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The Riemann hypothesis for Weng's zeta function of Sp(4) over $\mathbb{Q}$ ☆

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As a generalization of the Dedekind zeta function, Weng defined the high rank zeta functions and proved that they have standard properties of zeta functions, namely, meromorphic continuation, functional equation, and having only two simple poles. The rank one zeta function is the Dedekind zeta function. For the rank two case, the Riemann hypothesis is proved for a general number field. Recently, he defined more general new zeta function associated to a pair of a semi-simple reductive algebraic group and its maximal parabolic subgroup. As well as the high rank zeta function, the new zeta function satisfies standard properties of zeta functions. In this paper, we prove that the Riemann hypothesis for Weng's zeta function attached to the symplectic group of degree four. This paper includes an appendix written by L. Weng, in which he explains a general construction for zeta functions associated to Sp($2n$).

1. Introduction

Let C be a nonsingular projective algebraic curve over a finite field $\mathbb{F}_q$. The zeta function $\zeta_C(s)$ is defined by the Euler product $\zeta_C(s) = \prod_{P \in C_0}(1 - q^{-s\deg(P)})^{-1}$, where C_0 is the set of all closed points of C. The theory of $\zeta_C(s)$ is one of the most beautiful and successful in Number theory. A lot of nice theory was established by modeling on the theory of $\zeta_C(s)$. Recently, Lin Weng defined a class of general zeta functions starting from the following formula of $\zeta_C(s)$:

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[1] With Appendix by Lin Weng.

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$$\zeta_C(s)=\sum_{[L]\in\mathrm{Pic}(C)}\frac{q^{h^0(L)}-1}{q-1}q^{-s\deg(L)},\quad \Re(s)>1,\tag{1}$$

where $\mathrm{Pic}(C)$ is the moduli space of isomorphic classes $[L]$ of the line bundle L on C, and $h^0(L)$ is the dimension of the 0th cohomology $H^0(C,L)$ over $\mathbb{F}_q$. First he generalized $\zeta_C(s)$ to the rank $r\geqslant 1$ zeta function $\zeta_{C,r}(s)$ by replacing $\mathrm{Pic}(C)$ by the moduli space of isomorphic classes of semi-stable vector bundles on C of rank r. The semi-stability (introduced by Mumford) is needed to get a reasonable structure of the moduli space, and it is reflected to the standard properties of the zeta function $\zeta_{C,r}(s)$.

On the other hand, for the number field case, Iwasawa's interpretation of the Dedekind zeta function allows us to get an arithmetic analogue of (1) for the Dedekind zeta function. Let F be a number field with discriminant Δ_F, $\mathbb{A}_F^\times$ be the idele group of F and w_F be the number of roots of unity in F. We denote by $\xi_F(s)$ the completed Dedekind zeta function of F. An idele $a=(a_v)_v\in\mathbb{A}_F^\times$ defines an Arakelov line bundle L_a on $\overline{X}_F=\mathrm{Spec}\,\mathcal{O}_F\cup X_{F,\infty}$, and every Arakelov line bundles are obtained by this manner. (More precisely, the non-archimedean part of $(a_v)_v$ defines an invertible sheaf on $\mathrm{Spec}\,\mathcal{O}_F$ which is nothing but a projective $\mathcal{O}_F$-module of rank one, and the archimedean part of $(a_v)_v$ defines a Hermitian metric on $X_{F,\infty}$.) Moreover we have the topological group isomorphism $\mathbb{A}_F^\times/F^*\simeq\mathrm{Pic}(F)$, where $\mathrm{Pic}(F)$ is the moduli space of isomorphic classes of Arakelov line bundles on $\overline{X}_F$. Throughout this relation between ideles and Arakelov line bundles, Iwasawa's interpretation gives

$$\xi_F(s)=w_F^{-1}|\Delta_F|^{\frac{s}{2}}\int\limits_{\mathrm{Pic}(F)}\left(e^{h^0(L)}-1\right)e^{-s\deg(L)}\,d\mu(L),\quad \Re(s)>1,\tag{2}$$

where $\deg(L)$ is the arithmetic degree of the Arakelov line bundle L and $d\mu$ is a certain natural measure on $\mathrm{Pic}(F)$. In (2), the definition of the arithmetic dimension of the cohomology $h^0(L)=h^0(\overline{X_F},L)$ is naturally given along the line of Iwasawa's interpretation (see [10, B.2.1], and also [1]), and Tate's Fourier analysis on $\mathbb{A}_F$ gives the Riemann–Roch theorem as a consequence of the Pontryagin duality and the Poisson summation formula. The Pontryagin duality is interpreted as the Serre duality under suitable definition of h^1, see [10, B.2.3.3, B.2.3.4], and also [11, Section 1.8].

Standing on the arithmetic–geometric formula (2) and Tate's Fourier analysis on $\mathbb{A}_F$, Weng defined the (completed) rank r zeta function $\xi_{F,r}(s)$ of F by

$$\xi_{F,r}(s)=|\Delta_F|^{\frac{rs}{2}}\int\limits_{\mathcal{M}_{F,r}}\left(e^{h^0(E)}-1\right)e^{-s\deg(E)}\,d\mu(E),\quad \Re(s)>1,\tag{3}$$

where $\mathcal{M}_{F,r}$ is the moduli space of isomorphic classes $[E]$ of the rank r semi-stable Arakelov vector bundle E on $\overline{X}_F$, and $d\mu$ is its associated Tamagawa measure, and $\deg(E)$ is the arithmetic degree of E. The arithmetic analogue of h^0 and h^1 are defined as a natural extension of the line bundle case having in mind Tate's method ([10, B.2.3.4], see also [Ar1]). We have $\xi_{F,1}(s)=w_F\xi_F(s)$, since every Arakelov line bundles are semi-stable. Thanks to the semi-stability the right-hand side of (3) converges absolutely for $\Re(s)>1$. The arithmetic–geometric Riemann–Roch theorem is proved by using the Fourier analysis on $\mathbb{A}_F^n$ following Tate [10, B.2.3.4]. As a consequence of the Serre duality and the Riemann–Roch theorem, the rank r zeta function $\xi_{F,r}(s)$ is continued meromorphically to $\mathbb{C}$ with only two simple poles at $s=0,1$, and satisfies the standard functional equation $\xi_{F,r}(s)=\xi_{F,r}(1-s)$ [10, B.2.4.2]. Hence the Riemann hypothesis for $\xi_{F,r}(s)$ is stated that all zeros of $\xi_{F,r}(s)$ lie on the central line $\Re(s)=1/2$.

Remarkable fact for $\xi_{F,r}(s)$ is that the Riemann hypothesis for the rank 2 zeta function $\xi_{F,2}(s)$ is *proved* for all algebraic number fields F ([6,11], see also [2]). Moreover all zeros of $\xi_{\mathbb{Q},2}(s)$ are simple [6,4]. Now it is expected that the Riemann hypothesis holds for all $r\geqslant 1$ [12]. Note that the case $r=1$ is the Riemann hypothesis for the Dedekind zeta function. The study of high rank zeta functions $\xi_{F,r}(s)$ lead us to more general zeta functions.

In general, $\xi_{\mathbb{Q},r}(s)$ is expressed as an integral of the Eisenstein series [12]. For example, the rank two case is

$$\xi_{\mathbb{Q},2}(s) = \int\limits_{\mathcal{M}_{\mathbb{Q},2}[1]} \hat{E}(z,s)\, d\mu(z), \tag{4}$$

where $\mathcal{M}_{\mathbb{Q},2}[1] \simeq D_0$, $D_T = \{z = x + iy \mid |x| \leqslant 1/2,\ 0 < y \leqslant \exp(T),\ |z| \geqslant 1\}$ and

$$\hat{E}(z,s) = \xi(2s) \sum_{\gamma \in P(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \Im(\gamma z)^s \quad \big(\Re(s) > 1\big), \qquad P = B = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}.$$

Recall the group theoretic description of the upper half plane

$$D_T \hookrightarrow \mathcal{H} \simeq \mathrm{SL}(2,\mathbb{Z}) \setminus \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2),$$

and the fact that P is a maximal parabolic subgroup of SL(2). These lead us to a zeta function attached to a pair (G, P) of a semi-simple reductive algebraic group G and a maximal parabolic subgroup P of G motivated by (4).

The first important point of our generalization is the relation between the geometric truncation and the analytic truncation. In the rank two case, we define the analytic truncation $\wedge^T \hat{E}(z,s)$ by

$$\wedge^T \hat{E}(z,s) = \begin{cases} \hat{E}(z,s), & \text{if } y \leqslant \exp(T),\ z \in D_\infty, \\ \hat{E}(z,s) - a_0(y,s), & \text{if } y > \exp(T),\ z \in D_\infty, \end{cases}$$

where $a_0(y,s)$ is the constant term of the Fourier expansion of $\hat{E}(z,s)$. Then we have

$$\int\limits_{D_T} \hat{E}(z,s)\, d\mu(z) = \int\limits_{\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)} \wedge^T \hat{E}(z,s)\, d\mu(z). \tag{5}$$

Namely, the geometric truncation D_T is equal to the analytic truncation $\wedge^T$ in the rank two case. Taking $T = 0$, we obtain the rank two zeta function $\xi_{\mathbb{Q},2}(s)$. This kind of equality holds in widely general situation if T is sufficiently regular (see A.2.2 of Appendix [13]).

To explain the second point of our generalization, we restrict the case as $G = \mathrm{Sp}(4)$ and $F = \mathbb{Q}$ for the simplicity. See Appendix [13] for the detailed theory of $\mathrm{Sp}(2n)$, [9] for the zeta functions of the exceptional group G_2, and [14] for the general theory of zeta functions attached to the pair (G, P) of the reductive group G and its maximal parabolic subgroup P.

Let $\mathfrak{S}_2$ be the Siegel upper half-space of degree 2, and let $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})/\{\pm I_4\}$ be the Siegel modular group of degree 2, where I_4 is the identity matrix of size 4. Any $Z \in \mathfrak{S}_2$ is written as $Z = X + \sqrt{-1}\,Y$ with $X, Y \in \mathrm{Sp}(4,\mathbb{R})$ such that $Y = \Im(Z)$ is positive definite. For $Z \in \mathfrak{S}_2$ and $\Re(s) \ll 0$, the Siegel–Maaß Eisenstein series is defined by

$$E_2(Z,s) = \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \mathfrak{P}_2 \backslash \Gamma_2} \frac{|Y|^{-s}}{\|CZ + D\|^{-2s}}$$

where $\mathfrak{P}_2$ is the maximal parabolic subgroup $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \cap \Gamma_2$ attached to the partition $2 = 1 + 1$, $|Y| = \det Y$ and $\|CZ + D\| = |\det(CZ + D)|$. Let P_0 be the upper triangular Borel subgroup in Sp(4), M_0 be the Levi component of P_0, $X(M_0)$ be the group of characters of M_0 defined over $\mathbb{Q}$, $\Delta_0 \subset \mathfrak{a}_0^*$ be the set of simple roots attached to P_0, $\mathfrak{a}_0 = \mathrm{Hom}_{\mathbb{Z}}(X(M_0), \mathbb{R})$ and $\mathfrak{a}_0^+ = \{T \in \mathfrak{a}_0 \mid \langle \alpha, T\rangle > 0,\ \forall \alpha \in \Delta_0\}$.

