J. Ramanujan Math. Soc. **21,** No.3 (2006) 1–62

A rank two zeta and its zeros

Lin Weng
Graduate School of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
e-mail: weng@math.kyushu-u.ac.jp

Communicated by: V. Kumar Murty

Received:

Abstract.

Introduction

Theory of zeta functions plays a central role in arithmetic. In this paper, we use a new approach to study them. More precisely, we first reveal an intrinsic relation between higher rank zeta functions and Epstein zeta functions, and expose a fundamental relation between stability of lattices and distances of the corresponding modular points to cusps. Applying to rank two, we then explicitly express the associated zeta functions in terms of Dedekind zetas. Based on such an expression, finally, we show that all zeros of rank two zetas are entirely sitting on the critical line whose real part equals to $\frac{1}{2}$.

As such, this work is built up on classics of number theory. Many fine pieces of algebraic and analytic number theory are beautifully unified under our zetas:

1) New Geo-Arithmetic cohomology for lattices over number fields, by further developing Tate's fundamental work, known as Tate's Thesis;
2) A definition of new zeta functions for number fields, as a natural generalization (and hence offering a natural framework) for the classical Dedekind zeta functions;
3) A relation between our zeta and Epstein type zeta functions, via the well-known Mellin transformation;
4) A classification of lattices first according to their volumes and unit twists, in connection with an intrinsic relation between GL_n and SL_n over a number field K using Dirichlet's Unit Theorem; and hence a relation between the space of isometry classes of rank two lattices over ring of integers and the upper half space model;

5) A construction of a fundamental domain for the action of special automorphism group of rank two lattices on the associated upper half space using normalized Siegel type distances to cusps, by generalizing Siegel's construction for totally real fields;
6) An intrinsic relation between stability of lattices and distances to cusps, i.e., a lattice is semi-stable if and only if its distances to all cusps are at least one, by deepening an algebraic result of T. Hayashi;
7) A Fourier expansion for Epstein zeta function, along with the classical line;
8) An explicit expression of rank two zeta in terms of the associated Dedekind zeta function, as an application of Rankin-Selberg & Zagier method;
9) An analogue of the Riemann Hypothesis for rank two zetas, based on a result of M. Suzuki and J. Lagarias.

1. High rank zeta functions and Eisenstein series

1.1 Projective $\mathcal{O}_K$-modules

Let K be an algebraic number field with $\mathcal{O}_K$ its integer ring. An $\mathcal{O}_K$-module M is called *projective* if there exists an $\mathcal{O}_K$-module N such that $M \oplus N$ is free. We have

(i) For an exact sequence $0 \to M_0 \to M_1 \to M \to 0$, $M_1 \simeq M_0 \oplus M$.
(ii) All fractional $\mathcal{O}_K$-ideals are projective; and
(iii) Rank 1 projective $\mathcal{O}_K$-submodules in K are simply fractional $\mathcal{O}_K$-ideals.

Thus, by finiteness of the ideal class group of K, up to isomorphism, there are only finitely many rank 1 projective $\mathcal{O}_K$-modules in K. Choose integral $\mathcal{O}_K$-ideals $\mathfrak{a}_i, i = 1, \dots, h$ with $h = h(K)$, the class number of K, such that

(a) Any rank 1 projective $\mathcal{O}_K$-module is isomorphic to one of the $\mathfrak{a}_i$; while
(b) None of the $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are isomorphic to each other if $i \neq j$, $i, j = 1, \dots, h$.

Fix a choice of $\mathfrak{a}_i$, $1 \leq i \leq h$ satisfying (a) and (b) above and use $\mathfrak{a}$ as a running symbol for them.

Clearly for a fractional ideal $\mathfrak{a}$, $P_{\mathfrak{a}} := P_{r;\mathfrak{a}} := \mathcal{O}_K^{r-1} \oplus \mathfrak{a}$ is a rank r projective $\mathcal{O}_K$-module. Conversely, we have

Proposition. (See e.g. [8]).

(1) *For fractional ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$, $P_{r;\mathfrak{a}} \simeq P_{r;\mathfrak{b}}$ *iff* $\mathfrak{a} \simeq \mathfrak{b}$*;*
(2) *For any projective* $\mathcal{O}_K$*-module* P*, there exists a fractional ideal* $\mathfrak{a}$ *such that* $P \simeq P_{\mathfrak{a}}$.

In this paper, we use the natural inclusion of fractional ideals in K to embed $P_{r;\mathfrak{a}}$ into K^r, and write an element in K^r as a column vector.

Lemma. *For an $\mathcal{O}_K$-isomorphism $A : P_{r;\mathfrak{a}} \to P_{r;\mathfrak{b}}$ induced from $A \in GL(r,F)$, $\mathfrak{b} \simeq (\det A) \cdot \mathfrak{a}$. In particular, if $\mathfrak{a}$, $\mathfrak{b}$ are integral, (i) $\det A \in U_K$, the group of units of K; (ii) $\mathfrak{a} = \mathfrak{b}$; and (iii) $A \in \mathrm{Aut}_{\mathcal{O}_K}(P_{r;\mathfrak{a}})$.*

1.2 Semi-stable $\mathcal{O}_K$-lattices

Let σ be an Archimedean place of K, and K_σ its σ-completion. Then K_σ equals to either $\mathbb{R}$ or $\mathbb{C}$. Accordingly, we call σ real or complex and write $\sigma : \mathbb{R}$ or $\sigma : \mathbb{C}$.

By definition, an *$\mathcal{O}_K$-lattice* Λ consists of (1) a projective $\mathcal{O}_K$-module $P = P(\Lambda)$ of finite rank; and (2) an inner product on the vector space $V_\sigma := P \otimes_{\mathcal{O}_K} K_\sigma$ for each of the Archmidean place σ of K. Set $V = P \otimes_{\mathbb{Z}} \mathbb{R}$ so that $V = \prod_{\sigma \in S_\infty} V_\sigma$, where S_∞ denotes the collection of all Archimedean places, since as a $\mathbb{Z}$-module, an $\mathcal{O}_K$-ideal is of rank $n = r_1 + 2r_2$. Here $n = [K : \mathbb{Q}]$, r_1 (resp. r_2) denotes the number of real (resp. complex) places.

Let P be a rank r projective $\mathcal{O}_K$-module. Denote by $GL(P) := \mathrm{Aut}_{\mathcal{O}_K}(P)$. Let $\widetilde{\mathbf{\Lambda}} := \widetilde{\mathbf{\Lambda}}(P)$ be the space of ($\mathcal{O}_K$-)lattices Λ whose underlying $\mathcal{O}_K$-module is P. For $\sigma \in S_\infty$, let $\widetilde{\mathbf{\Lambda}}_\sigma$ be the space of inner products on V_σ; if a basis is chosen for V_σ as a real or a complex vector space accordingly, $\widetilde{\mathbf{\Lambda}}_\sigma$ may be realized as an open set of a real or complex vector space. (See 1.3 below.) We have $\widetilde{\mathbf{\Lambda}} = \prod_{\sigma \in S_\infty} \widetilde{\mathbf{\Lambda}}_\sigma$ from which we obtain a natural topology on $\widetilde{\mathbf{\Lambda}}$.

Given $\Lambda \in \widetilde{\mathbf{\Lambda}}$ and $u, w \in V_\sigma$, let $\langle u, w\rangle_{\Lambda,\sigma}$ or $\langle u, w\rangle_{\rho_\Lambda(\sigma)}$ denote the value of the inner product on the vectors u and w associated to the lattice Λ. As such, if $A \in GL(P)$, we may define a new lattice $A \cdot \Lambda$ in $\widetilde{\mathbf{\Lambda}}$ by $\langle u, w\rangle_{A\cdot\Lambda,\sigma} := \langle A^{-1} \cdot u, A^{-1} \cdot w\rangle_{\Lambda,\sigma}$. Clearly, the map $v \mapsto Av$ gives an isometry $\Lambda \cong A\cdot\Lambda$ of the lattices. Conversely, suppose that $A : \Lambda_1 \cong \Lambda_2$ is an isometry of $\mathcal{O}_K$-lattices, each of which is in $\widetilde{\mathbf{\Lambda}}$. Then, A defines an element, also denoted by A, of $GL(P)$. Clearly $\Lambda_2 \cong A \cdot \Lambda_1$. Therefore, the orbit set $GL(P)\backslash\widetilde{\mathbf{\Lambda}}(P)$ can be regarded as the set of isometry classes of $\mathcal{O}_K$-lattices whose underlying $\mathcal{O}_K$-modules are all isomorphic to the fixed P.

