\infty \frac{x\,(x+2)\, dx}{(x+1)^{\widetilde{N}+2}} = \frac{2\,(\widetilde{N}+m)!}{(\widetilde{N}+1)!},$$

so Eq. (2) in the two cases when all d_i are strictly positive reduces to

$$T_{N,m}(1, \underbrace{0, \ldots, 0}_{N-1}; \underbrace{1, \ldots, 1}_{m})$$

$$= C_1(\widetilde{N}; 1, \ldots, 1)\,\Bigl(\ell_1 - \frac{\Delta}{\widetilde{N}}\Bigr) + C_2(\widetilde{N}; 1, \ldots, 1)\,\Delta = \frac{(\widetilde{N}+m)!}{\widetilde{N}!}\,\ell_1$$

and

$$T_{N,m}(\underbrace{0,\ldots,0}_{N};2,\underbrace{1,\ldots,1}_{m-1})$$
$$=\frac{1}{2!}\,C_2(\widetilde{N};2,1,\ldots,1)\,\Delta=\frac{(\widetilde{N}+m)!}{(\widetilde{N}+1)!}\,\Delta,$$

in accordance with the initial values in Proposition 5.

To prove the recursion formula of Proposition 4 (string equation), we will show that it is equivalent to a pair of recurrences for the coefficients $C_\nu(\mathbf{d})$ (Eq. (11) below) and then prove these recurrences using the integral representation (9). Denote the right-hand side of (2) by $t_{N,m}(a_1,\ldots,a_{N+m})$ or $t_{N,m}(a_1,\ldots,a_N;d_1,\ldots,d_m)$, with $d_j=a_{N+j}$ for $1\le j\le m$. Since we want $T_{N,m}(a_1,\ldots,a_{N+m})=t_{N,m}(a_1,\ldots,a_{N+m})/\prod_{i=1}^{N+m}a_i!$, we have to prove the recursion

$$t_{N,m+1}(a_1,\ldots,a_{N+m},0)=\sum_{i=1}^{N+m}a_i\,t_{N,m}(a_1,\ldots,a_i-1,\ldots,a_{N+m}).$$

(The extra factor a_i in front of $t_{N,m}(a_1,\ldots,a_i-1,\ldots,a_{N+m})$ comes from the change in $\prod_{i=1}^{N+m}a_i!$ when a_i is decreased by 1.) Now again separating the a_i $(1\le i\le N)$ and the $d_j=a_{N+j}$ $(1\le j\le m)$, we can write this out more explicitly as

$$\begin{aligned}&t_{N,m+1}(a_1,\ldots,a_N;\ d_1,\ldots,d_m,0)\\&\quad=\sum_{i=1}^{N}a_i\,t_{N,m}(a_1,\ldots,a_i-1,\ldots,a_N;\ d_1,\ldots,d_m)\\&\qquad+\sum_{j=1}^{m}d_j\,t_{N,m}(a_1,\ldots,a_N;\ d_1,\ldots,d_j-1,\ldots,d_m).\end{aligned}\tag{10}$$

We can write the definition of $t_{N,m}(a_1,\ldots,a_N;\ d_1,\ldots,d_m)$ in an abbreviated notation as

$$t_{N,m}(a_1,\ldots,a_N;\ d_1,\ldots,d_m)=C_1(\mathbf{d})\,\sum_{k=1}^{N}a_k\widehat{\ell_k}+C_2(\mathbf{d})\,\Delta,$$

where $C_\nu(\mathbf{d})=C_\nu(\widetilde{N};d_1,\ldots,d_m)$ as before ($\widetilde{N}$ does not change when we change m by 1, so can be omitted from the notation) and where $\widehat{\ell_k}$ is the element $\ell_k-\Delta/\widetilde{N}$ of $\mathrm{Pic}(\mathcal{M}_{g,N})\otimes\mathbb{Q}$. Then the left-hand side of (10) equals

$$C_1(\mathbf{d},0)\,\sum_{i=1}^{N}a_k\widehat{\ell_k}+C_2(\mathbf{d},0)\,\Delta,$$

while the right-hand side equals

$$\begin{aligned}&\sum_{i=1}^{N}a_i\left[C_1(\mathbf{d})\,\sum_{k=1}^{N}\bigl(a_k-\delta_{ki}\bigr)\,\widehat{\ell_k}+C_2(\mathbf{d})\,\Delta\right]\\&\quad+\sum_{j=1}^{m}d_j\left[C_1(d_1,\ldots,d_j-1,\ldots,d_m)\sum_{k=1}^{N}a_k\widehat{\ell_k}+C_2(d_1,\ldots,d_j-1,\ldots,d_m)\Delta\right]\end{aligned}$$

$$= \sum_{k=1}^{N} \left[C_1(\mathbf{d}) \Bigl(\sum_{i=1}^{N} a_i - 1 \Bigr) + \sum_{j=1}^{m} d_j \, C_1(d_1, \dots, d_j - 1, \dots, d_m) \right] a_k \, \widehat{\ell_k}$$
$$+ \left[C_2(\mathbf{d}) \Bigl(\sum_{i=1}^{N} a_i \Bigr) + \sum_{j=1}^{m} d_j \, C_2(d_1, \dots, d_j - 1, \dots, d_m) \right] \Delta .$$