An element $T \in \mathfrak{a}_0^+$ is called sufficiently regular if $\langle \alpha, T \rangle > 0$ is large enough for every $\alpha \in \Delta_0$. For sufficiently regular $T \in \mathfrak{a}_0^+$, we define

$$Z^T_{\mathrm{Sp}(4),\mathbb{Q}}(s) := \int\limits_{\Gamma_2 \backslash \mathfrak{S}_2} \wedge^T E_2(Z; s)\, d\mu(Z), \tag{6}$$

where $\wedge^T$ is Arthur's truncation operator with respect to T. Denote by $\mathfrak{F}(T)$ the compact subset of $\Gamma_2 \setminus \mathfrak{S}_2$ whose characteristic function is given by $\wedge^T \mathbf{1}$. Then we have

$$Z^T_{\mathrm{Sp}(4),\mathbb{Q}}(s) = \int\limits_{\Gamma_2 \backslash \mathfrak{S}_2} \wedge^T E_2(Z; s)\, d\mu(Z) = \int\limits_{\mathfrak{F}(T)} E_2(Z; s)\, d\mu(Z) \tag{7}$$

for every sufficiently regular $T \in \mathfrak{a}_0^+$ by Corollary 2.2.1 of Appendix [13, A.2.2, A.2.3]. Remind the formula (4) of the high rank zeta function $\xi_{\mathbb{Q},r}(s)$. Then we understand that $Z^{T=0}_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ is a natural analogue of $\xi_{\mathbb{Q},r}(s)$, if we can give a reasonable arithmetic or geometric meaning of the set $\mathfrak{F}(0)$ [13, A.2.7]. Here we have two problems:

(1) How to calculate the integral in (7)?
(2) Can we take $T = 0$ in (7)?

To attack these problems, we note the formula

$$E_2(Z, s) = \operatorname*{Res}_{z=s+\frac{1}{2}} E\left(\mathbf{1}; z, s - \frac{1}{2}; Z\right),$$

where $E(\varphi; z_1, z_2; Z)$ is the Langlands–Eisenstein series associated to the Borel subgroup P_0, $(z_1, z_2) \in \mathfrak{a}_0^* \otimes \mathbb{C} \simeq \mathbb{C}^2$ and a cuspidal automorphic form φ of level M_0. The constant function $\mathbf{1}$ is cuspidal at the level of M_0. (See A.2.5 of Appendix [13] for the definition of $E(\varphi; z_1, z_2; Z)$, and [9, Section 2], or better, [12, Sections 2, 3] for the general theory of this one.) Hence we have

$$Z^T_{\mathrm{Sp}(4),\mathbb{Q}}(s) = \int\limits_{\mathfrak{F}(T)} \operatorname*{Res}_{z=s+\frac{1}{2}} E\left(\mathbf{1}; z, s - \frac{1}{2}; Z\right) d\mu(Z). \tag{8}$$

Now we *assume* that

($\star$) we can exchange the integration and the taking the residue in (8).

Then we have

$$\begin{aligned} Z^T_{\mathrm{Sp}(4),\mathbb{Q}}(s) &= \operatorname*{Res}_{z=s+\frac{1}{2}} \int\limits_{\mathfrak{F}(T)} E\left(\mathbf{1}; z, s - \frac{1}{2}; Z\right) d\mu(Z) \\ &= \operatorname*{Res}_{z=s+\frac{1}{2}} \int\limits_{\Gamma_2 \backslash \mathfrak{S}_2} \wedge^T E\left(\mathbf{1}; z, s - \frac{1}{2}; Z\right) d\mu(Z). \end{aligned} \tag{9}$$

The integral on the right-hand side is calculated *explicitly* in terms of the Weyl group and the Riemann zeta function by the method of Jacquet, Lapid and Rogawski [3, Corollary 17]. Moreover, in such explicit formula of $Z^T_{\mathrm{Sp}(4),\mathbb{Q}}(s)$, we can take $T = 0$! (see A.2.6 of Appendix [13]).

Unfortunately, assumption ($\star$) is *not* allowed in general, even if $G = \mathrm{SL}(n)$ ($n \geqslant 3$). In other words, we cannot define a *direct* analogue $Z^{T=0}_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ of $\xi_{\mathbb{Q},r}(s)$ at present (more precisely, see A.2.7 of Appendix [13]). Here we turn the consideration. We define the *new* zeta function by using the right-hand side of (9). This is the second point of the generalization mentioned above. We define the new zeta function $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ by

$$\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s) := \xi(2)\xi(s+1)\xi(2s) \cdot \left[\operatorname*{Res}_{z=s-\frac{1}{2}} \int_{\Gamma_2\backslash\mathfrak{S}_2} \wedge^T E\left(\mathbf{1}; z, s-\frac{3}{2}; Z\right) d\mu(Z) \right]_{T=0}. \tag{10}$$

Here the operation $T = 0$ is justified via the explicit calculation of the integral by using [3, Corollary 17] (see A.2.7 of Appendix [13]), the factor $\xi(2)\xi(s+1)\xi(2s)$ is introduced to clearance the denominator of that explicit formula, and the shift $s - \frac{1}{2} \mapsto s - \frac{3}{2}$ is introduced to normalize the form of functional equation.

By a way similar to the above, the new zeta functions are defined for more general semi-simple reductive algebraic group G defined over $\mathbb{Q}$ except for the part of taking residues. To specify the way of taking residues we need the maximal parabolic subgroup P of G. Then the new zeta function $\xi_{\mathbb{Q}}^{G/P}(s)$ is defined for the pair (G, P) ([14], see also [9]), and the above $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ is nothing but $\xi_{\mathbb{Q}}^{\mathrm{Sp}(4)/\mathfrak{P}_2}(s)$. As well as the high rank zeta function, it is expected that the zeta functions $\xi_{\mathbb{Q}}^{G/P}(s)$ satisfy several standard properties of zeta functions, even if $G \neq \mathrm{SL}(n), \mathrm{Sp}(2n)$. In particular, it is conjectured that $\xi_{\mathbb{Q}}^{G/P}(s)$ has a standard functional equation, and the conjectural functional equation is proved for a few concrete examples. This is in fact the next most important yet doable question in this direction.

For Sp(4), the zeta function $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ is calculated explicitly as

$$\begin{aligned}
\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s) = {} & \frac{1}{s-2}\xi(2) \cdot \xi(s+1)\xi(2s) - \frac{1}{s+1}\xi(2) \cdot \xi(s-1)\xi(2s-1) \\
& - \frac{1}{2s-2} \cdot \xi(s+1)\xi(2s) + \frac{1}{2s} \cdot \xi(s-1)\xi(2s-1) \\
& - \frac{1}{(2s-2)(s+1)} \cdot \xi(s)\xi(2s) - \frac{1}{(2s)(s-2)} \cdot \xi(s)\xi(2s-1),
\end{aligned} \tag{11}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function (see A.3.1 of Appendix [13]). By formula (11), the zeta function $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ is continued meromorphically to $\mathbb{C}$, and satisfies the functional equation

$$\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s) = \xi_{\mathrm{Sp}(4),\mathbb{Q}}(1-s). \tag{12}$$

All poles of $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ are four simple poles $s = 0, 1$ and $s = -1, 2$. For the zeros of $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$, we have the following significant result.

Theorem 1 *(RH for* $\mathrm{Sp}(4)/\mathbb{Q}$*). All zeros of the zeta function* $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ *lie on the line* $\Re(s) = 1/2$.

The entire function

$$Z(s) := 4s^2(s-1)^2 \cdot (s+1)(2s-1)(s-2) \cdot \xi_{\mathrm{Sp}(4),\mathbb{Q}}(s) \tag{13}$$

is more useful than $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ itself for the proof of Theorem 1. We have

$$\begin{aligned}
Z(s) = {} & (s-1)(As - A + 1) \cdot \chi(s+1)\chi(2s) - (s-2) \cdot \chi(s)\chi(2s) \\
& - s(As-1) \cdot \chi(s-1)\chi(2s-1) - (s+1) \cdot \chi(s)\chi(2s-1),
\end{aligned} \tag{14}$$

where

$$A = 2\xi(2) - 1 = \pi/3 - 1 > 0, \qquad \chi(s) = s(s-1)\xi(s). \tag{15}$$

Note that $Z(s)$ has real zeros at $s = 0, 1$ and $s = 1/2$, since $s = 0, 1$ are simple poles of $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ and $s = 1/2$ is its regular point. Thus Theorem 1 is equivalent to the following.

Theorem 2. *All zeros of the entire function $Z(s)$ lie on the line $\Re(s) = 1/2$ except for two simple zeros at $s = 0, 1$.*

Corollary 3. *The Riemann zeta function is a factor of the difference of two entire functions which satisfy the Riemann hypothesis. More precisely, we have*

$$s(s-1)\xi(s) \cdot U(s) = V(s) - Z(s), \tag{16}$$

where

$$\begin{aligned} U(s) &= (s+1)\chi(2s-1) - (s-2)\chi(2s), \\ V(s) &= (s-1)(As - A + 1)\chi(s+1)\chi(2s) - s(As-1)\chi(s-1)\chi(2s-1). \end{aligned} \tag{17}$$

To prove Theorem 2, we use two auxiliary functions $f(s)$ and $g(s)$ defined by

$$\begin{aligned} f(s) &= (s-1)(As - A + 1) \cdot \chi(s+1) - (s-2) \cdot \chi(s), \\ g(s) &= f(s) \cdot \chi(2s). \end{aligned} \tag{18}$$

By definitions of $f(s)$ and $g(s)$, and (14), we have

$$Z(s) = g(s) - g(1-s). \tag{19}$$

Here we used the functional equation $\chi(s) = \chi(1-s)$ of the Riemann zeta function.

Roughly, the proof of Theorem 2 is divided into two steps. First, we prove that all zeros of $f(s)$ lie in a vertical strip $\sigma_0 < \Re(s) < 0$ except for finitely many exceptional zeros (Section 2). Then we obtain a nice product formula of $f(s)$ by a variant of Lemma 3 in [8] (Lemma 5, it is proved in Section 4). Second, by using the product formula of $f(s)$, we prove that all zeros of $Z(s)$ lie on the line $\Re(s) = 1/2$ except for two simple zeros $s = 0, 1$ (Section 3). In this process, we use the result of Lagarias [5] concerning the explicit upper bound for the difference of the imaginary parts of the zeros of the Riemann zeta function. In the final section (Section 5) we give the proof of Corollary 3.

Finally, we comment on high rank zeta functions $\xi_{\mathbb{Q},r}(s)$ and new zeta functions $\xi_{\mathrm{SL}(n),\mathbb{Q}}(s) := \xi_{\mathbb{Q}}^{G/P}(s)$ attached to $(G, P) = (\mathrm{SL}(n), P_{n-1,1})$. Roughly, $\xi_{\mathbb{Q},r}(s)$ corresponds to $(\mathrm{Res} \to \int)$-ordered construction, and new zeta function $\xi_{\mathrm{SL}(n),\mathbb{Q}}(s)$ corresponds to $(\int \to \mathrm{Res})$-ordered construction. Here "$(\mathrm{Res} \to \int)$-ordered" means that we first take residues then take the integral, similarly, "$(\int \to \mathrm{Res})$-ordered" means that we first take the integral then take residues. We have $\xi_{\mathbb{Q},2}(s) = \xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$, since we do not need taking residue. However, in general, there is a discrepancy between $\xi_{\mathbb{Q},r}(s)$ and $\xi_{\mathrm{SL}(n),\mathbb{Q}}(s)$, because of the obstruction for the exchanging of $\int$ and Res. For example, $\xi_{\mathbb{Q},3}(s)$ has only two singularities at $s = 0, 1$, but $\xi_{\mathrm{SL}(3),\mathbb{Q}}(s)$ has four singularities at $s = 0, 1/3, 2/3, 1$. However, we expect that the distribution of the zeros of $\xi_{\mathrm{SL}(n),\mathbb{Q}}(s)$ is quite regular as well as $\xi_{\mathbb{Q},r}(s)$. In fact, we have the Riemann hypothesis for $\xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$ by $\xi_{\mathbb{Q},2}(s) = \xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$ and the result of [6], and the author proved the Riemann hypothesis for $\xi_{\mathrm{SL}(3),\mathbb{Q}}(s)$ in [8]. Moreover the author checked the Riemann hypothesis for $\xi_{\mathrm{SL}(4),\mathbb{Q}}(s)$ and $\xi_{\mathrm{SL}(5),\mathbb{Q}}(s)$ computationally. As observed in [8], the study of $\xi_{\mathrm{SL}(3),\mathbb{Q}}(s)$ gives an information for the Riemann zeta function. In addition, Corollary 3 gives a relation between

$\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ and the Riemann zeta function. As these, the study of $\xi_F^{G/P}(s)$ is not only interesting itself but also suggestive for the study of the Dedekind zeta function.