Also if $T \in \mathbb{R}_{>0}$, then from Λ, we can produce a new $\mathcal{O}_K$-lattice called $\Lambda[T]$ by multiplying each of the inner products on Λ, or better, on Λ_σ for $\sigma \in S_\infty$, by T^2. Let then $\mathbf{\Lambda} = \mathbf{\Lambda}(P)$ be the quotient of $\widetilde{\mathbf{\Lambda}}$ by the equivalence relation $\Lambda \sim \Lambda[T]$. As such, $\mathbf{\Lambda}$ admits a natural topological structure as well. Furthermore, as it becomes clear later, the construction of $\mathbf{\Lambda}$ from $\widetilde{\mathbf{\Lambda}}$ plays a key role when we want to get the compactness statement for our moduli spaces.

Let Λ be an $\mathcal{O}_K$-lattice with underlying $\mathcal{O}_K$-module P. Then via restriction of inner products, any submodule $P_1 \subset P$ can be made into an $\mathcal{O}_K$-lattice. Call the resulting $\mathcal{O}_K$-lattice $\Lambda_1 := \Lambda \cap P_1$ and write $\Lambda_1 \subset \Lambda$. If moreover, P/P_1 is projective, we say that Λ_1 is a *sublattice* of Λ. Via orthogonal projections, we can give P/P_1 an $\mathcal{O}_K$-lattice structure, the *quotient lattice* Λ/Λ_1 of Λ by Λ_1.

Restriction of scalars makes an $\mathcal{O}_K$-lattice into a $\mathbb{Z}$-lattice. Recall that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \prod_{\sigma \in S_\infty} V_\sigma$. Define an inner product on the real vector space V by $\langle u, w\rangle_\infty := \sum_{\sigma:\mathbb{R}} \langle u_\sigma, w_\sigma\rangle_\sigma + \sum_{\sigma:\mathbb{C}} \mathrm{Re}\,\langle u_\sigma, w_\sigma\rangle_\sigma$. Let $\mathrm{Res}_{K/\mathbb{Q}}\Lambda$ be the $\mathbb{Z}$-lattice obtained by equipped P, regarding as a $\mathbb{Z}$-module, with this inner product (at the unique infinite place ∞ of $\mathbb{Q}$).

We let $\mathrm{rk}(\Lambda)$ denote the $\mathcal{O}_K$-module rank of P (or of Λ) and let $\dim(\Lambda)$ denote the rank of P as $\mathbb{Z}$-module. Define the *Lebesgue volume* of Λ, denoted by $\mathrm{Vol}_{\mathrm{Leb}}(\Lambda)$, to be the (co)volume of the lattice $\mathrm{Res}_{K/\mathbb{Q}}\Lambda$ inside its inner product space V.

Example. Take $P = \mathcal{O}_K$ and for each place σ, let $\{1\}$ be an orthonormal basis of $V_\sigma = K_\sigma$. This makes $\mathcal{O}_K$ into an $\mathcal{O}_K$-lattice $\overline{\mathcal{O}_K} = (\mathcal{O}_K, \mathbf{1})$ in a natural way and $\mathrm{Vol}_{\mathrm{Leb}}(\overline{\mathcal{O}_K}) = 2^{-r_2} \cdot \sqrt{\Delta_F}$, where Δ_F denotes the absolute value of the discriminant of K.

More generally, take $P = \mathfrak{a}$ an fractional idea of K and equip the same inner product as above on V_σ. Then $\mathfrak{a}$ becomes an $\mathcal{O}_K$-lattice $\overline{\mathfrak{a}} = (\mathfrak{a}, \mathbf{1})$ in a natural way with $\mathrm{rk}(\mathfrak{a}) = 1$, and $\mathrm{Vol}_{\mathrm{Leb}}(\overline{\mathfrak{a}}) = 2^{-r_2} \cdot (N(\mathfrak{a}) \cdot \sqrt{\Delta_K})$, where $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$.

Due to the appearence of the factor 2^{-r_2}, we also define the *canonical volume* of Λ, denoted by $\mathrm{Vol}_{\mathrm{can}}(\Lambda)$ or simply by $\mathrm{Vol}(\Lambda)$, to be $2^{r_2 \mathrm{rk}(\Lambda)}\mathrm{Vol}_{\mathrm{Leb}}(\Lambda)$. So in particular, $\mathrm{Vol}(\overline{\mathfrak{a}}) = N(\mathfrak{a}) \cdot \sqrt{\Delta_K}$, with $\mathrm{Vol}(\overline{\mathcal{O}_K}) = \sqrt{\Delta_K}$ as its special case.

The canonical measure has an advantage theoretically.

Arakelov-Riemann-Roch Formula: For an $\mathcal{O}_K$-lattice Λ of rank r,

$$-\log(\mathrm{Vol}(\Lambda)) = \deg(\Lambda) - \frac{r}{2}\log \Delta_K.$$

(For the reader who does not know the definition of Arakelov degree, he or she may simply take this relation as a definition.)

Definition. *An $\mathcal{O}_K$ lattice Λ is called semi-stable (resp. stable) if for any proper sublattice Λ_1 of Λ,* $\mathrm{Vol}(\Lambda_1)^{\mathrm{rk}(\Lambda)} \geq$ (resp. $>$) $\mathrm{Vol}(\Lambda)^{\mathrm{Vol}(\Lambda_1)}$.

The last inequality is equivalent to $\mathrm{Vol}_{\mathrm{Leb}}(\Lambda_1)^{\mathrm{rk}(\Lambda)} \geq \mathrm{Vol}_{\mathrm{Leb}}(\Lambda)^{\mathrm{Vol}(\Lambda_1)}$. *So it does not matter which volume, the canonical one or the Lebesgue one, we use.*

Remark. Even we introduce the stability for lattices independently, many others, notably Stuhler, introduced the stability earlier. (See e.g. [Gr], [St].)

1.3 Space of $\mathcal{O}_K$-lattices via special linear groups

Via the Minkowski embedding $K \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, we obtain a natural embedding for P: $P := \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a} \hookrightarrow K^{(r)} \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r \cong (\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$, which is simply the space $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ above. Thus, our lattice Λ then is determined by a metric structure on $V = \prod_{\sigma \in S_\infty} V_\sigma$, or better, on $(\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$. Hence, we need to determine all metrized structures on $(\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$.

For this, let us start with each component $\mathbb{R}^r$ (resp. $\mathbb{C}^r$).

(i) For any $g \in GL(r, \mathbb{R})$ (resp. $g \in GL(r, \mathbb{C})$), there is an associated metric structure $\rho(g)$ or simply g on $\mathbb{R}^r$ (resp. on $\mathbb{C}^r$) given by the matrix $g \cdot g^t$ (resp. $g \cdot \overline{g^t}$). More precisely, for $x,\ y \in \mathbb{R}^r$ (resp. $\mathbb{C}^r$) $\langle x, y \rangle_g := \langle x, y \rangle_{\rho(g)} := x \cdot (g\overline{g^t}) \cdot y^t = (xg) \cdot \overline{(yg)}^t$;

(ii) Two matrices g and g' in $GL(r, \mathbb{R})$ (resp. in $GL(r, \mathbb{C})$) correspond to the same metrized structure on $\mathbb{R}^r$ (resp. $\mathbb{C}^r$) iff there is $A \in GL(r, \mathbb{R})$ (resp. $GL(r, \mathbb{C})$) s.t. $g' = g \cdot A$ and $A \cdot A^t = E_r$ (resp. $A \cdot \overline{A^t} = E_r$). That is to say, g and g' differ from each other by a matrix A from the orthogonal group $O(r)$ (resp. from the unitary group $U(r)$).