Comparing these two expressions, and recalling that $\sum_{i=1}^{N} a_i = m+2-\sigma_1(\mathbf{d})$ (because the sum of all the indices $a_1, \dots, a_N, d_1, \dots, d_m, 0$ in Eq. (10) must equal $m+2$), we find that the theorem will follow from the two identities:

$$C_1(\mathbf{d}, 0) = \bigl(m + 1 - \sigma_1\bigr)\, C_1(\mathbf{d}) + \sum_{j=1}^{m} d_j \, C_1(d_1, \dots, d_j - 1, \dots, d_m), \qquad (11a)$$

$$C_2(\mathbf{d}, 0) = \bigl(m + 2 - \sigma_1\bigr)\, C_2(\mathbf{d}) + \sum_{j=1}^{m} d_j \, C_2(d_1, \dots, d_j - 1, \dots, d_m). \qquad (11b)$$

To prove the first of these, we use Eq. (9a). Replacing $\mathbf{d} = (d_1, \dots, d_m)$ by $(\mathbf{d}, 0) = (d_1, \dots, d_m, 0)$ increases m by 1 and replaces the polynomial $F(x)$ by $xF(x)$, whereas replacing $\mathbf{d}$ by $(d_1, \dots, d_j - 1, \dots, d_m)$ leaves m unchanged and replaces $F(x)$ by $F(x)(x+d_j-1)/(x+d_j)$. Therefore substituting (9a) into (11a) and dividing both sides by $(\widetilde{N}+m)!/(\widetilde{N}-1)!$ gives

$$(\widetilde{N} + m + 1) \int_0^\infty \frac{x\, F(x)\, dx}{(x+1)^{\widetilde{N}+m+2}}$$
$$= \int_0^\infty \left[m + 1 + \sum_{j=1}^{m} d_j \Bigl(-1 + \frac{x + d_j - 1}{x + d_j} \Bigr) \right] \frac{F(x)\, dx}{(x+1)^{\widetilde{N}+m+1}}$$

as the identity to be proved. But this is immediate by integration by parts, since the expression in square brackets equals $1 + \sum_{j=1}^{m} \frac{x}{x+d_j} = 1 + x\frac{F'(x)}{F(x)}$. The proof of Eq. (11b) is exactly the same, using (9b) instead of (9a), with $F(x)$ replaced by $xF(x)$. This completes the proof of the theorem.

7. Final Remark

A formula very similar to Eq. (2) (in the case when all $a_i = 0$) appears in §4.6 of [8], but in a somewhat different situation: the formula there is for the moduli space of curves of genus 1 with N marked points and deals with the Gromov-Witten invariants, which are integers, whereas our formula is for arbitary genus (although in the final result the genus does not appear except in the shift from N to $\widetilde{N}$) and gives the Deligne products, which take values in $\mathrm{Pic}(\mathcal{M}_{g,N})$. Both proofs are based on the string and dilaton equations, which are valid in both contexts. This suggests a possible common generalization. Our situation concerns codimension one cycles, while Gromov-Witten invariants have to do with zero-dimensional cycles. It therefore seems reasonable to ask whether (2) and the equation in [8] are special cases of a more general result valid for intermediate dimensions, for which the string and dilaton equations still hold.

References

1. Arbarello, E., Cornalba, M.: The Picard groups of the moduli spaces of curves. Topology **26**(2), 153–171 (1987)
2. Deligne, P.: Le déterminant de la cohomologie. In: *Current trends in arithmetic algebraic geometry*, Contemporary Math. Vol. **67**, Providence, RI: Amer. Math. Soc., pp. 93–178, 1987
3. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. IHES Publ. Math. **36**, 75–109 (1969)
4. Harer, J.: The second homology group of the mapping class group of an orientable surface. Invent. Math. **72**(2), 221–239 (1983)
5. Kaufmann, R., Manin, Yu., Zagier, D.: Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves. Commun. Math. Phys. **181**(3), 763–787 (1996)
6. Knudsen, F.: The projectivity of the moduli space of stable curves, II. Math. Scand. **52**, 161–199 (1983)
7. Knudsen, F., Mumford, D.: The projectivity of the moduli space of stable curves. I. Math. Scand. **39**, 19–55 (1976)
8. Lando, S.K., Zvonkin, A.K.: Graphs on Surfaces and Their Applications, In: *Encyclopedia of Mathematical Sciences*, Berlin–Heidelberg: Springer-Verlag, 2004
9. Takhtajan, L., Zograf, P.: The Selberg zeta function and a new Kähler metric on the moduli space of punctured Riemann surfaces. J. Geo. Phys. **5**, 551–570 (1988)
10. Takhtajan, L., Zograf, P.: A local index theorem for families of $\bar{\partial}$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces. Commun. Math. Phys. **137**, 399–426 (1991)
11. Weng, L.: *Hyperbolic Metrics, Selberg Zeta Functions and Arakelov Theory for Punctured Riemann Surfaces*. Lecture Note Series in Mathematics **6**, Osaka: Osaka University, 1998
12. Weng, L.: Ω-admissible theory, II.. Math. Ann. **320**, 239–283 (2001)

Communicated by L. Takhtajan