2. First step for Theorem 2

The aim of this section is to prove the following proposition.

Proposition 4. *Let $f(s)$ be the function defined in (18). Then $f(s)$ has the product formula*

$$f(s) = f(0)e^{B's}\left(1 - \frac{s}{\rho_0}\right)\left(1 - \frac{s}{\bar{\rho}_0}\right) \cdot \Pi(s) \quad (B' \geqslant 0), \tag{20}$$

where

$$\Pi(s) = \prod_{\substack{\beta<1/2 \\ 0\neq\beta\in\mathbb{R}}}\left(1 - \frac{s}{\beta}\right) \prod_{\substack{\beta<1/2 \\ \gamma>0}}\left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right)\right] \quad (\rho = \beta + i\gamma), \tag{21}$$

where ρ_0 is a complex zero of $f(s)$ with $\Re(\rho_0) > 1/2$, β is the real zeros of $f(s)$ and $\rho = \beta + i\gamma$ are other zeros of $f(s)$. The product converges absolutely on every compact subset of $\mathbb{C}$ if we take the product with the bracket.

This is proved by checking that $f(s)$ satisfies all conditions of Lemma 5 below.

Lemma 5. *Let $F(s)$ be an entire function of genus zero or one. Suppose that*

(i) *$F(s)$ is real on the real axis,*
(ii) *there exists $\sigma_0 > 0$ such that all zeros of $F(s)$ lie in the vertical strip*

$$\sigma_0 < \Re(s) < 1/2 \tag{22}$$

except for finitely many zeros,
(iii) *the number of zeros of $F(s)$ on the right half-plane $\Re(s) \geqslant 1/2$ is finite,*
(iv) *there exists $C > 0$ such that*

$$N(T) \leqslant CT\log T \quad \text{as } T \to \infty, \tag{23}$$

where $N(T)$ is the number of zeros of $F(s)$ satisfying $0 \leqslant \Im(\rho) < T$.
(v) *$F(1-\sigma)/F(\sigma)$ is positive for sufficiently large $\sigma > 0$, and*

$$F(1-\sigma)/F(\sigma) \to 0 \quad \text{as } \sigma \to \infty. \tag{24}$$

Then $F(s)$ has the product formula

$$F(s) = Cs^m e^{B's} \prod_{0\neq\rho\in\mathbb{R}}\left(1 - \frac{s}{\rho}\right) \prod_{\Im(\rho)>0}\left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right)\right] \tag{25}$$

with

$$B' \geqslant 0. \tag{26}$$

The product on the right-hand side converges absolutely on every compact subset of $\mathbb{C}$ if we take the product with the bracket.

Remark 6. This lemma is a variant of Lemma 3 of [8]. Differently from Lemma 3 of [8], we allow $F(s)$ to have finitely many zeros in $\Re(s) \geqslant 1/2$. The most important part of this lemma is the nonnegativity of B'.

We prove Lemma 5 in Section 4. For $f(s)$ in (18), condition (i) is trivial. Under (ii), condition (iv) is easily proved by using the well-known estimate $|\chi(s)| \leqslant \exp(C|s|\log|s|)$ and Jensen's formula (see [8, §4.1] for example). Therefore it remains to prove (ii), (iii) and (v) for $f(s)$.

2.1. Proof of (v)

First we show that $f(1-\sigma)/f(\sigma)$ is positive for sufficiently large $\sigma > 0$. Using the functional equation of $\chi(s)$, we have

$$\frac{f(1-\sigma)}{f(\sigma)} = \frac{\sigma(A\sigma-1)\cdot\chi(\sigma-1)+(\sigma+1)\cdot\chi(\sigma)}{(\sigma-1)(A\sigma-A+1)\cdot\chi(\sigma+1)-(\sigma-2)\cdot\chi(\sigma)}.$$

Clearly the numerator is positive for large $\sigma > 0$. By [6, pp. 109–110], we have $|\chi(2\sigma-1)/\chi(2\sigma)| < 1$ for $\sigma > 1/2$. Replacing $2\sigma - 1$ by σ

$$\bigl|\chi(\sigma)/\chi(\sigma+1)\bigr| < 1 \quad (\sigma > 0). \tag{27}$$

Hence the denominator is also positive for large $\sigma > 0$, since $A = \pi/3 - 1 > 0$. Now we prove (24). We have

$$\begin{aligned}\frac{f(1-\sigma)}{f(\sigma)} &= \frac{\sigma(A\sigma-1)}{(\sigma-1)(A\sigma-A+1)}\cdot\frac{\chi(\sigma-1)}{\chi(\sigma+1)}\cdot\frac{1+g(\sigma)}{1-h(\sigma)}\\ &= \bigl(1+O\bigl(\sigma^{-1}\bigr)\bigr)\cdot\frac{\chi(\sigma-1)}{\chi(\sigma+1)}\cdot\frac{1+g(\sigma)}{1-h(\sigma)},\end{aligned}$$

where

$$g(\sigma) = \frac{\sigma+1}{\sigma(A\sigma-1)}\cdot\frac{\chi(\sigma)}{\chi(\sigma-1)}, \qquad h(\sigma) = \frac{\sigma-2}{(\sigma-1)(A\sigma-A+1)}\cdot\frac{\chi(\sigma)}{\chi(\sigma+1)}.$$

We have

$$\begin{aligned}\frac{\chi(\sigma-1)}{\chi(\sigma+1)} &= \bigl(1+O\bigl(\sigma^{-1}\bigr)\bigr)\frac{\xi(\sigma-1)}{\xi(\sigma+1)} = \bigl(1+O\bigl(\sigma^{-1}\bigr)\bigr)\cdot\pi\cdot\frac{\Gamma((\sigma-1)/2)\zeta(\sigma-1)}{\Gamma((\sigma+1)/2)\zeta(\sigma+1)}\\ &= \bigl(1+O\bigl(\sigma^{-1}\bigr)\bigr)\cdot\frac{\Gamma((\sigma-1)/2)}{\Gamma((\sigma+1)/2)}\cdot O(1)\end{aligned}$$

for large $\sigma > 0$. Using the Stirling formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^{z}\bigl(1+O\bigl(|z|^{-1}\bigr)\bigr) \quad \bigl(|z| \geqslant 1,\ |\arg z| < \pi-\varepsilon\bigr),$$

we obtain

$$\frac{\chi(\sigma-1)}{\chi(\sigma+1)} = O\bigl(\sigma^{-1}\bigr) \quad \text{as } \sigma \to +\infty. \tag{28}$$

For $g(\sigma)$, we have

$$\begin{aligned} g(\sigma) &= \frac{\sigma+1}{\sigma(A\sigma-1)} \cdot \left(1+O\left(\sigma^{-1}\right)\right) \cdot \pi^{-1/2} \frac{\Gamma(\sigma/2)\zeta(\sigma)}{\Gamma((\sigma-1)/2)\zeta(\sigma-1)} \\ &= \frac{\sigma+1}{\sigma(A\sigma-1)} \cdot \left(1+O\left(\sigma^{-1}\right)\right) \cdot \frac{\Gamma(\sigma/2)}{\Gamma((\sigma-1)/2)} \cdot O(1) \\ &= \frac{\sigma+1}{\sigma(A\sigma-1)} \cdot \left(1+O\left(\sigma^{-1}\right)\right) \cdot O\left(\sqrt{\sigma}\right). \end{aligned}$$

Here we used the Stirling formula in the third equation. Thus

$$g(\sigma) = O\left(\sigma^{-1/2}\right) \quad \text{as } \sigma \to +\infty. \tag{29}$$

For $h(\sigma)$, by using (27), we have

$$h(\sigma) = O\left(\sigma^{-1}\right) \quad \text{as } \sigma \to +\infty. \tag{30}$$

From (28)–(30), we obtain

$$\frac{f(1-\sigma)}{f(\sigma)} = O\left(\sigma^{-1}\right) \quad \text{as } \sigma \to +\infty.$$

This shows that $f(s)$ satisfies (v).

2.2. Proof of (ii) and (iii)

Lemma 7. *There exists $\sigma_1 < 0$ such that the function $f(s)$ in* (18) *has no zero on the left half-plane $\Re(s) < \sigma_1$.*

Proof. Assume that $\sigma = \Re(s) < 0$. We have

$$f(s) = (s-1)(As-A+1)\chi(s+1)\left[1-R(s)\right]$$

with

$$R(s) = \frac{s-2}{(s-1)(As-A+1)} \cdot \frac{\chi(s)}{\chi(s+1)}.$$

The factor $(s-1)(As-A+1)\chi(s+1)$ has no zero in the left half-plane $\Re(s) < 1-(1/A) \simeq -20.187$, because of the Euler product of $\chi(s)$. Use the functional equation of $\chi(s)$,

$$\begin{aligned} R(s) &= \frac{s-2}{(s-1)(As-A+1)} \frac{\chi(1-s)}{\chi(-s)} = \frac{s-2}{(s+1)(As-A+1)} \frac{\xi(1-s)}{\xi(-s)} \\ &= \frac{s-2}{(s+1)(As-A+1)} \frac{\Gamma((1-s)/2)}{\sqrt{\pi}\,\Gamma(-s/2)} \frac{\zeta(1-s)}{\zeta(-s)}. \end{aligned}$$

Therefore

$$\left|R(s)\right| \leqslant \frac{1}{\sqrt{\pi}} \left|\frac{s-2}{(s+1)(As-A+1)}\right| \left|\frac{\Gamma((1-s)/2)}{\Gamma(-s/2)}\right| \zeta(-\sigma)\zeta(1-\sigma).$$

If $\sigma = \Re(s) < 0$, we have $|\arg(-s/2)| < \pi/2$ and $|\arg(1-s)/2| < \pi/2$. Hence we can apply the Stirling formula for $\Re(s) < 0$. We obtain

$$\left|\frac{\Gamma((1-s)/2)}{\Gamma(-s/2)}\right| = \frac{|s|^{1/2}}{\sqrt{2}} \cdot \left|1-\frac{1}{s}\right|^{-1} \cdot \frac{1+O(|s|^{-1})}{1+O(|s|^{-1})} = O\left(|s|^{1/2}\right).$$

On the other hand

$$\zeta(-\sigma)\zeta(1-\sigma) \to 1 \quad (\sigma \to -\infty).$$

Therefore

$$\bigl|R(s)\bigr| = \frac{1}{\sqrt{2\pi}} \cdot \frac{|s|^{1/2}}{|(As-A+1)|}\bigl(1+o(1)\bigr) = \frac{1}{A\sqrt{2\pi|s|}}\bigl(1+o(1)\bigr),$$

if $\sigma = \Re(s) < 0$, and $|s|$, $|\sigma|$ are both large. This implies Lemma 7. □

Lemma 8. *The entire function $f(s)$ in (18) has only finitely many zeros on the right half-plane $\Re(s) > 0$. In particular, the number of zeros of $f(s)$ on $\Re(s) \geqslant 1/2$ is finite.*

Proof. We have

$$f(s) = (s-1)(As-A+1)\chi(s+1)\left[1 - \frac{s-2}{(s-1)(As-A+1)} \cdot \frac{\chi(s)}{\chi(s+1)}\right]. \tag{31}$$

By [6, pp. 109–110], we have $|\chi(2s-1)/\chi(2s)| < 1$ for any $\Re(s) > 1/2$. Replacing $2s-1$ by s, we obtain

$$\left|\frac{\chi(s)}{\chi(s+1)}\right| < 1 \quad \bigl(\Re(s) > 0\bigr). \tag{32}$$

Let D be the region

$$D := \left\{ s \in \mathbb{C} \;\middle|\; \Re(s) \geqslant 0,\ \left|\frac{s-2}{(s-1)(As-A+1)}\right| \geqslant 1 \right\}.$$

Then $f(s) \neq 0$ if $s \notin D$ and $\Re(s) \geqslant 0$, because of (31) and (32). The region D is bounded, since

$$\left|\frac{s-2}{(s-1)(As-A+1)}\right| < 1$$

for large $|s|$. Hence the number of zeros of $f(s)$ in $\Re(s) \geqslant 0$ is finite. □

Proof of (ii) and (iii). Lemmas 7 and 8 show that (ii) and (iii) hold for $f(s)$. □

2.3. Proof of Proposition 4

By the results in Sections 2.1 and 2.2, we can apply Lemma 5 to $f(s)$. On the other hand, we have $f(0) \simeq 1.047 \neq 0$. Hence the proof of Proposition 4 is completed by the following lemma.