Therefore, metrized structures on $(\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$ are parametrized by the space

$$(GL(r, \mathbb{R})/O(r))^{r_1} \times (GL(r, \mathbb{C})/U(r))^{r_2}.$$

Next we will shift from the general linear group GL to the special linear group SL. We start with a local discussion on $\mathcal{O}_K$-lattice structures. For complex places τ, clearly, by fixing a branch of the n-th root, we get natural identifications

$$\begin{array}{ccccc} GL(r, \mathbb{C}) & \to & SL(r, \mathbb{C}) \times \mathbb{C}^* & \to & SL(r, \mathbb{C}) \times S^1 \times \mathbb{R}^*_+ \\ g & \mapsto & \left(\frac{1}{\sqrt[r]{\det g}} g, \det g \right) & \mapsto & \left(\frac{1}{\sqrt[r]{\det g}} g, \frac{\det g}{|\det g|}, |\det g| \right) \end{array}$$

and $U(r) \to SU(r) \times S^1, U \mapsto (\frac{1}{\sqrt[r]{\det U}} U, \det U)$, where SL (resp. SU) denotes the special linear group (resp. the special unitary group) and S^1 denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in $\mathbb{C}^*$. Consequently, we obtain $GL(r, \mathbb{C})/U(r) \cong (SL(r, \mathbb{C})/SU(r)) \times \mathbb{R}^*_+$.

For real places σ, one might try to use the same approach for $\mathbb{C}$ above. But $\sqrt[r]{\det g}$ is not always well-defined in the reals. Accordingly, we modify our approach by using the subgroups $GL^+(r, \mathbb{R}) := \{g \in GL(r, \mathbb{R}) : \det g > 0\}$ and $O^+(r) := \{A \in O(r, \mathbb{R}) : \det g > 0\}$. Clearly, (i) $O^+(r) = SO(r)$, the special orthogonal group consisting of these A's in $O(r)$ whose determinants are exactly 1; (ii) $GL(r, \mathbb{R})/O(r) \cong GL^+(r, \mathbb{R})/SO(r)$; and (iii) There is an identification $GL^+(r, \mathbb{R}) \to SL(r, \mathbb{R}) \times \mathbb{R}^*_+, g \mapsto$

$\left(\frac{1}{\sqrt[r]{\det g}}g, \det g\right)$. Consequently, we have $GL(r,\mathbb{R})/O(r) \cong (SL(r,\mathbb{R})/SO(r)) \times \mathbb{R}_+^*$.

Hence, metrized structures on $V = \prod_{\sigma\in S_\infty} V_\sigma \simeq (\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$ are parametrized by the space $((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}) \times (\mathbb{R}_+^*)^{r_1+r_2}$. Furthermore, when we really work with $\mathcal{O}_K$-lattice structures on P, i.e., with the space $\mathbf{\Lambda} = \mathbf{\Lambda}(P)$, from the above parametrized space of metric structures on $V = \prod_{\sigma\in S_\infty} V_\sigma$, we need to further factor out $GL(P)$, i.e., the automorphism group $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ of $\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$ as $\mathcal{O}_K$-modules.

As such, naturally, now we want (a) to study the structure of the group $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ in terms of SL and units; and (b) to see how this group acts on the space of metrized structures

$$((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}) \times (\mathbb{R}_+^*)^{r_1+r_2}.$$

View $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ as a subgroup of $GL(r,K)$. Easily, for an element $A = (a_{ij}) \in \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$, $\det A \in U_K$. Moreover $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) = GL(r, \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$

$$:= \left\{ (a_{ij}) \in GL(r,K) : \begin{array}{l} a_{rr} \& a_{ij} \in \mathcal{O}_K, \\ a_{ir} \in \mathfrak{a},\ a_{rj} \in \mathfrak{a}^{-1}, \end{array} i,j = 1,\dots,r-1; \right.$$
$$\left. \det(a_{ij}) \in U_K \right\}.$$

In other words,

$$\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) = \left\{ A \in GL(r,K) \cap \begin{pmatrix} & & & \mathfrak{a} \\ & \mathcal{O}_K & & \vdots \\ & & & \mathfrak{a} \\ \mathfrak{a}^{-1} & \dots & \mathfrak{a}^{-1} & \mathcal{O}_K \end{pmatrix} : \det A \in U_K \right\}.$$

To go further, we still need to see how $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ decomposes with respect to the shift from GL to SL adopted in the discussion on metrized structures. For this purpose, we first introduce the subgroup $\mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ of $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ consisting of these elements whose local determinants at real places are all positive. Clearly, $\mathrm{diag}(-1,1,\dots,1)$ is an element of $O(r)$, which is supposed to be factored out in our final discussion. Note also that $GL(r,\mathbb{R})/O(r) \simeq GL^+(r,\mathbb{R})/O^+(r)$ and $O^+(r) = SO(r)$. Consequenly, we obtain a natural identification of quotient spaces between

$$\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \backslash ((GL(r,\mathbb{R})/O(r))^{r_1} \times (GL(r,\mathbb{C})/U(r))^{r_2})$$

and

$$\mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((GL^+(r,\mathbb{R})/O^+(r))^{r_1} \times (GL(r,\mathbb{C})/U(r))^{r_2}).$$

As such, to shift further to the special linear group SL, we need Dirichlet's Unit Theorem, i.e., finiteness of the group of units. Locally,

$$\begin{aligned} GL(r,\mathbb{R})/O(r) \to & GL^+(r,\mathbb{R})/SO(r) \\ \to & (SL(r,\mathbb{R})/SO(r)) \times (\mathbb{R}_+^* \cdot \mathrm{diag}(1,\ldots,1)) \\ & \simeq (SL(r,\mathbb{R})/SO(r)) \times \mathbb{R}_+^* \end{aligned}$$

via

$$\begin{aligned} [A] \mapsto [A^+] \mapsto & \left(\frac{1}{\sqrt[r]{\det A^+}} A^+, \mathrm{diag}(\sqrt[r]{\det A^+}, \ldots, \sqrt[r]{\det A^+})\right) \\ \mapsto & \left(\frac{1}{\sqrt[r]{\det A^+}} A^+, \sqrt[r]{\det A^+}\right) \end{aligned}$$

for real places, and

$$\begin{aligned} GL(r,\mathbb{C})/U(r) \to & (SL(r,\mathbb{C}) \times \mathbb{C})/(SU(r) \times S^1) \\ \to & (SL(r,\mathbb{C})/SU(r)) \times (\mathbb{R}_+^* \cdot \mathrm{diag}(1,\ldots,1)) \\ & \simeq \Big(SL(r,\mathbb{C})/SU(r)) \times \mathbb{R}_+^* \end{aligned}$$

via

$$\begin{aligned} [A] \mapsto [A] \mapsto & \left(\frac{1}{\sqrt[r]{\det A}} A, \mathrm{diag}(\sqrt[r]{\det A}, \ldots, \sqrt[r]{\det A})\right) \\ \mapsto & \left(\frac{1}{\sqrt[r]{\det A}} A, \sqrt[r]{\det A}\right), \end{aligned}$$

for complex places. Ideally, we want to have corresponding identifications for elements in $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$. However, this cannot be achieved in general, since the r-th roots of a unit in K lie only in a finite extension of K.

Recall that for a unit $\varepsilon \in U_K$, (a) $\mathrm{diag}(\varepsilon,\ldots,\varepsilon) \in \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$; and (b) $\det \mathrm{diag}(\varepsilon,\ldots,\varepsilon) = \varepsilon^r \in U_F^r := \{\varepsilon^r : \varepsilon \in U_K\}$. So to begin with, note that to pass from GL to SL over K, we need to use the intermediate subgroup GL^+. Consequently, we introduce a subgroup U_K^+ of U_K by setting

$$U_K^+ := \{\varepsilon \in U_K : \varepsilon_\sigma > 0, \forall \sigma \text{ real}\}$$

so as to get a well-controlled subgroup $U_K^{r,+} := U_K^+ \cap U_K^r$. Indeed, by Dirichlet's Unit Theorem, the quotient group $U_K^+/(U_K^+ \cap U_K^r)$ is finite.

With this said, next we use $U_K^+ \cap U_K^r$ to decompose the group $\mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$. Thus, choose elements $u_1, \dots, u_{\mu(r,F)} \in U_K^+$ such that $\{[u_1], \dots, [u_{\mu(r,F)}]\}$ gives a completed representatives of the finite quotient group $U_K^+/(U_K^+ \cap U_K^r)$, where $\mu(r,K)$ denotes the cardinality of the group $U_K^+/(U_K^+ \cap U_K^r)$. Set $SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) := SL(r,K) \cap GL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ as well.