Lemma 9. *The number of zeros of $f(s)$ in $\Re(s) \geqslant 1/2$ is just two. They are nonreal zeros and conjugate each other. The values of them are about $s \simeq 0.927 \pm i \cdot 3.20$.*

Remark 10. This lemma is used to simplify the proof of Theorem 2. However the explicit values of exceptional zeros are not essential in the proof. We already know explicitly the region that $f(s)$ possibly have a zero. From this fact, to prove Theorem 2, it is sufficient that we know the explicit number of zeros of $f(s)$ in that region.

Proof. The domain $D \cap \{\Re(s) \leqslant 1/2\}$ is contained in the rectangle $R = [1/2, 2] \times [-10, 10]$, where D is the region in the proof of Lemma 8. Because of the argument principle, the number of zeros of $f(s)$ in R is given by

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f'}{f}(s)\, ds.$$

In particular, the value of this integral is an integer. Therefore we can check computationally that the value of this integral is just two (for example Mathematica, Maple, PARI/GP, etc.). Hence we conclude that $f(s)$ has just two zeros in the rectangle R. By another computational way, we find the approximated values of these two zeros are $s \simeq 0.927 \pm i \cdot 3.20$. □

3. Second step for Theorem 2

Proof of Theorem 2. We have the following three assertions.

Proposition 11. *$Z(s)$ has no zero on the right half-plane $\Re(s) \geqslant 20$.*

Proposition 12. *$Z(s)$ has no zero in the region $1/2 < \sigma < 20$, $|t| \geqslant 22$.*

Proposition 13. *$Z(s)$ has only one simple zero $s = 1$ in the region $1/2 < \sigma < 20$, $|t| \leqslant 22$.*

Then, as a consequence of these three results and the functional equation of $Z(s)$, all zeros of $Z(s)$ lie on the line $\Re(s) = 1/2$ except for simple zeros $s = 0, 1/2, 1$. □

Among the above three assertions, the hardest part is in the proof of Proposition 12. To prove Proposition 12, we use the results in the first step and a result of Lagarias [5] concerning the distribution of the zeros of the Riemann zeta function.

3.1. Proof of Proposition 11

We have

$$Z(s) = (s-1)(As - A + 1) \cdot \chi(s+1)\chi(2s)\big(1 - R_1(s) - R_2(s) - R_3(s)\big), \tag{33}$$

where

$$\begin{aligned} R_1(s) &= \frac{(s-2) \cdot \chi(s)}{(s-1)(As - A + 1) \cdot \chi(s+1)}, \\ R_2(s) &= \frac{s(As-1) \cdot \chi(s-1)\chi(2s-1)}{(s-1)(As - A + 1) \cdot \chi(s+1)\chi(2s)}, \\ R_3(s) &= \frac{(s+1) \cdot \chi(s)\chi(2s-1)}{(s-1)(As - A + 1) \cdot \chi(s+1)\chi(2s)}. \end{aligned} \tag{34}$$

For each $R_i(s)$ $(i = 1, 2, 3)$, we have

$$\begin{aligned} \big|R_1(s)\big| &= \left| \frac{s-2}{(s+1)(As - A + 1)} \frac{\sqrt{\pi}\, \Gamma(s/2)}{\Gamma((s+1)/2)} \frac{\zeta(s)}{\zeta(s+1)} \right| \\ &\leqslant \pi^{1/2} \left| \frac{s-2}{(s+1)(As - A + 1)} \right| \left| \frac{\Gamma(s/2)}{\Gamma((s+1)/2)} \right| \zeta(\sigma)\zeta(\sigma+1), \end{aligned} \tag{35}$$

$$\begin{aligned}\left|R_2(s)\right| &= \pi^{3/2}\left|\frac{(s-1)(s-2)(As-1)}{s(s+1)(As-A+1)}\frac{\Gamma((s-1)/2)\Gamma(s-1/2)}{\Gamma((s+1)/2)\Gamma(s)}\frac{\zeta(s-1)\zeta(2s-1)}{\zeta(s+1)\zeta(2s)}\right| \\ &\leqslant 2\pi^{3/2}\left|\frac{(s-2)(As-1)}{s(s+1)(As-A+1)}\right|\left|\frac{\Gamma(s-1/2)}{\Gamma(s)}\right|\zeta(\sigma+1)\zeta(\sigma-1)\zeta(2\sigma)\zeta(2\sigma-1), \end{aligned} \tag{36}$$

and

$$\begin{aligned}\left|R_3(s)\right| &= \pi\left|\frac{s-1}{s(As-A+1)}\frac{\Gamma(s/2)\Gamma(s-1/2)}{\Gamma((s+1)/2)\Gamma(s)}\frac{\zeta(s)\zeta(2s-1)}{\zeta(s+1)\zeta(2s)}\right| \\ &\leqslant \pi\left|\frac{s-1}{s(As-A+1)}\right|\left|\frac{\Gamma(s-1/2)}{\Gamma(s)}\right|\zeta(\sigma+1)\zeta(\sigma)\zeta(2\sigma)\zeta(2\sigma-1). \end{aligned} \tag{37}$$

Using the Stirling formula, we obtain

$$\left|R_1(s)\right| = O\left(|s|^{-1/2}\right), \qquad \left|R_2(s)\right| = \left(|s|^{-3/2}\right), \qquad \left|R_3(s)\right| = \left(|s|^{-1}\right) \tag{38}$$

as $|s| \to \infty$ on the right half-plane. Therefore $Z(s) \neq 0$ for some right half-plane $\Re(s) \geqslant \sigma_2$. Using the monotone decreasing property of $\zeta(\sigma)$ as $\sigma \to +\infty$ and the effective version of Stirling's formula [7]

$$\Gamma(s) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}}\left(\frac{s}{e}\right)^{s}\left\{1+\Theta\left(\frac{1}{8|s|}\right)\right\} \quad \left(\Re(s) > 1\right), \tag{39}$$

where the notation $f = \Theta(g)$ means $|f| \leqslant g$, we have

$$\left|R_1(s)\right| \leqslant 0.5, \qquad \left|R_2(s)\right| \leqslant 0.1, \qquad \left|R_3(s)\right| \leqslant 0.3 \tag{40}$$

for $\Re(s) \geqslant 20$ (in fact, these bounds already hold for $\Re(s) \geqslant 10$). These estimates imply $Z(s) \neq 0$ for $\Re(s) \geqslant 20$ by (33), since $(s-1)(As-A+1)\cdot\chi(s+1)\chi(2s)$ has no zero in the right half-plane $\Re(s) \geqslant 20$.

3.2. *Proof of Proposition 12*

Let $\rho_0 = \beta_0 + i\gamma_0$ ($\gamma_0 > 0$) be the zero of $f(s)$ in Lemma 9. By Proposition 4 we have

$$f(s) = f(0)e^{B's}\left(1-\frac{s}{\rho_0}\right)\left(1-\frac{s}{\bar{\rho}_0}\right)\cdot\Pi(s) \quad (B' \geqslant 0), \tag{41}$$

where

$$\Pi(s) = \prod_{\substack{\beta<1/2 \\ 0\neq\beta\in\mathbb{R}}}\left(1-\frac{s}{\beta}\right)\prod_{\substack{\rho=\beta+i\gamma \\ \beta<1/2,\,\gamma>0}}\left[\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{\bar{\rho}}\right)\right] \quad (\rho=\beta+i\gamma). \tag{42}$$

Have in mind that all zeros of $\Pi(s)$ lie in $\sigma_0 < \Re(s) < 1/2$ for some σ_0 by Lemmas 8 and 9. By definition (18) of $g(s)$,

$$Z(s) = g(s)\cdot\left(1-\frac{g(1-s)}{g(s)}\right) \quad \left(g(s) = f(s)\cdot\chi(2s)\right). \tag{43}$$

On the other hand, by (41),

$$\left|\frac{g(1-s)}{g(s)}\right| = e^{B'(1-2\sigma)}\cdot\left|\frac{\Pi(1-s)}{\Pi(s)}\right|\cdot\left|\frac{s-1+\rho_0}{s-\rho_0}\cdot\frac{s-1+\bar{\rho}_0}{s-\bar{\rho}_0}\right|\cdot\left|\frac{\chi(2s-1)}{\chi(2s)}\right|. \tag{44}$$

Because $B' \geqslant 0$ by Lemma 5,

$$e^{B'(1-2\sigma)} \leqslant 1 \quad \big(\Re(s) > 1/2\big). \tag{45}$$

For the ratio of $\Pi(s)$ in (44), we have

$$\left|\frac{\Pi(1-s)}{\Pi(s)}\right| = \prod_{\substack{\rho=\beta+i\gamma \\ \beta<1/2,\, \gamma>0}} \left(\left|\frac{1-s-\overline{\rho}}{s-\rho}\right| \cdot \left|\frac{1-s-\rho}{s-\overline{\rho}}\right|\right) < 1 \quad \big(\Re(s) > 1/2\big), \tag{46}$$

by term-by-term argument as in [6] by using $\beta < 1/2$ and

$$\left|\frac{1-s-\overline{\rho}}{s-\rho}\right|^2 = 1 - \frac{(2\sigma-1)(1-2\beta)}{(\sigma-\beta)^2+(t-\gamma)^2},$$

where $\rho = \beta + i\gamma$ is a zero of $f(s)$. It remains to give an estimate for

$$r(s) := \left|\frac{s-1+\rho_0}{s-\rho_0} \cdot \frac{s-1+\overline{\rho}_0}{s-\overline{\rho}_0}\right| \cdot \left|\frac{\chi(2s-1)}{\chi(2s)}\right|. \tag{47}$$

To evaluate $r(s)$, we use the following lemma essentially.

Lemma 14. *(See [5].) For any real t with $|t| \geqslant 12$ there exist at least two distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leqslant 1/2$ and*

$$|t-\gamma| \leqslant 10.1. \tag{48}$$

Proof. Suppose $t \geqslant 25$. Then there exist at least two distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ satisfying $0 < \beta \leqslant 1/2$ and $|t-\gamma| < 10.1$ by applying Lemma 5 in [8] to $t+5.1$ and $t-5.1$ (Lemma 5 in [8] is essentially Lemma 3.5 of [5]). For $12 \leqslant t < 25$, estimate (48) also holds for two zeros because $\xi(s)$ has zeros at $s = \pm 14.13, \pm 21.02, \pm 25.01$. □

Using Lemma 14 we show the following.

Lemma 15. *Let $\rho_0 = \beta_0 + i\gamma_0 \simeq 0.927 + i \cdot 3.20$ be the zero of $f(s)$ in Lemma 9. Let $s = \sigma + it$ with $1/2 < \sigma \leqslant 20$ and $t \geqslant 22$. Then there exist at least two distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leqslant 1/2$, $|t-\gamma| \leqslant 10.1$,*

$$\left|\frac{s-1+\overline{\rho_0}}{s-\rho_0}\right| \cdot \left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right| < 1 \tag{49}$$

and

$$\left|\frac{s-1+\rho_0}{s-\overline{\rho_0}}\right| \cdot \left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right| < 1. \tag{50}$$

Proof. By squaring (49) and (50) we have

$$\frac{(\sigma+\beta_0-1)^2+(t\pm\gamma_0)^2}{(\sigma-\beta_0)^2+(t\pm\gamma_0)^2} \cdot \frac{(2\sigma+\beta-2)^2+(t-\gamma)^2}{(2\sigma-\beta)^2+(t-\gamma)^2} < 1. \tag{51}$$

To prove Lemma 15 it is sufficient that (51) holds for $0 < \beta \leqslant 1/2$, $|t-\gamma| < 10.1$, $1/2 < \sigma \leqslant 20$ and $t \geqslant 22$, because of Lemma 14. To establish (51) in that conditions it suffices to show that

$$\frac{(\sigma+\beta_0-1)^2+(t\pm\gamma_0)^2}{(\sigma-\beta_0)^2+(t\pm\gamma_0)^2}\cdot\frac{(2\sigma-\frac{3}{2})^2+10^2}{(2\sigma-\frac{1}{2})^2+10^2}<1 \tag{52}$$

by a similar reason in the later half of §4.3 in [8]. This inequality is equivalent to

$$(2\sigma-1)\big(8(t\pm\gamma_0)^2-P(\sigma)\big)>0, \tag{53}$$

where $P(\sigma)=8(4\beta_0-3)\sigma^2-8(4\beta_0-3)\sigma-8\beta_0^2+818\beta_0-409$. Using the value $\beta_0 \simeq 0.927$ we find that $P(\sigma) < 2580$ for $1/2 < \sigma < 20$. On the other hand, using the value $\gamma_0 \simeq 3.20$ we find that $8(t\pm\gamma_0)^2 > 2590$ for $t \geqslant 22$ since $|t\pm\gamma_0| = t\pm\gamma_0 > 18$ for $t \geqslant 22$. Hence (53) holds, and it implies (51). □

Lemma 15 and $\overline{Z(s)} = Z(\bar{s})$ imply

$$\big|r(s)\big|<1 \quad \text{for } 1/2<\sigma\leqslant 20,\ |t|\geqslant 22 \tag{54}$$

by taking two distinct zeros of $\xi(s)$ in this region, since we have the inequality

$$\left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right|<1 \quad \big(\Re(s)>1/2\big), \tag{55}$$

for other terms in $r(s)$, where ρ is a zero of $\xi(s)$ $(0<\Re(\rho)<1)$. Estimates (45), (46) and (54) show that

$$\left|\frac{g(1-s)}{g(s)}\right|<1 \tag{56}$$

for $1/2 < \sigma \leqslant 20$, $|t| \geqslant 22$. By (43) this estimate implies Proposition 12, since $g(s)$ has no zero in $1/2 < \sigma \leqslant 20$, $|t| \geqslant 22$ by Lemmas 8 and 9.

3.3. *Proof of Proposition 13*

Because the region $1/2 < \sigma \leqslant 20$, $|t| \leqslant 22$ is finite, we can check Proposition 13 by using the computer as in the proof of Lemma 9.

4. Proof of Lemma 5

We prove the lemma only if $F(s)$ has genus one, since if $F(s)$ has genus zero it is easily proved by a way similar to the case of genus one. The genus one assumption is equivalent to the Hadamard product factorization

$$F(s)=e^{A+Bs}s^m\prod_{\rho}\left(1-\frac{s}{\rho}\right)\exp(s/\rho) \quad (m\in\mathbb{Z}_{\geqslant 0}) \tag{57}$$

converges absolutely and uniformly on every compact subsets of $\mathbb{C}$. It is also equivalent to $\sum_\rho |\rho|^{-2} < \infty$. Assumption (i) implies the symmetry of the set of zeros under the conjugation $\rho \mapsto \overline{\rho}$. It follows that the set of zeros $\rho = \beta + i\gamma$, counted with multiplicity, is partitioned into blocks $B(\rho)$ comprising

$\{\rho, \overline{\rho}\}$ if $\gamma > 0$ and $\{\rho\}$ if $\beta \neq 0$ and $\gamma = 0$. Each block is labeled with the unique zero in it having $\gamma \geqslant 0$. Using assumption (ii), we show

$$F(s) = s^m e^{A+B's} \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left(1 - \frac{s}{\rho}\right)\right) \tag{58}$$

where the outer product on the right-hand side converges absolutely and uniformly on every compact subsets of $\mathbb{C}$. This assertion holds because the block convergence factors $\exp(c(B(\rho))s)$ are given by $c(B(\rho)) = 2\beta|\rho|^{-2}$ for $\gamma > 0$. Assumption (ii) implies $|\beta - 1/2| < \sigma_0$. Hence

$$\sum_{B(\rho)} \left|c\big(B(\rho)\big)\right| \leqslant \sum_{0 \neq \rho:\, \mathrm{real}} |\rho|^{-1} + (2\sigma_0 + 1) \sum_{\rho} |\rho|^{-2} < \infty.$$

Thus the convergence factors $\exp(c(B(\rho))s)$ can be pulled out of the product. Hence we have (58) with

$$B' = B + \sum_{B(\rho)} c\big(B(\rho)\big). \tag{59}$$

Using assumptions (iii)–(v) we show

$$B' \geqslant 0. \tag{60}$$

By (24) in assumption (v) we have

$$\mathbb{R} \ni \log\left(\frac{F(1-\sigma)}{F(\sigma)}\right) \to -\infty \quad \text{as } \sigma \to +\infty. \tag{61}$$

Using (58) we have

$$\frac{F(1-\sigma)}{F(\sigma)} = e^{B'(1-2\sigma)} \left(\frac{\sigma - 1}{\sigma}\right)^m \prod_{\rho = \beta \in \mathbb{R}} \frac{\sigma - 1 + \beta}{\sigma - \beta} \prod_{\substack{\rho = \beta + i\gamma \\ \gamma > 0}} \frac{(\sigma - 1 + \beta)^2 + \gamma^2}{(\sigma - \beta)^2 + \gamma^2}.$$

Thus

$$\log\left(\frac{F(1-\sigma)}{F(\sigma)}\right) = B'(1-2\sigma) + m \log\left(1 - \frac{1}{\sigma}\right) + \sum_{\rho = \beta \in \mathbb{R}} \log\left(1 - \frac{1 - 2\beta}{\sigma - \beta}\right)$$
$$+ \sum_{\substack{\rho = \beta + i\gamma \\ \gamma > 0}} \log\left(1 - \frac{(1-2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2}\right). \tag{62}$$

Note that

$$\log\left(1 - \frac{(1-2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2}\right) < 0 \quad \text{for } \sigma > 1/2 \tag{63}$$

if $\beta < 1/2$, and

$$\log\left(1 - \frac{1}{\sigma}\right), \log\left(1 - \frac{1 - 2\beta}{\sigma - \beta}\right), \log\left(1 - \frac{(1-2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2}\right) \to 0 \quad \text{as } \sigma \to +\infty$$

for any fixed $\rho = \beta + i\gamma$. By assumption (iii), (63) holds except for finitely many zeros. Hence if we suppose $B' < 0$, (61) and (62) imply

$$\left|\sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| \geqslant 2|B'|\sigma \tag{64}$$

for large $\sigma > 1/2$, because the number of real zeros is also finite by assumptions (ii) and (iii). On the other hand, for sufficiently large $\sigma > 1/2$, we have

$$\begin{aligned}\left|\sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| &\leqslant \left|\sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\sigma_0)(2\sigma-1)}{(\sigma-1/2)^2+\gamma^2}\right)\right| \\ &\ll (2\sigma-1) \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \frac{1}{(\sigma-1/2)^2+\gamma^2}.\end{aligned}$$

The sum on the right-hand side is written as the Stieltjes integral

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2+t^2}.$$

Using (23) in (iv) we have

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2+t^2} \ll \int_{\gamma_0}^{\infty} \frac{(\log t)\,dt}{(\sigma-1/2)^2+t^2} \ll \frac{\log(\sigma+\gamma_0)}{\sigma-1/2}.$$

Hence we obtain

$$\left|\sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| \ll \log(\sigma+\gamma_0) \tag{65}$$

for sufficiently large $\sigma > 1/2$. This contradicts (64). Thus (60) holds.

5. Proof of Corollary 3

By Theorem 2, it remains to show that all zeros of $V(s)$ in (16) lie on the line $\Re(s) = 1/2$. Taking $v(s) = (s-1)(As - A + 1)\chi(s+1)$ and using the functional equation of $\chi(s)$, we have $V(s) = v(s)\chi(2s) - v(1-s)\chi(2s-1)$. All zeros of $v(s)$ lie in the strip $1 - (1/A) < \Re(s) < 0$ except for the simple zero $s = 1$. Then we find that all zeros of $V(s)$ lie on the line $\Re(s) = 1/2$ by a way similar to Section 2 replacing ρ_0 by 1.

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Appendix: Zeta functions for *Sp*(2*n*) ☆

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A.1. Introduction

Associated to a number field F is the genuine high rank zeta function $\xi_{F,r}(s)$ for every fixed $r \in \mathbb{Z}_{>0}$. Being natural generalizations of (completed) Dedekind zeta functions, these functions satisfy canonical properties for zetas as well. Namely, they admit meromorphic continuations to the whole complex s-plane, satisfy the functional equation $\xi_{F,r}(1-s) = \xi_{F,r}(s)$ and have only two singularities, all simple poles, at $s = 0, 1$. Moreover, it is known that all zeros of $\xi_{F,2}(s)$ lie on the central line $\mathrm{Re}(s) = \frac{1}{2}$. (We in fact now expect that the Riemann Hypothesis holds for all $\xi_{F,r}(s)$.)

Recall that $\xi_{F,r}(s)$ is defined by

$$\xi_{F,r}(s) := \big(|\Delta_F|\big)^{\frac{rs}{2}} \int\limits_{\mathcal{M}_{F,r}} \big(e^{h^0(F,\Lambda)} - 1\big) \cdot \big(e^{-s}\big)^{\deg(\Lambda)}\, d\mu(\Lambda), \quad \mathrm{Re}(s) > 1$$

where Δ_F denotes the discriminant of F, $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_F$-lattices of rank r (here $\mathcal{O}_F$ denotes the ring of integers), $h^0(F,\Lambda)$ and $\deg(\Lambda)$ denote the 0th geo-arithmetic cohomology and the arithmetic degree of the lattice Λ, and $d\mu(\Lambda)$ a certain naturally associated Tamagawa type measure on $\mathcal{M}_{F,r}$. (For details, see [W0,W1,W3] for basic theory, and [LS,W2,S], see also [H], for the Riemann Hypothesis arguments.)

Algebraic groups associated to $\mathcal{O}_F$-lattices are general linear group GL and special linear group SL. A natural question then is whether principal lattices associated to other reductive groups G and

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their associated zeta functions can be introduced and studied. In this paper we start with symplectic group Sp. In contrasting with a geo-arithmetic method used for high rank zetas, the one adopted in this paper is rather analytic. And to avoid further complication, we here only work out the full details for $Sp(4)$ over $\mathbb{Q}$, even a general framework for Sp over F is outlined. As a concrete result, we obtain a precise formula for the zeta function $\xi_{Sp(4),\mathbb{Q}}(s)$ associated to $Sp(4)$ over $\mathbb{Q}$, by studying a certain Siegel–Eisenstein period.

The newly obtained zeta $\xi_{Sp(4),\mathbb{Q}}(s)$ for $Sp(4)$ over $\mathbb{Q}$ proves to be canonical as well. For example, $\xi_{Sp(4),\mathbb{Q}}(s)$ can be meromorphically extended to the whole complex s-plane, satisfies the standard functional equation

$$\xi_{Sp(4),\mathbb{Q}}(1-s) = \xi_{Sp(4),\mathbb{Q}}(s)$$

and admits only four singularities, all simple poles, at $s = -1, 0, 1, 2$. Most importantly, $\xi_{Sp(4),\mathbb{Q}}(s)$ satisfies the RH, a result due to Suzuki [S2]:

All zeros of $\xi_{Sp(4),\mathbb{Q}}(s)$ lie on the central line $\mathrm{Re}(s) = 1/2$.