Lemma. *There exist elements $A_1, \dots, A_{\mu(r,K)}$ in $GL^+(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ such that*

(i) $\det A_i = u_i, i = 1, \dots, \mu(r,K)$;
(ii) $A_1, \dots, A_{\mu(r,K)}$ *consist of a completed representatives of the quotient* $\mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ *by* $SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \times (U_K^{r,+} \cdot \mathrm{diag}(1, \dots, 1))$.

That is to say, for automorphism groups,

(a) $\mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ is naturally identified with the disjoint union

$$\cup_{i=1}^{\mu(r,K)} A_i \cdot (SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \times (U_K^{r,+} \cdot \mathrm{diag}(1, \dots, 1)));$$

(b) The $\mathcal{O}_K$-lattice structures $\mathbf{\Lambda}(P)$ on the projective $\mathcal{O}_K$-module $P = \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$ are parametrized by the disjoint union

$$\begin{aligned}\cup_{i=1}^{\mu(r,K)} & A_i \backslash ((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \backslash ((SL(r,\mathbb{R})/SO(r))^{r_1} \\ & \times (SL(r,\mathbb{C})/SU(r))^{r_2})) \\ & \times (|U_K^r \cap U_K^+| \backslash (\mathbb{R}_+^*)^{r_1+r_2})).\end{aligned}$$

Proof. This is a direct consequence of (1) For all $\varepsilon \in U_K^+$, $\mathrm{diag}(\varepsilon, \dots, \varepsilon) \in \mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$ and its determinant belongs to $U_K^+ \cap U_K^r$; and (2) For $A \in \mathrm{Aut}^+_{\mathcal{O}_K}(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})$, by definition, $\det A \in U_K^+$.

Therefore, to understand the space of $\mathcal{O}_K$-lattice structures, beyond the spaces $SL(r,\mathbb{R})/SO(r)$ and $SL(r,\mathbb{C})/SU(r)$, we need to further study

(i) the quotient space $|U_K^r \cap U_K^+| \backslash (\mathbb{R}_+^*)^{r_1+r_2}$; and more importantly,
(ii) the (modular) space

$$SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \backslash ((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}).$$

Now denote by $\widetilde{\mathcal{M}}_{K,r}(\mathfrak{a})$ the moduli space of rank r semi-stable $\mathcal{O}_K$-lattices with underlying projective module $\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$. For our own convenience, for a set X of (isometry classes of) lattices, denote by X_{ss} the subset of X consisting of lattices which are semi-stable. □

Proposition. *There is a natural identification between the moduli space* $\widetilde{\mathcal{M}}_{K,r}(\mathfrak{a})$ *of rank* r *semi-stable* $\mathcal{O}_K$*-lattices on the projective module* $\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$ *and the disjoint union of (the ss part of) the quotient spaces*

$$\begin{aligned}\cup_{i=1}^{\mu(r,K)} A_i\backslash((SL(&\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1}\\ &\times (SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}}\\ &\times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2})).\end{aligned}$$

Proof. By definition and the previous lemma,

$$\begin{aligned}\widetilde{\mathcal{M}}_{K,r}(\mathfrak{a}) \cong{} & [\cup_{i=1}^{\mu(r,K)} A_i\backslash((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1}\\ &\times (SL(r,\mathbb{C})/SU(r))^{r_2}))\\ &\times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2}))]_{\mathrm{ss}}.\end{aligned}$$

Moreover, by definition, we can interchange the subindex ss with the disjoint union symbol. With this said, it is sufficient to show that

$$\begin{aligned}&[A_i\backslash(SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}))\\ &\qquad\qquad\qquad\qquad \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2}))]_{\mathrm{ss}}\\ =&A_i\backslash((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}}\\ &\qquad\qquad\qquad\qquad \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2})).\end{aligned}$$

Clearly, an action of an automorphism of a lattice does not change the semi-stability. Hence we need to check whether

$$\begin{aligned}&[(SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}))\\ &\qquad\qquad\qquad\qquad \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2})]_{\mathrm{ss}}\\ =&[SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2})]_{\mathrm{ss}}\\ &\qquad\qquad\qquad\qquad \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2}).\end{aligned}$$

This is simple since a lattice Λ is semi-stable if and only if its $[T]$-modifications $\Lambda[T]$ are semi-stable for all $T > 0$. □

1.4 Structure of moduli space: action of $\mathcal{O}_K$-units

To further understand the structure of moduli space of semi-stable $\mathcal{O}_K$-lattices, let us consider the quotient space $|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2}$.

We start with $U_K^r \cap U_K^+$. Clearly, $U_K^2 \subset U_K^+$. On the other hand, by Dirichlet's Unit Theorem, up to a finite torsion subgroup consisting of the roots of unity in K, the image $|U_K|$ of U_K (under the natural logarithm map) is a $\mathbb{Z}$-lattice of rank r_1+r_2-1 in $\mathbb{R}^{r_1+r_2}$. As such, the image $|U_K^r|$ of U_K^r corresponds simply to the sublattice $r|U_K|$, i.e., the one consists of all elements in the lattice $|U_K|$ which are r-times of elements in $|U_K|$. Consequenly, U_K^+ as well as $U_K^+\cap U_K^r$ are all finite index subgroups of U_K.

Next, let us look at the quotient $|U_K^r\cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2}$. For this, we adopt Neukirch's [19] presentation.

Let X be a finite $G(\mathbb{C}|\mathbb{R})$-set, i.e., a finite set with an involution $\tau\mapsto\bar\tau,\forall\tau\in X$, and let $n=\#X$. Consider the n-dimensional $\mathbb{C}$-algebra $\mathbf{C}:=\prod_{\tau\in X}\mathbb{C}$ of all tuples $z:=(z_\tau)_{\tau\in X}, z_\tau\in\mathbb{C}$, with componentwise addition and multiplication. Set involutions $z\mapsto\bar z\in\mathbf{C}$ (resp. $z\mapsto z^*$, resp., $z\mapsto {}^*z$) as follows; for $z=(z_\tau)\in\mathbf{C}$, the element $\bar z\in\mathbf{C}$ (resp. $z^*\in\mathbf{C}$, resp. ${}^*z\in\mathbf{C}$) is defined to be the element of $\mathbf{C}$ having the following components: $(\bar z)_\tau=\bar z_{\bar\tau}$, (resp. $z_\tau^*=z_{\bar\tau}$, resp. ${}^*z_\tau=\overline{z_\tau}$). Clearly, $\bar z={}^*z^*$. As such, the invariant subset $\mathbf{R}:=[\prod_{\tau\in X}\mathbb{C}]^+:=\{z\in\mathbf{C}:z=\bar z\}$ forms an n-dimensional commutative $\mathbb{R}$-algebra, and $\mathbf{C}=\mathbf{R}\otimes_\mathbb{R}\mathbb{C}$. For example, for a number field K of degree n with $X=\mathrm{Hom}(K,\mathbb{C})$, $\mathbf{R}$ coincides with the Minkowski space $K_\mathbb{R}:=K\otimes_\mathbb{Q}\mathbb{R}$.

For the additive, resp. multiplicative group $\mathbf{C}$, resp. $\mathbf{C}^*$, we have the homomorphism $\mathrm{Tr}:\mathbf{C}\to\mathbb{C}, z\mapsto\sum_\tau z_\tau$, resp. $N:\mathbf{C}^*\to\mathbb{C}^*, z\mapsto\prod_\tau z_\tau$. Furthermore, we have on $\mathbf{C}$ the hermitian scalar product $\langle x,y\rangle:=\sum_\tau x_\tau\overline{y_\tau}=\mathrm{Tr}(x\cdot{}^*y)$ which is invariant under conjugation, i.e., $\overline{\langle x,y\rangle}=\langle\bar x,\bar y\rangle$. Thus, by restricting it to $\mathbf{R}$, we get a scalar product $\langle\cdot,\cdot\rangle$, i.e., an Euclidean metric, on the $\mathbb{R}$-vector space $\mathbf{R}$.