A.2. Periods for $Sp(2n)$

A.2.1. Siegel–Maaß Eisenstein series

Let $G = Sp(2n)$ with $G(\mathbb{R}) = Sp(2n, \mathbb{R})$ the symplectic group of degree n over $\mathbb{R}$. Denote by $\mathfrak{S} := \mathfrak{S}_n$ the so-called Siegel upper half-space of size n, and for any $Z \in \mathfrak{S}$, write $Z = X + \sqrt{-1}Y$ according to its real and imaginary parts, so that $Y = \mathrm{Im}\, Z > 0$ and $Z^t = Z$ is symmetric. Moreover, for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2n, \mathbb{R})$, as usual, set $M\langle Z\rangle := (AZ + B) \cdot (CZ + D)^{-1}$ and write $Y(M) := \mathrm{Im}\, M\langle Z\rangle$. This defines a natural transitive action of $Sp(2n, \mathbb{R})$ on $\mathfrak{S}_n$. Note that the stabilizer of $\sqrt{-1}E_n$ is simply $SO(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R})$, consequently, we obtain the following well-known identification $Sp(2n, \mathbb{R})/SO(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R}) \simeq \mathfrak{S}_n$.

Introduce the Siegel modular group $\Gamma_n := \{\mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)\} \setminus Sp(n, \mathbb{Z})$. Let $\mathfrak{P}_n := \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \Gamma\right\}$ be a subgroup of Γ (associated to a certain standard maximal parabolic subgroup).

Fix $Z \in Sp(2n, \mathbb{R})$, define then the associated Siegel–Maaß Eisenstein series, or the same, the Siegel–Epstein zeta function by

$$E_n(Z; s) := \sum_{\gamma \in \mathfrak{P}_n \setminus \Gamma_n} \frac{|Y|^{-s}}{\|CZ + D\|^{-2s}}.$$

A.2.2. Arthur's analytic truncation and Eisenstein period

As usual, for symplectic group $G = Sp(2n)$, and for a parabolic group P with Levi decomposition $P = MN$ with M the Levi and N the nilpotent, denote by $\mathfrak{a}_0$ (resp. $\mathfrak{a}_P$) the space of characters associated to the Borel (resp. to P). Denote by Δ_0 the corresponding collection of simple roots. By definition, an element $T \in \mathfrak{a}_0$ is said to be *sufficiently regular* and denoted by $T \gg 0$ if $\langle \alpha, T\rangle \gg 0$ are large enough for all $\alpha \in \Delta_0$. Fix such a T. Let $\phi : \Gamma \setminus \mathfrak{S}_n \to \mathbb{C}$ be a smooth function. We define Arthur's analytic truncation $\wedge^T \phi$ (for ϕ with respect to the parameter T) to be the function on $Sp(2n, \mathbb{Z}) \setminus Sp(2n, \mathbb{R})$ given by

$$\left(\wedge^T \phi\right)(Z) := \sum_{P:\,\mathrm{standard}} (-1)^{\mathrm{rank}(P)} \sum_{\delta \in P(\mathbb{Z}) \setminus Sp(2n,\mathbb{Z})} \phi_P(\delta g) \cdot \hat{\tau}_P\left(H_P(\delta g) - T\right),$$

where ϕ_P denotes the constant term of ϕ along with the standard parabolic subgroup P, $\hat{\tau}_P$ is the characteristic function of the so-called positive cone in $\mathfrak{a}_P$, and $H_P(Z) := \log_M m_P(Z)$ is an element in $\mathfrak{a}_P$. (For unknown notation, all standard, see e.g., [Ar1,Ar2,JLR], and/or [We-1,W3].)

Fundamental properties of Arthur's truncation may be summarized in the following:

Theorem 2.2.1. *([Ar1,Ar2], see also [OW].) For a sufficiently positive T in $\mathfrak{a}_0$, we have*

(1) $\wedge^T \phi$ *is rapidly decreasing, if* ϕ *is an automorphic form on* $G(\mathbb{Z}) \setminus \mathfrak{S}_n$*,*
(2) $\wedge^T \circ \wedge^T = \wedge^T$*,*
(3) $\wedge^T$ *is self-adjoint,*
(4) $\wedge^T \mathbf{1}$ *is a characteristic function of a compact subset of* $G(\mathbb{Z}) \setminus \mathfrak{S}_n$*.*

Denote by $\mathfrak{F}(T)$ the compact subset of $G(\mathbb{Z}) \setminus \mathfrak{S}_n$ whose characteristic function is given by $\wedge^T \mathbf{1}$ in (4).

Corollary 2.2.1. *([W1,W3], see also [KW].) Let* $T \gg 0$ *be a fixed element in* $\mathfrak{a}_0$*. For an automorphic form* ϕ *on* $G(\mathbb{Z}) \setminus \mathfrak{S}_n$*,*

$$\int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \wedge^T \phi(g)\, dg = \int_{\mathfrak{F}(T)} \phi(g)\, dg.$$

Proof. By (1), $\int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \wedge^T \phi(g)\, dg$ is well defined. Moreover,

$$\begin{aligned}
&\int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \wedge^T \phi(g)\, dg \\
&\quad = \int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \mathbf{1}(g) \cdot \left(\wedge^T \circ \wedge^T\right)\phi(g)\, dg \quad \big(\text{by (2) above}\big) \\
&\quad = \int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \wedge^T \mathbf{1}(g) \cdot \wedge^T \phi(g)\, dg \quad \big(\text{by (3) above since } \wedge^T \phi(g) \text{ is rapidly decreasing}\big) \\
&\quad = \int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \left(\wedge^T \circ \wedge^T\right)\mathbf{1}(g) \cdot \phi(g)\, dg \\
&\qquad \big(\text{by (3) again since } \phi \text{ is of moderate growth and } \wedge^T \mathbf{1} \text{ is compactly supported}\big) \\
&\quad = \int_{G(\mathbb{Z})\setminus\mathfrak{S}_n} \wedge^T \mathbf{1}(g) \cdot \phi(g)\, dg \quad \big(\text{by (2) again}\big) \\
&\quad = \int_{\mathfrak{F}(T)} \phi(g)\, dg \quad \big(\text{by (4)}\big). \qquad \square
\end{aligned}$$

A.2.3. Siegel–Maaß-period: an analog of high rank zeta

Motivated by our study on the high rank zeta associated to $SL(n)$ in [W1,W3], we define the *Siegel–Maaß-period for* $Sp(n)$ *over* $\mathbb{Q}$, a special kind of the so-called Eisenstein period, by

$$Z^T_{Sp(2n),\mathbb{Q}}(s) := \int_{\Gamma\setminus\mathfrak{S}_n} \wedge^T E_n(Z; s)\, d\mu(Z).$$

This is then a function on s depending also on the parameter T. By Corollary 2.2.1,

$$Z^T_{Sp(2n),\mathbb{Q}}(s) = \int_{\mathfrak{F}(T)} E_n(Z; s)\, d\mu(Z).$$

Thus the study of $Z^T_{Sp(2n),\mathbb{Q}}(s)$ may be carried out from that for the Siegel–Maaß series $E_n(Z; s)$.

A.2.4. Siegel–Eisenstein series

In general, it is *very difficult*, most of the time, *quite impossible*, to calculate Eisenstein periods. However, if the original automorphic form from which the Eisenstein series in use is defined is *cuspidal*, then an advanced version of Rankin–Selberg method can be applied to evaluate them. Motivated by this, we in this subsection explain a method due to Diehl to realize the Siegel–Maaß series, which may be viewed as an Eisenstein series associated to the constant function one on a certain *maximal parabolic*, as the residue of the so-called Siegel–Eisenstein series associated to the constant function one on the *Borel*.

As usual, corresponding to the partition $n = r + 1 + 1 + \cdots + 1$, introduce the standard parabolic subgroup $\mathfrak{P}_r := \{\left(\begin{smallmatrix} A & * \\ 0 & B\end{smallmatrix}\right) \in \Gamma\}$, where $A = \begin{pmatrix} H^t & & & \\ & 1 & 0 & \\ & * & \ddots & \\ & & & 1\end{pmatrix}$, $B = \begin{pmatrix} H^{-1} & & & \\ & 1 & * & \\ & 0 & \ddots & \\ & & & 1\end{pmatrix}$ with $H = H^{(r)}$, $|H| = 1$. Accordingly, define the associated Siegel–Eisenstein series by

$$E_r(Z; s_r, \dots, s_n) := \sum_{\gamma \in \mathfrak{P}_r \backslash \Gamma} \prod_{\nu=r}^{n} \left| Y(\gamma)_\nu \right|^{-s_\nu}.$$

Here for a matrix $A = (a_{ij})_{i,j=1}^n$, denote by A_ν the matrix $A_\nu := (a_{ij})_{i,j=1}^\nu$, $1 \leqslant \nu \leqslant n$.

It is known that such Siegel–Eisenstein series are naturally related to the Siegel zeta functions associated to the standard parabolic subgroup $\mathfrak{Q}_r$ of $SL(n)$. More precisely, let $\mathfrak{R} := \{\mathrm{diag}(\pm 1, \dots, \pm 1)\} \backslash SL(n, \mathbb{Z})$ and $\mathfrak{Q}_r$ the standard parabolic subgroup associated to the partition $n = r + 1 + 1 + \cdots + 1$ consisting of matrices in $SL(n, \mathbb{Z})$ of the form $\begin{pmatrix} H & & & \\ & 1 & * & \\ & 0 & \ddots & \\ & & & 1\end{pmatrix}$ with $H = H^{(r)}$, $|H| = 1$. Define the associated Siegel zeta functions by

$$\xi_r^*(Y; s_r, \dots, s_{n-1}) := \sum_{N \in \mathfrak{Q}_r \backslash \mathfrak{R}} \prod_{\nu=r}^{n-1} \left| Y[N]_\nu \right|^{-s_\nu}$$

for all $1 \leqslant r \leqslant n-1$. Then, we have

Lemma 2.4.1. *(See [D].) With the same notation as above,*

(i)
$$E_r(Z; s_r, \dots, s_n) = \sum_{\gamma \in \mathfrak{B} \backslash \Gamma} \left| Y(\gamma) \right|^{-s_n} \cdot \xi_r^*\bigl(Y(\gamma); s_r, \dots, s_{n-1}\bigr);$$

(ii) *There exists a constant c depending only on r such that*

$$\mathrm{Res}_{s_r = \frac{r+1}{2}} \bigl(\xi_r^*(Y; s_r, \dots, s_{n-1})\bigr) = c_r \cdot \xi_{r+1}^*\left(Y; s_{r+1} + \frac{r}{2}, s_{r+2}, \dots, s_{n-1}\right).$$

Consequently, we have, up to constant factors,

$$\begin{aligned}\mathrm{Res}_{s_{r+1} + \frac{r}{2} = \frac{(r+1)+1}{2}} \mathrm{Res}_{s_r = \frac{r+1}{2}} \xi_r^*(Y; s_r, \dots, s_{n-1}) &= \mathrm{Res}_{s_{r+1} + \frac{r}{2} = \frac{(r+1)+1}{2}} \xi_{r+1}^*\left(Y; s_{r+1} + \frac{r}{2}, \dots, s_{n-1}\right) \\ &= \xi_{r+2}^*\left(Y; s_{r+2} + \frac{r+1}{2}, \dots, s_{n-1}\right).\end{aligned}$$

Thus, by taking $r = 1$ and repeating this process, we obtain the following

$$\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \big(\xi_1^*(Y; s_1, s_2, \dots, s_{n-1})\big) = |Y|^{-\frac{n-1}{2}}$$

up to a constant factor. In particular, we get the following

Lemma 2.4.2. *Up to a constant factor,*

$$\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \big(E_r(Z; s_r, \dots, s_n)\big) = E_n\left(Z; s_n + \frac{n-1}{2}\right).$$

Proof. Indeed, up to constant factors,

$$\begin{aligned}
&\mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} E_r(Z; s_r, \dots, s_n) \\
&\quad = \sum_{\gamma \in \mathfrak{B} \backslash \Gamma} \big|Y(\gamma)\big|^{-s_n} \cdot \mathrm{Res}_{s_{n-1}=1} \cdots \mathrm{Res}_{s_2=1} \mathrm{Res}_{s_1=1} \xi_r^*\big(Y(\gamma); s_r, \dots, s_{n-1}\big) \\
&\quad = \sum_{\gamma \in \mathfrak{B} \backslash \Gamma} \big|Y(\gamma)\big|^{-s_n} \cdot \big|Y(\gamma)\big|^{-\frac{n-1}{2}} = E_n\left(Z; s_n + \frac{n-1}{2}\right). \qquad \square
\end{aligned}$$

A.2.5. Relation with Langlands' Eisenstein series

To facilitate further discussions, we next write classical Siegel–Eisenstein series in terms of Langlands' language [L].