In $\mathbf{R}$, consider the subspace $\mathbf{R}_\pm:=\{x\in\mathbf{R}:x=x^*\}=[\prod_\tau\mathbf{R}]^+$. Clearly, for $x=(x_\tau)\in\mathbf{R}_\pm$, its components satisfy $x_{\bar\tau}=x_\tau\in\mathbf{R}$. For our convenience, for $\delta\in\mathbb{R}$, we simply write $x>\delta$ to signify that $x_\tau>\sigma$ for all τ. With this, then we introduce the multiplicative group $\mathbf{R}_+^*:=\{x\in\mathbf{R}_\pm: x>0\}=[\prod_\tau\mathbf{R}_+^*]^+$. Clearly, $\mathbf{R}_+^*$ consists of the tuples $x=(x_\tau)$ of positive real numbers x_τ such that $x_{\bar\tau}=x_\tau$, and admits two homomorphisms: $|\ |:\mathbf{R}^*\to\mathbf{R}_+^*, x=(x_\tau)\mapsto|x|=(|x_\tau|)$, and $\log:\mathbf{R}_+^*\to\mathbf{R}_\pm, x=(x_\tau)\mapsto\log x=(\log x_\tau)$. For example, when $X=\mathrm{Hom}(K,\mathbb{C})$, $\mathbf{R}_+^*=\mathbb{R}_{>0}^{r_1+r_2}$ is exactly the (unit) factor appeared in our description of the moduli space of semi-stable lattices above. Moreover, the $G(\mathbb{C}|\mathbb{R})$-set $X=\mathrm{Hom}(K,\mathbb{C})$ then corresponds to the Minkowski space $K_\mathbb{R}=\mathbf{R}=[\prod_\tau C]^+$, in which the field K may be naturally embedded. In particular, $N((a))=|N_{K/\mathbb{Q}}(a)|=|N(a)|$, where N denotes the norm on $\mathbf{R}^*$.

Now let $\mathfrak{p}=\{\tau,\bar\tau\}$ be a conjugation class in X. We call $\mathfrak{p}$ real or complex according to $\#\mathfrak{p}=1$ or 2. Accordingly, $\mathbf{R}_+^*=\prod_\mathfrak{p}\mathbf{R}_{+\mathfrak{p}}^*$ with $\mathbf{R}_{+\mathfrak{p}}^*=\mathbb{R}_+^*$ when $\mathfrak{p}$ is real and $\mathbf{R}_{+\mathfrak{p}}^*=(\mathbb{R}_+^*\times\mathbb{R}_+^*)^+=\{(y,y):y\in\mathbb{R}_+^*\}$. Further, define isomorphisms $\mathbf{R}_{+\mathfrak{p}}^*\simeq\mathbb{R}_+^*$ by $y\mapsto y$ resp. $(y,y)\mapsto y^2$ for $\mathfrak{p}$ real resp. complex, so as to obtain a natural isomorphism $\alpha:\mathbf{R}_+^*\simeq\prod_\mathfrak{p}\mathbb{R}_+^*$.

With this, by $\frac{dy}{y}$ the Haar measure on $\mathbf{R}_+^*$, we mean that one corresponding to the product measure $\prod_{\mathfrak{p}} \frac{dt}{t}$, where $\frac{dt}{t}$ is the usual Haar measure on $\mathbb{R}_+^*$. We call the Haar measure thus defined the *canonical measure* on $\mathbf{R}_+^*$. Under the logarithm map $\log : \mathbb{R}_+^* \to \mathbf{R}_\pm$, it is mapped to the Haar measure dx on $\mathbf{R}_\pm$ which under the isomorphism $\mathbf{R}_\pm = \prod_{\mathfrak{p}} \mathbb{R}_{\pm p} \to \prod_{\mathfrak{p}} \mathbb{R}$ (componentwisely given by $x_{\mathfrak{p}} \mapsto x_{\mathfrak{p}}$ resp. $(x_{\mathfrak{p}}, x_{\mathfrak{p}}) \mapsto 2x_{\mathfrak{p}}$ for $\mathfrak{p}$ real resp. complex) corresponds to the standard Lebesgue measure on $\prod_{\mathfrak{p}} \mathbb{R}$.

Obviously, for a unit ε in U_K, its $K/\mathbb{Q}$-norm gives ± 1 (in $\mathbb{Q}$). Hence, the image $|U_K|$ of the unit group U_K under the map $|\ | : \mathbf{R}^* \to \mathbf{R}_+^*$ is contained in the norm-one hypersurface $\mathbf{S} := \{x \in \mathbf{R}_+^* : N(x) = 1\}$. Write every $y \in \mathbf{R}_+^*$ in the form $y = xt^{\frac{1}{n}}$, where $x = \frac{y}{N(y)^{\frac{1}{n}}}$, $t = N(y)$. We then obtain a direct decomposition $\mathbf{R}_+^* = \mathbf{S} \times \mathbb{R}_+^*$. Let d^*x be the unique Haar measure on the mulitplicative group $\mathbf{S}$ such that the canonical Haar measure $\frac{dy}{y}$ on $\mathbf{R}_+^*$ becomes the product measure $\frac{dy}{y} = d^*x \times \frac{dt}{t}$.

The logarithm map log takes $\mathbf{S}$ to the trace-zero space $\mathbf{H} := \{x \in \mathbf{R}_\pm : \mathrm{Tr}(x) = 0\}$ and the group $|U_K|$ is taken to a full ($\mathbb{Z}$-)latice $G = G_K$ in $\mathbf{H}$ (Dirichlet's Unit Theorem). We claim that the group $|U_K^+|$ of U_K^+ is also a full lattice $G^+ = G_K^+$ in $\mathbf{H}$. Indeed, it is clear that $|U_K^2| \subset |U_K^+| \subset |U_K|$. But $|U_K^2| = 2|U_K|$ is a finite index subgroup of $|U_K|$. Thus, $[G : G^+]$ is finite, and being a subgroup of G, a full rank lattice, of finite index, G^+ has to be a full rank lattice. Similarly, one sees that the image $G_{K,r}^+$ of the group $|U_K^r \cap U_K^+|$ is a full rank lattice in $\mathbf{H}$ as well.

Choose now $F_{K,r}^+$ to be the preimage of an arbitrary fundamental parallelopiped $\mathfrak{D}_{K,r}^+$ of the lattice $G_{K,r}^+$ in $\mathbf{H}$, then the fundamental domain $F_{K,r}^+$ cuts up the norm-one hypersurface $\mathbf{S}$ into the disjoint union $\mathbf{S} = \cup_{\eta \in U_F^+} \eta^r F_{r,K}^+$.

Lemma. *The fundamental domain $F_{r,K}^+$ of $U_K^r \cap U_K^+$ in $\mathbf{S}$ has the following volume with respect to d^*x:* $\mathrm{Vol}(F_{r,K}^+) = r^{r_1+r_2-1} R_K^+$ *where R_K^+ is the narrow regulator of K.*

Proof. Since $I := \{t \in \mathbb{R}_+^* : 1 \le t \le e\}$ has measure 1 with respect to $\frac{dt}{t}$, the quantity $\mathrm{Vol}(F_{r,K}^+)$ is also the volume of $F_{r,K}^+ \times I$ with respect to $d^*x \times \frac{dt}{t}$, i.e., the volume of $\alpha(F_{r,K}^+ \times I)$ with respect to $\frac{dy}{y}$. The composition ψ of the isomorphisms $\mathbf{R}_+^* \xrightarrow{\log} \mathbf{R}_\pm \xrightarrow{\phi} \prod_{\mathfrak{p}|\infty} \mathbb{R} = \mathbb{R}^{r_1+r_2}$ transforms $\frac{dy}{y}$ into the Lebesgue measure of $\mathbb{R}^{r_1+r_2}$, $\mathrm{Vol}(F_{r,K}^+) = \mathrm{Vol}_{\mathbb{R}^{r_1+r_2}}((\psi \circ \alpha)(F_{r,K}^+ \times I))$. Let us compute the image $(\psi \circ \alpha)(F_{r,K}^+ \times I)$. Let $\mathbf{1} := (1, 1, \dots, 1) \in \mathbf{S}$. Then we find $(\psi \circ \alpha)(F_{r,K}^+ \times I) = \mathfrak{e} \cdot \log t^{1/n} = \frac{1}{n}\mathfrak{e} \log t$ with the vector $\mathfrak{e} = (e_{\mathfrak{p}_1}, \dots, e_{\mathfrak{p}_{r_1+r_2}}) \in \mathbb{R}^{r_1+r_2}$, $e_{\mathfrak{p}_i} = 1$ or 2 depending whether $\mathfrak{p}_i$ is real or complex. By definition of $F_{r,K}^+$, we also have $(\psi \circ \alpha)(F_{r,K}^+ \times \{1\}) =$

$r\Phi_K^+$ where Φ_K^+ denotes the fundamental parallelopiped of the totally positive unit lattice G_K^+ in the trace-zero space $\mathbf{H}$. This gives $(\psi\circ\alpha)(F_{r,K}^+\times I) = r\Phi_K^+\times[0,\frac{1}{n}]\mathfrak{e}$, the parallelopiped spanned by the vectors $\mathfrak{e}_1,\dots,\mathfrak{e}_{r_1+r_2-1},\frac{1}{n}\mathfrak{e}$ if $\mathfrak{e}_1,\dots,\mathfrak{e}_{r_1+r_2-1}$ span the fundamental domain Φ_K^+. Its volume is $\frac{1}{n}r^{r_1+r_2-1}$ times the absolute value of the determinant

$$\det\begin{pmatrix}\mathfrak{e}_{1,1} & \dots & \mathfrak{e}_{r_1+r_2-1,1} & \mathfrak{e}_{\mathfrak{p}_1}\\ \dots & \dots & \dots & \dots\\ \mathfrak{e}_{1,r_1+r_2} & \dots & \mathfrak{e}_{r_1+r_2-1,r_1+r_2} & \mathfrak{e}_{\mathfrak{p}_{r_1+r_2-1}}\end{pmatrix}.$$

Adding the first r_1+r_2-1 lines to the last one, all entries of the last line becomes 0 except the last one, which is $\sum_i\mathfrak{e}_{\mathfrak{p}_i}=n$. By defintion, the absolute value of the determinant of the matrix above these zeros is equal to the narrow regulator R_K^+. Thus we get $\mathrm{Vol}(F_{r,K}^+) = r^{r_1+r_2-1}R_K^+$. This completes the proof. □

In summary, for (unit) factor $|U_K^r\cap U_K^+|\backslash\big(\mathbb{R}_+^*\big)^{r_1+r_2}$, its structure may be understood via natural decomposition $((U_K^r\cap U_K^+)\backslash\mathbf{S})\times\mathbb{R}_+^*$, with $\mathbf{S}$ the norm-one hypersurface $\mathbf{S} := \{x\in\mathbf{R}_+^* : N(x)=1\}$; together with a disjoint union $\mathbf{S}=\cup_{\eta\in U_F^+}\eta^r F_{r,K}^+$, where $F_{r,K}^+$ denotes a 'fundamental parallogram' of $U_K^r\cap U_K^+$ in $\mathbf{S}$ with $r^{r_1+r_2-1}R_K^+$ as its volume.

1.5 High rank zeta functions for number fields

Let K be an algebraic number field (of finite degree n) with Δ_K the absolute value of its discriminant and $\mathcal{O}_K$ the integer ring. For a fixed positive integer $r\in\mathbb{N}$, denote by $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_K$-lattices of rank r. Denote by $d\mu$ the natural associated (Tamagawa type) measure (induced from that on GL). For each $\Lambda\in\mathcal{M}_{K,r}$, define the associated *0-th geo-arithmetical cohomology* $h^0(K,\Lambda)$ by

$$h^0(K,\Lambda) := \log\left(\sum_{x\in\Lambda}\exp\left(-\pi\sum_{\sigma:\mathbb{R}}\|x_\sigma\|_{\rho_\sigma}^2 - 2\pi\sum_{\sigma:\mathbb{C}}\|x_\sigma\|_{\rho_\sigma}^2\right)\right)$$

where $x=(x_\sigma)_{\sigma\in S_\infty}$ and $(\rho_\sigma)_{\sigma\in S_\infty}$ denote the σ-component of the metric $\rho=\rho_\Lambda$ determinet by the lattice Λ with S_∞ a collection of inequivalent Archimedean places of K. (See e.g. [9], [3] and [28, 29].) And following [28, 29], we introduce the following

Definition. *Define the rank r zeta function $\xi_{K,r}(s)$ of number field K by*

$$\xi_{K,r}(s) := \int_{\Lambda\in\mathcal{M}_{K,r}}(e^{h^0(K,\Lambda)-1})\cdot(e^{-s})^{-\log\mathrm{Vol}(\Lambda)}\,d\mu(\Lambda),\qquad \Re(s)>1.$$

By the Arakelov–Riemann–Roch Formula, one can write the rank r zeta function in the following form which fits more for practical purpose

$$\xi_{K,r}(s) := \left(\Delta_K^{\frac{r}{2}}\right)^s \cdot \int_{\Lambda\in\mathcal{M}_{K,r}} (e^{h^0(K,\Lambda)-1})\cdot(e^{-s})^{\deg(\Lambda)}\, d\mu(\Lambda), \qquad \Re(s) > 1.$$

In [28, 29], we, for an $\mathcal{O}_K$-lattice Λ, construct two geo-arithmetical cohomology groups

$$H^0(K,\Lambda) := \Lambda, \qquad \text{and} \qquad H^1(K,\Lambda) := V(\Lambda)/\Lambda$$

where $V(\Lambda)$ denotes $V := \prod_{\sigma\in S_\infty} V_\sigma$ for $V_\sigma := \Lambda \otimes_{\mathcal{O}_K} K_\sigma$ equipped with the canonical measures. In such a way, both $H^0(K,\Lambda)$ and $H^1(K,\Lambda)$ are topological groups. More precisely, H^0 is discrete, while H^1 is compact. As a direct consequence, then the corresponding geo-arithmetical counts for these locally compact groups can be done by using Fourier analysis on them so as to naturally get not only the above h^0 but also a new h^1 in a very natural way for lattices. Moreover, fundamental results corresponding to the Serre duality and Riemann–Roch Theorem hold for these newly defined $h^i, i = 0, 1$ as well. To state them more clearly, as usual, introduce the *dualizing lattice* $\mathcal{K}_K$ of K as the dual of the so-called different lattice $\overline{\mathfrak{D}}_K$ of K. (Here by the different lattice $\overline{\mathfrak{D}}_K$, we mean the rank one $\mathcal{O}_K$-lattice whose underlying module is given by the different $\mathfrak{d}_K$ of K and whose metric is induced from the canonical one via the natural embedding $\mathfrak{d}_K \hookrightarrow K_\mathbb{R}$, the Minkowski space.) Also as usual, denote the (Arakelov) dual lattice of Λ by $\Lambda^\vee$. Then we have the following

(1) (**Serre Duality** $=$ **Pontragin Duality**)
   (a) (Topologically) $\widehat{H^1(K,\Lambda)} \cong H^0(K, \mathcal{K}_K \otimes \Lambda^\vee)$, where $\widehat{\ }$ denotes the Pontragin dual of a topological group;
   (b) (Analytically) $h^1(K,\Lambda) = h^0(K, \mathcal{K}_K \otimes \Lambda^\vee)$;

(2) (**Riemann–Roch Theorem**) $h^0(K,\Lambda) - h^1(K,\Lambda) =: \chi(K,\Lambda) = \deg(\Lambda) - \frac{r}{2}\log \Delta_K$.

Remarks.

(1) While H^0 and H^1 together with h^0 and h^1 are quite similar to those for function fields via an adelic approach (see e.g., [4], [13], [20] or [27]), two major differences should be noticed: (a) For number fields, H^0 is discrete and H^1 is compact, while for function fields, both H^0 and H^1 are linearly compact, i.e., are finite dimensional vector spaces over the base field; (b) For number fields, h^i are defined using Fourier analysis, say, a weight of Gauss distribution is attached to each element of H^0 in defining h^0. But for function fields, h^i are defined using a much simpler count. Say, when the base fields are finite, the counts are carried out by

a direct counting process, i.e., every element in H^i is counted with the naive weight 1.