Let $\lambda = (z_1, z_2, \dots, z_n) \in \mathfrak{a}_0$, then for $Z = X + \sqrt{-1}Y \in \mathfrak{S}$, set

$$\mathbf{a}^\lambda(Z) = \prod_{v=1}^{n} a_v^{-z_v} \quad \text{with } a_v = |Y_v|/|Y_{v-1}|$$

and the Langlands–Eisenstein series associated to the constant function one on the Borel is defined by

$$E(\mathbf{1}; \lambda; Z) := \sum_{\delta \in B(\mathbb{Z}) \backslash \Gamma} \mathbf{a}^\lambda(\delta Z).$$

As such, then the so-called power function

$$\mathbf{p}_{-\mathbf{s}}(Y) := \prod_{\mu=1}^{n} |Y_\mu|^{-s_\mu}$$

is given by

$$\begin{aligned}
\prod_{\mu=1}^{n} |Y_\mu|^{-s_\mu} = \mathbf{p}_{-\mathbf{s}}(Y) = \mathbf{a}^\lambda(Y) = \prod_{v=1}^{n} a_v^{-z_v} \\
= |Y_1|^{-z_1+z_2} |Y_2|^{-z_2+z_3} \cdots |Y_{n-1}|^{-z_{n-1}+z_n} |Y_n|^{-z_n}.
\end{aligned}$$

This then gives the following relations among z_i's and s_j's:

$$\begin{cases} s_1 = z_1 - z_2, \\ s_2 = z_2 - z_3, \\ \dots \\ s_{n-1} = z_{n-1} - z_n, \\ s_n = z_n. \end{cases}$$

Consequently, by Lemma 2.4.2, we have the following

Lemma 2.5.1. *With the same notation as above,*

(i) $E(\mathbf{1}; z_1, z_2, \dots, z_n; Z) = E_1(Z; s_1, s_2, \dots, s_n)$;
(ii) *up to a suitable constant factor,*

$$E_n\left(Z, z_n + \frac{n-1}{2}\right) = \mathrm{Res}_{z_{n-1}-z_n=1} \cdots \mathrm{Res}_{z_2-z_3=1} \mathrm{Res}_{z_1-z_2=1}\big(E(\mathbf{1}; z_1, z_2, \dots, z_n; Z)\big).$$

In particular, in the case $n = 2$, i.e, for $Sp(4)$, we have

$$\mathrm{Res}_{z_1-z_2=1}\big(E(\mathbf{1}; z_1, z_2; Z)\big) = E_2\left(Z, z_2 + \frac{1}{2}\right).$$

A.2.6. Advanced Rankin–Selberg & Zagier method

The advantage of using $E(\mathbf{1}; z_1, z_2, \dots, z_n; Z)$ instead of directly using $E_n(Z, s)$ is that the Eisenstein periods for $E(\mathbf{1}; z_1, z_2, \dots, z_n; Z)$ can be evaluated. Indeed, if φ is an automorphic form of $P' = M'N'$-level, where P' is a standard parabolic subgroup of a reductive group G, then we can form the associated Eisenstein series

$$E(\varphi; \lambda; g) := \sum_{\delta \in P'(\mathbb{Z})\backslash G(\mathbb{Z})} m_{P'}(\delta g)^{\lambda+\rho} \cdot \varphi(\delta g), \quad \lambda \in \mathcal{C}_{P'}.$$

Theorem 2.6.1. *(See [JLR, Corollary 17].) Let $P = MN$ be a minimal parabolic subgroup and let φ be a P-level cusp form. Let $E(\varphi; \lambda; g)$ be the Eisenstein series associated to φ. Then*

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})} \wedge^T E(g, \varphi, \lambda)\, dg$$

is equal to

$$v \sum_{w \in W} \frac{e^{\langle w\lambda - \rho, T\rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee\rangle} \int_{M(\mathbb{Z})\backslash M(\mathbb{R})^1 \times K} M(w, \lambda)\varphi(mk)\, dm\, dk,$$

where $v = \mathrm{Vol}(\{\sum_{\alpha \in \Delta_0} a_\alpha \alpha^\vee \colon 0 \leqslant a_\alpha < 1\})$, W denotes the Weyl group and $M(w, \lambda)$ denotes the so-called intertwining operator.

This is an advanced version of Rankin–Selberg & Zagier methods [Z,W0].

In particular, we have the following

Corollary 2.6.1. *With the same notation as above, up to a constant factor,*

$$\int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}} \left(\wedge^T E(\mathbf{1}; z_1, z_2, \ldots, z_n; M)\right) d\mu(M) = \sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,\ w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}.$$

Proof. In fact, by the Gindikin–Karpelevich formula, we have

$$M(w,\lambda) = \prod_{\alpha>0,\ w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}.$$

Here $\xi(s)$ is the completed Riemann zeta function with the usual Γ-factors, namely, $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. □

A.2.7. Periods for Sp(2n) over $\mathbb{Q}$

Recall that the Siegel–Maaß-period $Z^T_{Sp(2n),\mathbb{Q}}(s)$, an analog of high rank zeta for $Sp(2n)$, is equal to

$$\int_{\mathfrak{F}(T)} E_n(Z; s)\, d\mu(Z).$$

Thus to evaluate the Siegel–Maaß-period $Z^T_{Sp(2n),\mathbb{Q}}(s)$, it suffices to evaluate the integration

$$\int_{\mathfrak{F}(T)} \mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}\left(E(\mathbf{1}; z_1, z_2, \ldots, z_n; Z)\right) d\mu(Z).$$

Consequently, if we were able to freely make an interchange between

(i) the operation of taking integration $\int_{\mathfrak{F}(T)}$ and
(ii) the operation of taking residues $\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}$,

it would be sufficient for us to evaluate

$$\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}\left(\int_{\mathfrak{F}(T)} E(\mathbf{1}; z_1, z_2, \ldots, z_n; Z)\, d\mu(Z)\right),$$

or better, to evaluate the expression

$$\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}\left(\sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,\ w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

since by Corollary 2.2.1,

$$\int_{\mathfrak{F}(T)} E(\mathbf{1}; z_1, z_2, \ldots, z_n; Z)\, d\mu(Z) = \int_{Sp(2n,\mathbb{Z})\backslash\mathfrak{S}_n} \Lambda^T E(\mathbf{1}; z_1, z_2, \ldots, z_n; Z)\, d\mu(Z).$$

Unfortunately, this interchange of orders of two operations is not allowed in general. As examples, one can observe this by working on $SL(n)$ and by comparing the poles for the resulting expressions.

On the other hand, even with the existence of such discrepancies, the function

$$\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}\left(\sum_{w\in W}\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

proves to be extremely *natural and nice*. To see this, and to make the discussion simpler, let now concentrate to the case when $T=0$.

Definition.

(1) The period $\omega_{Sp(2n),\mathbb{Q}}(\lambda)$ associated to $Sp(2n)$ over $\mathbb{Q}$ is defined by

$$\omega_{Sp(2n),\mathbb{Q}}(\lambda):=\sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right).$$

(2) The period $Z_{\mathbb{Q}}^{Sp(2n)}(z_n)$ associated to $Sp(2n)$ over $\mathbb{Q}$ is defined by

$$Z_{\mathbb{Q}}^{Sp(2n)}(z_n):=\mathrm{Res}_{z_{n-1}-z_n=1}\cdots\mathrm{Res}_{z_2-z_3=1}\mathrm{Res}_{z_1-z_2=1}$$
$$\times\left(\sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}\right)\right).$$

Remarks.

(1) Recall that in the original discussions in Sections A.2.2, A.2.3 and A.2.6, T was assumed to be sufficiently positive, for the reason of Corollary 2.2.1. However, with the use of the above concrete expression, T can be chosen to be any element in $\mathfrak{a}_0$.
(2) For high rank zetas, i.e., in the case where the corresponding algebraic group is $SL(n)$, the corresponding $\mathfrak{F}(T)$ makes sense also for $T=0$. Indeed, $\mathfrak{F}(0)$ coincides with the moduli space of semi-stable lattices of volume one and rank n. Consequently, the corresponding period after putting $T=0$ gives essentially the high rank zeta $\xi_{\mathbb{Q},r}(s)$ there. So to obtain an analogue of high rank zeta for $Sp(2n)$ geo-arithmetically, we need to understand $\mathfrak{F}(0)$ geo-arithmetically.

A.3. Zetas for $Sp(2n)$

A.3.1. Zeta for $Sp(4)$

Now we focus on the case $G=Sp(4)$. Then $\Delta_0=\{e_1-e_2,2e_2\}$. There are 4 positive roots $\{e_1\pm e_2,2e_1,2e_2\}$ and the Weyl group consists of 8 Weyl elements

$$\{1,(12),c_1,c_2,(12)c_1,(12)c_2,(12)c_1c_2,c_1c_2\},$$

where (12) is the lower indices change and c_i's are sign changes. Also $\rho = 2e_1 + e_2$. Consequently, we have the following table for the Weyl action on positive roots:

	e_1-e_2	$2e_2$	e_1+e_2	$2e_1$
1	e_1-e_2	$2e_2$	e_1+e_2	$2e_1$
(12)	e_2-e_1	$2e_1$	e_1+e_2	$2e_2$
c_1	$-e_1-e_2$	$2e_2$	$-e_1+e_2$	$-2e_1$
c_2	e_1+e_2	$-2e_2$	e_1-e_2	$2e_1$
$(12)c_1$	$-e_1-e_2$	$2e_1$	e_1-e_2	$2e_2$
$(12)c_2$	e_1+e_2	$-2e_1$	e_2-e_1	$2e_2$
$(12)c_1c_2$	e_1-e_2	$-2e_1$	$-e_1-e_2$	$-2e_2$
c_1c_2	e_2-e_1	$-2e_2$	$-e_1-e_2$	$2e_1$

So to calculate the part $\frac{1}{\langle\lambda, w^{-1}\alpha_1\rangle-1} \cdot \frac{1}{(\langle\lambda, w^{-1}\alpha_2\rangle-1}$, we need to use the left half

	e_1-e_2	$2e_2$
$1 = 1^{-1}$	e_1-e_2	$2e_2$
$(12) = (12)^{-1}$	e_2-e_1	$2e_1$
$c_1 = c_1^{-1}$	$-e_1-e_2$	$2e_2$
$c_2 = c_2^{-1}$	e_1+e_2	$-2e_2$
$(12)c_1 = ((12)c_2)^{-1}$	$-e_1-e_2$	$2e_1$
$(12)c_2 = ((12)c_1)^{-1}$	e_1+e_2	$-2e_1$
$(12)c_1c_2$	e_1-e_2	$-2e_1$
c_1c_2	e_2-e_1	$-2e_2$

and to calculate the zeta part $\prod_{\alpha>0, w\alpha<0} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}$, we need to use the following table:

	e_1-e_2	$2e_2$	e_1+e_2	$2e_1$
1	$\times$	$\times$	$\times$	$\times$
(12)	0	$\times$	$\times$	$\times$
c_1	0	$\times$	0	0
c_2	$\times$	0	$\times$	$\times$
$(12)c_1$	0	$\times$	$\times$	0
$(12)c_2$	$\times$	0	0	$\times$
$(12)c_1c_2$	$\times$	0	0	0
c_1c_2	0	0	0	0

where '$\times$' means the corresponding positive root will not contribute and '0' means the corresponding positive root will contribute as it changes to a negative root under the corresponding Weyl action.