(2) It is remarkable to see that the analogue of Serre Duality has a certain topological counterpart via Pontragin Duality for topological groups and an analytic counterpart via the Plancherel Formula, a special kind of Fourier Inversion Formula.

(3) The Riemann–Roch Theorem is a direct consequence of the Serre Duality and the Poisson Summation Formula. So the above constructions and results are almost in Tate's Thesis, but not quite yet there.

(4) A two dimensional analogue of such a theory seems to be very much in demanding – Such a two dimensional theory is closely related with the Riemann Hypothesis via an intersection approach proposed in [28].

(5) The reader may learn how to appreciate the treatment here for H^i's and h^i's by consulting Weil's *Basic Number Theory* and Neukirch's *Algebraic Number Theory*. For the first one, mainly due to the lack of the construction above, Weil, unlike in the rest of his book, treated zeta functions for number fields separately from that for function fields, while for the second, Neukirch introduced a different type of h^i for which no duality is satisfied.

With all this well-prepared cohomology theory, standard yet fundamental properties for high rank zeta functions can be easily deduced. It works exactly as that for Artin zeta functions for curves over finite fields, as done by H. L. Schmid. Indeed, it is now a standard procedure to deduce the meromorphic continuation from the Riemann–Roch, to establish the functional equation from the Serre Duality and to locate the singularities from both Riemann–Roch and Serre Duality. (For details, please see Moreno [18] and/or Weil [27] and/or [28, 29].) That is to say, we have the following

Fact

(I) (Meromorphic Continuation) *The rank r zeta function $\xi_{K,r}(s)$ is well-defined when $\Re(s) > 1$ and admits a meromorphic continuation, denoted also by $\xi_{K,r}(s)$, to the whole complex s-plane;*

(II) (Functional Equation) $\xi_{K,r}(1-s) = \xi_{K,r}(s)$;

(III) (Singularities & Residues) *$\xi_{K,r}(s)$ has only two singularities, all are simple poles, at $s = 0, 1$, with the same residues $\mathrm{Vol}\Big(\mathcal{M}_{K,r}\Big(\Big[\Delta_K^{\frac{r}{2}}\Big]\Big)\Big)$, where $\mathcal{M}_{K,r}\Big(\Big[\Delta_K^{\frac{r}{2}}\Big]\Big)$ denotes the moduli space of rank r semi-stable $\mathcal{O}_K$-lattices whose volumes are fixed to be $\Delta_K^{\frac{r}{2}}$.*

Remarks.

(1) Due to the fact that the volumes of lattices are fixed, the semi-stable condition implies that the first Minkowski successive minimums of the lattices involved admit a natural lower bound away from 0 (depending only on r). Hence by the standard reduction theory, see e.g., Borel [1, 2], $\mathcal{M}_{K,r}\Big(\Big[\Delta_K^{\frac{r}{2}}\Big]\Big)$ is compact. Consequently, the volume $\mathrm{Vol}\Big(\mathcal{M}_{K,r}\Big(\Big[\Delta_K^{\frac{r}{2}}\Big]\Big)\Big)$ appeared above does make sense.

(2) The Tamagawa type of volume $\mathrm{Vol}\Big(\mathcal{M}_{K,r}\Big(\Big[\Delta_K^{\frac{r}{2}}\Big]\Big)\Big)$ is a new intrinsic non-abelian invariant for the number field K.

1.6 High rank zeta functions and Epstein zeta functions

Recall that we can choose integral $\mathcal{O}_K$-ideals $\mathfrak{a}_1 = \mathcal{O}_K, \mathfrak{a}_2, \ldots, \mathfrak{a}_h$ such that the ideal class group $CL(K)$ is given by $\{[\mathfrak{a}_1], \ldots, [\mathfrak{a}_h]\}$, and that any rank r projective $\mathcal{O}_K$-module P is isomorphic to $P_{\mathfrak{a}_i}$ for a certain $i, 1 \leq i \leq h$. Here, $P_\mathfrak{a} := P_{r,\mathfrak{a}} := \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$ for a fractional $\mathcal{O}_K$-ideal $\mathfrak{a}$. (Quite often, we use $\mathfrak{a}$ as a running symbol for $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_h$.) Consequently, $\mathcal{M}_{K,r} = \cup_{i=1}^h \widetilde{\mathcal{M}}_{K,r}(\mathfrak{a}_i)$ with $\widetilde{\mathcal{M}}_{K,r}(\mathfrak{a}_i) =: (\widetilde{\mathbf{\Lambda}}(P_{\mathfrak{a}_i}))_{\mathrm{ss}}$, the part of $\widetilde{\mathbf{\Lambda}}(P_{\mathfrak{a}_i})$ consisting of only semi-stable $\mathcal{O}_K$-lattices.

As such, introduce a partial high rank zeta function $\widetilde{\xi}_{K,r;\mathfrak{a}}(s)$ by setting

$$\widetilde{\xi}_{K,r;\mathfrak{a}}(s) := \int_{\widetilde{\mathcal{M}}_{K,r}(\mathfrak{a})} (e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)} d\mu(\Lambda), \qquad \Re(s) > 1.$$

Consequently, $\widetilde{\xi}_{K,r;\mathfrak{a}}(s) = \sum_{j=1}^{\mu(r,K)} \xi_{K,r;\mathfrak{a};A_j}(s)$ where

$$\xi_{K,r;\mathfrak{a};A_j}(s) := \int_{\Lambda \in \mathcal{M}_{K,r;A_j}(\mathfrak{a})} (e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)} d\mu(\Lambda), \ \ \Re(s) > 1$$

with $\mathcal{M}_{K,r;A_j}(\mathfrak{a})$ the component of the moduli space of semi-stable $\mathcal{O}_K$-lattices whose points corresponding to these in

$$\begin{aligned} &[A_i \backslash ((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \backslash ((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2})) \\ &\quad \times (|U_K^r \cap U_K^+| \backslash (\mathbb{R}_+^*)^{r_1+r_2}))]_{\mathrm{ss}} \\ &= A_i \backslash ((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}) \backslash ((SL(r,\mathbb{R})/SO(r))^{r_1} \\ &\quad \times (SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}} \times (|U_K^r \cap U_K^+| \backslash (\mathbb{R}_+^*)^{r_1+r_2})). \end{aligned}$$

under the natural identification above. Moreover, since A_i is simply an automorphism, its action does not change the total volumes as well as the h^0 of the lattices. We then introduce the (genuine) partial zeta function $\xi_{K,r;\mathfrak{a}}(s)$ by setting

$$\xi_{K,r;\mathfrak{a}}(s) := \int_{\mathcal{M}_{K,r}(\mathfrak{a})} (e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)} d\mu(\Lambda), \qquad \Re(s) > 1$$

where $\mathcal{M}_{K,r}(\mathfrak{a})$ denotes the part of the moduli space of semi-stable $\mathcal{O}_K$-lattices whose points corresponding to these in

$$((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}} \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2})).$$

Consequently we then obtain the following

Proposition. $\xi_{K,r;\mathfrak{a};A_j}(s) = \xi_{K,r;\mathfrak{a}}(s), \forall j = 1, \ldots, \mu(r,K)$. *In particular,*

$$\xi_{K,r}(s) = \mu(r,K) \cdot \sum_{i=1}^{h} \xi_{K,r;\mathfrak{a}_i}(s).$$

This been said, to further understand the structure of our zeta function $\xi_{K,r}(s)$, we next investigate how the integrand $(e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)} d\mu(\Lambda)$ behaves over the space

$$((SL(\mathcal{O}_K^{(r-1)} \oplus \mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1} \times (SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}} \times (|U_K^r \cap U_K^+|\backslash(\mathbb{R}_+^*)^{r_1+r_2})).$$

By definition, $e^{h^0(K,\Lambda)} - 1 = \sum_{x \in \Lambda\backslash\{0\}} \exp(-\pi \sum_{\sigma:\mathbb{R}} \|x_\sigma\|_{\rho_\sigma} - 2\pi \sum_{\sigma:\mathbb{C}} \|x_\sigma\|_{\rho_\sigma})$. Also, in terms of the embedding $z \in \Lambda = \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a} \hookrightarrow K^{(r)} \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r \simeq (\mathbb{R}^r)^{r_1} \times (\mathbb{C}^r)^{r_2}$, z maps to the corresponding point (z_σ) and $\|z_\sigma\|_{\rho_\sigma} = \|g_\sigma z_\sigma\|$, where the metric ρ_σ is defined by $g_\sigma \cdot g_\sigma^t$ for certain $g_\sigma \in GL(r,\mathbb{R})$ when σ is real, and by $g_\sigma \cdot \bar{g}_\sigma^t$ for certain $g_\sigma \in GL(r,\mathbb{C})$ when σ is complex.