From all this, by a routine but direct calculation, which we decide to omit, we obtain the following table

	$\frac{1}{\langle\lambda,w^{-1}\alpha_1\rangle-1}\cdot\frac{1}{\langle\lambda,w^{-1}\alpha_2\rangle-1}$	$\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)}$
1	$\frac{1}{(z_1-z_2-1)}\frac{1}{(z_2-1)}$	1
(12)	$\frac{1}{(z_2-z_1-1)}\frac{1}{(z_1-1)}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}$
c_1	$\frac{1}{(-z_1-z_2-1)}\frac{1}{(z_2-1)}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\cdot\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\cdot\frac{\xi(z_1)}{\xi(z_1+1)}$
c_2	$\frac{1}{(z_1+z_2-1)}\frac{1}{(-z_2-1)}$	$\frac{\xi(z_2)}{\xi(z_2+1)}$
$(12)c_1$	$\frac{1}{(z_1+z_2-1)}\frac{1}{(-z_1-1)}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\cdot\frac{\xi(z_1)}{\xi(z_1+1)}$
$(12)c_2$	$\frac{1}{(-z_1-z_2-1)}\frac{1}{(z_1-1)}$	$\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\cdot\frac{\xi(z_2)}{\xi(z_2+1)}$
$(12)c_1c_2$	$\frac{1}{(z_1-z_2-1)}\frac{1}{(-z_1-1)}$	$\frac{\xi(z_2)}{\xi(z_2+1)}\cdot\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\cdot\frac{\xi(z_1)}{\xi(z_1+1)}$
c_1c_2	$\frac{1}{(-z_1+z_2-1)}\frac{1}{(-z_2-1)}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\cdot\frac{\xi(z_2)}{\xi(z_2+1)}\cdot\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\cdot\frac{\xi(z_1)}{\xi(z_1+1)}.$

As such, by taking the residue along the singular line $z_1-z_2=1$ and setting $z_1=b+1, z_2=b$, we get, for the product

$$\frac{1}{\langle\lambda,w^{-1}\alpha_1\rangle-1}\cdot\frac{1}{\langle\lambda,w^{-1}\alpha_2\rangle-1}\cdot\prod_{\alpha>0,\,w\alpha<0}\frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(\langle\lambda,\alpha^\vee\rangle+1)},$$

the following table of contributions

1	$\frac{1}{b-1}$
(12)	$\frac{1}{-2}\frac{1}{b}\frac{1}{\xi(2)}$
c_1	$\frac{1}{-2b-2}\frac{1}{b-1}\frac{1}{\xi(2)}\frac{\xi(2b+1)}{2b+2)}\frac{\xi(b+1)}{\xi(b+2)}$
c_2	0
$(12)c_1$	$\frac{1}{2b}\frac{1}{-b-2}\frac{1}{\xi(2)}\frac{\xi(b+1)}{\xi(b+2)}$
$(12)c_2$	0
$(12)c_1c_2$	$\frac{1}{-b-2}\frac{\xi(b)}{\xi(b+1)}\frac{\xi(2b+1)}{\xi(2b+2)}\frac{\xi(b+1)}{\xi(b+2)}$
c_1c_2	$\frac{1}{-2}\frac{1}{-b-1}\frac{1}{\xi(2)}\frac{\xi(b)}{\xi(b+1)}\frac{\xi(2b+1)}{\xi(2b+2)}\frac{\xi(b+1)}{\xi(b+2)}$

This then leads to the following explicit expression of the period $Z_{\mathbb{Q}}^{Sp(4)}$:

$$\begin{aligned} Z_{\mathbb{Q}}^{Sp(4)}(b) = {} & \frac{1}{b-1}+\frac{1}{-2}\frac{1}{b}\frac{1}{\xi(2)}+\frac{1}{-2b-2}\frac{1}{b-1}\frac{1}{\xi(2)}\frac{\xi(2b+1)}{2b+2)}\frac{\xi(b+1)}{\xi(b+2)} \\ & +\frac{1}{2b}\frac{1}{-b-2}\frac{1}{\xi(2)}\frac{\xi(b+1)}{\xi(b+2)}+\frac{1}{-b-2}\frac{\xi(b)}{\xi(b+1)}\frac{\xi(2b+1)}{\xi(2b+2)}\frac{\xi(b+1)}{\xi(b+2)} \\ & +\frac{1}{-2}\frac{1}{-b-1}\frac{1}{\xi(2)}\frac{\xi(b)}{\xi(b+1)}\frac{\xi(2b+1)}{\xi(2b+2)}\frac{\xi(b+1)}{\xi(b+2)}. \end{aligned}$$

Now multiplying the period $Z_{\mathbb{Q}}^{Sp(4)}(b)$ with the factor $\xi(b+2)\cdot\xi(2b+2)$, for the purpose of clearing up the xi function factors appeared in the denominators, and with the factor $\xi(2)$ for the purpose of clearing up the xi special values appeared in the denominators, we then obtain the following function

$$\xi_{Sp(4),\mathbb{Q}}^{o}(s) := \bigl(\xi(2)\cdot\xi(s+2)\xi(2s+2)\bigr)\cdot Z_{\mathbb{Q}}^{Sp(4)}(s).$$

Consequently,

$$\begin{aligned}\xi^{o}_{Sp(4),\mathbb{Q}}(b) = {} & \frac{1}{b-1}\xi(2)\cdot\xi(b+2)\xi(2b+2) - \frac{1}{b+2}\xi(2)\cdot\xi(b)\xi(2b+1)\\ & - \frac{1}{2b}\cdot\xi(b+2)\xi(2b+2) + \frac{1}{2(b+1)}\cdot\xi(b)\xi(2b+1)\\ & - \frac{1}{(2b)(b+2)}\cdot\xi(b+1)\xi(2b+2) + \frac{1}{(-2b-2)(b-1)}\cdot\xi(2b+1)\xi(b+1),\end{aligned}$$

and $\xi^{o}_{Sp(4),\mathbb{Q}}(b)$ satisfies the following functional equation

$$\xi^{o}_{Sp(4),\mathbb{Q}}(-b-1) = \xi^{o}_{Sp(4),\mathbb{Q}}(b).$$

Definition and Proposition. Define the zeta function $\xi_{Sp(4),\mathbb{Q}}(s)$ for $Sp(4)$ over $\mathbb{Q}$ by

$$\xi_{Sp(4),\mathbb{Q}}(s) := \xi^{o}_{Sp(4),\mathbb{Q}}(s-1).$$

Then

(1) The zeta function $\xi_{Sp(4),\mathbb{Q}}(s)$ is given by

$$\begin{aligned}\xi_{Sp(4),\mathbb{Q}}(s) = {} & \frac{1}{s-2}\xi(2)\cdot\xi(s+1)\xi(2s) - \frac{1}{s+1}\xi(2)\cdot\xi(s-1)\xi(2s-1)\\ & - \frac{1}{2s-2}\cdot\xi(s+1)\xi(2s) + \frac{1}{2s}\cdot\xi(s-1)\xi(2s-1)\\ & - \frac{1}{(2s-2)(s+1)}\cdot\xi(s)\xi(2s) - \frac{1}{(2s)(s-2)}\cdot\xi(s)\xi(2s-1).\end{aligned}$$

(2) It satisfies the standard functional equation

$$\xi_{Sp(4),\mathbb{Q}}(1-s) = \xi_{Sp(4),\mathbb{Q}}(s).$$

(3) There are only four singularities, all simple poles, at $s = -1, 0, 1, 2$ and their residue at $s = 2$ coincides with the volume of the compact domain $\mathfrak{F}(0)$, that is,

$$\mathrm{Res}_{s=2}\xi_{Sp(4),\mathbb{Q}}(s) = \xi(2)\xi(4) - \frac{1}{4}\xi(2) - \frac{1}{3}\xi(2) + \frac{1}{4}.$$

Remark. The formula for the volume is arranged in the form to reflect the fact that $\mathfrak{F}(0)$ is obtained from the total fundamental domain $\mathfrak{F}(\infty)$ whose volume is given by $\xi(2)\xi(4)$, a result due to Siegel, by subtracting two cuspidal neighborhoods corresponding to two maximal parabolic subgroups whose volumes are $\frac{1}{4}\xi(2)$ and $\frac{1}{3}\xi(2)$ respectively and adding a cuspidal neighborhood corresponding to Borel subgroups whose volume is simple $\frac{1}{4}$. For related results, please refer to [KW].

Furthermore, we have the following result of M. Suzuki [S2]:

Riemann Hypothesis for $\xi_{Sp(4),\mathbb{Q}}(s)$. *All zeros of $\xi_{Sp(4),\mathbb{Q}}(s)$ lie on the central line* $\mathrm{Re}(s) = \frac{1}{2}$.

A.3.2. Zetas for $Sp(2n)$

Recall that we have introduced the associated period $Z_{\mathbb{Q}}^{Sp(2n)}(z_n)$ for $Sp(2n)$ over $\mathbb{Q}$ by

$$Z_{\mathbb{Q}}^{Sp(2n)}(z_n) := \mathrm{Res}_{z_{n-1}-z_n=1} \cdots \mathrm{Res}_{z_2-z_3=1} \mathrm{Res}_{z_1-z_2=1} \times \left(\sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \cdot \prod_{\alpha>0,\, w\alpha<0} \frac{\xi(\langle \lambda, \alpha^\vee \rangle)}{\xi(\langle \lambda, \alpha^\vee \rangle + 1)} \right).$$

Clearly, there are minimal integers I, J, constants a_i, b_i, $i = 1, 2, \ldots, I$, and c_j, $j = 1, 2, \ldots, J$, such that after multiplying the factor $\prod_{i=1}^{I} \xi(a_i z_n + b_i) \cdot \prod_{j=1}^{J} \xi(c_j)$, the resulting function

$$\xi^o_{Sp(2n),\mathbb{Q}}(z_n) := \left(\prod_{i=1}^{I} \xi(a_i z_n + b_i) \cdot \prod_{j=1}^{J} \xi(c_j) \right) \cdot Z_{\mathbb{Q}}^{Sp(2n)}(z_n)$$

admits only finitely many poles and there are no special ξ-values appeared in the denominators.

Conjecture *(Functional equation). There exists a constant c_n such that*

$$\xi^o_{Sp(2n),\mathbb{Q}}(c_n - s) = \xi^o_{Sp(2n),\mathbb{Q}}(s).$$

Definition. The zeta function $\xi_{Sp(2n),\mathbb{Q}}(s)$ of $Sp(2n)$ over $\mathbb{Q}$ is defined by

$$\xi_{Sp(2n),\mathbb{Q}}(s) := \xi^o_{Sp(2n),\mathbb{Q}}\left(s + \frac{c_n - 1}{2} \right).$$

Remark. This in fact is a special case of a more general construction. For details, please see [W4].

As a direct consequence of the above conjecture, $\xi_{Sp(2n),\mathbb{Q}}(s)$ satisfies the following functional equation

$$\xi_{Sp(2n),\mathbb{Q}}(1 - s) = \xi_{Sp(2n),\mathbb{Q}}(s).$$

With such a normalization, we then expect the following generalized RH for our zetas $\xi_{Sp(2n),\mathbb{Q}}(s)$.

The Riemann Hypothesis$_{\mathbf{Sp(2n),\,\mathbb{Q}}}$. *All zeros of $\xi_{Sp(2n),\mathbb{Q}}(s)$ lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

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