Recall that $\|g_\sigma z_\sigma\|$ is $O(r)$ resp. $U(r)$ invariant when σ is real resp. complex. Similarly, $\mathrm{Vol}(\Lambda)$ is invariant. Consequently, $(e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)}$ is well-defined over $(GL(r,\mathbb{R})/O(r))^{r_1} \times (GL(r,\mathbb{C})/U(r))^{r_2}$.

To go further, we next study how $(e^{h^0(K,\Lambda)} - 1) \cdot (e^{-s})^{-\log \mathrm{Vol}(\Lambda)}$ changes when we apply the operation $\Lambda \mapsto \Lambda[t]$ for $t > 0$. Clearly, in terms of each local component, $\rho_\sigma \mapsto t_\sigma \rho_\sigma$ with $t_\sigma \in \mathbb{R}_+^*$, we have $\|x_\sigma\|_{t_\sigma\rho_\sigma}^2 = t_\sigma^2 \cdot \|x_\sigma\|_{\rho_\sigma}^2$. Hence $(e^{h^0(K,\Lambda[t])} - 1)$ changes to $\sum_{x \in \Lambda\backslash\{0\}} \exp\Big(-\pi \sum_{\sigma:\mathbb{R}} \|x_\sigma\|_{\rho_\sigma} \cdot t_\sigma^{\frac{r}{2}} - 2\pi \sum_{\sigma:\mathbb{C}} \|x_\sigma\|_{\rho_\sigma} \cdot t_\sigma^{\frac{r}{2}}\Big)$, while $\mathrm{Vol}(\Lambda[t])$ decomposes to $\mathrm{Vol}(\Lambda) \cdot \prod_{\sigma \in S_\infty} t_\sigma^r$

for $t=(t_\sigma)$. On the other hand, by changing the volume in such a way, $d\mu(\Lambda)$ becomes $\prod_{\sigma\in S_\infty}\frac{dt_\sigma}{t_\sigma}\cdot d\mu_1(\Lambda_1)$, where $d\mu_1(\Lambda_1)$ denotes the corresponding volume form on the space of semi-stable lattices corresponding to the points in

$$\mathcal{M}_{F,r;\mathfrak{a}}\left[N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right]:=(SL(\mathcal{O}_K^{(r-1)}\oplus\mathfrak{a})\backslash((SL(r,\mathbb{R})/SO(r))^{r_1}\times(SL(r,\mathbb{C})/SU(r))^{r_2}))_{\mathrm{ss}},$$

due to the fact $\mathrm{Vol}(\overline{\mathcal{O}_K^{(r-1)}\oplus\mathfrak{a}})=\Delta_K^{\frac{r-1}{2}}\cdot\left(N(\mathfrak{a})\cdot\Delta_K^{\frac{1}{2}}\right)=N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}$. (As we are going to identify the moduli space of lattices with its realization in terms of SL, from now on we make no distinction between them.) Moreover, note that the $\mathcal{O}_K$-units have their (total rational) norm 1, hence $\mathcal{O}_K$-units do not really change the total volume of the lattice. All in all, then we get for $\Re(s)>1$,

$$\begin{aligned}\xi_{F,r;\mathfrak{a}}(s)=&\left(N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right)^s\cdot\int_{\mathbb{R}_{>0}^{r_1+r_2}}t_\sigma^{rs}\prod_{\sigma\in S_\infty}\frac{dt_\sigma}{t_\sigma}\\&\times\int_{\Lambda\in\mathcal{M}_{F,r;\mathfrak{a}}\left[N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right]}\sum_{x\in(\Lambda\backslash\{0\})/U_{r,F}^+}\\&\times\exp\left(-\pi\sum_{\sigma:\mathbb{R}}\|x_\sigma\|_{\rho_\sigma}\cdot t_\sigma^{\frac{r}{2}}-2\pi\sum_{\sigma:\mathbb{C}}\|x_\sigma\|_{\rho_\sigma}\cdot t_\sigma^{\frac{r}{2}}\right)d\mu_1(\Lambda).\end{aligned}$$

Therefore, by applying the Mellin transform and using $\int_0^\infty e^{-At^B}t^s\frac{dt}{t}=\frac{1}{B}\cdot A^{-\frac{s}{B}}\cdot\Gamma\left(\frac{s}{B}\right)$, we obtain that

$$\begin{aligned}\xi_{F,r;\mathfrak{a}}(s)=&\left(N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right)^s\cdot\int_{\Lambda\in\mathcal{M}_{F,r;\mathfrak{a}}\left[N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right]}\sum_{x\in(\Lambda\backslash\{0\})/U_{r,F}^+}\\&\times\left(\prod_{\sigma:\mathbb{R}}\left(\frac{r}{2}\cdot(\pi\|x_\sigma\|_{\rho_\sigma})^{-\frac{rs}{2}}\Gamma\left(\frac{rs}{2}\right)\right)\right.\\&\left.\times\cdot\prod_{\sigma:\mathbb{C}}\left(\frac{r}{2}\cdot(2\pi\|x_\sigma\|_{\rho_\sigma})^{-\frac{rs}{2}}\Gamma(rs)\right)\right)d\mu_1(\Lambda)\\=&\left(\frac{r}{2}\right)^{r_1+r_2}\cdot\left(\pi^{-\frac{rs}{2}}\Gamma\left(\frac{rs}{2}\right)\right)^{r_1}\cdot((2\pi)^{-rs}\Gamma(rs))^{r_2}\end{aligned}$$

$$\times \left(N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right)^s \cdot \int_{\Lambda\in\mathcal{M}_{F,r;\mathfrak{a}}\left[N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right]}$$

$$\times \left(\sum_{x\in(\Lambda\backslash\{0\})/U_{r,F}^+}\frac{1}{\|x\|_\Lambda^{rs}}\right) d\mu(\Lambda), \quad \Re(s)>1.$$

Accordingly, for $\Re(s) > 1$, define the *completed Epstein zeta function* $\hat{E}_{K,r;\mathfrak{a}}(s)$ by

$$\hat{E}_{K,r;\mathfrak{a}}(s) := \left(\pi^{-\frac{rs}{2}}\Gamma\left(\frac{rs}{2}\right)\right)^{r_1}\cdot\left((2\pi)^{-rs}\Gamma(rs)\right)^{r_2}$$

$$\cdot\left[\left(N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right)^s\cdot\sum_{x\in(\Lambda\backslash\{0\})/U_{r,F}^+}\frac{1}{\|x\|_\Lambda^{rs}}\right].$$

All in all, what we have just said exposes the following

Fact

(IV) (Decomposition) *The rank r zeta funtion of K admits a natural decomposition* $\xi_{K,r}(s) = \mu(r,K)\cdot\sum_{i=1}^h \xi_{K,r;\mathfrak{a}_i}(s)$;

(V) (High Rank Zeta = Integration of Epstein Zeta) *The partial rank r zeta function $\xi_{F,r;\mathfrak{a}}(s)$ of K associated to $\mathfrak{a}$ is given by an integration of a completed Epstein type zeta function:*

$$\xi_{F,r;\mathfrak{a}}(s) = \left(\frac{r}{2}\right)^{r_1+r_2}\cdot\int_{\mathcal{M}_{F,r;\mathfrak{a}}\left[N(\mathfrak{a})\cdot\Delta_K^{\frac{r}{2}}\right]}\hat{E}_{K,r;\mathfrak{a}}(s)\,d\mu, \qquad \Re(s)>1.$$

Remark. The relation between high rank zeta and Epstein zeta was first established for $\mathbb{Q}$ in our 'Analytic truncation and Rankin-Selberg versus algebraic truncation and non-abelian zeta', *Algebraic Number Theory and Related Topics*, RIMS Kokyuroku, No. 1324 (2003). Furthermore, in [28, 29], we develop a general theory of non-abelian L-functions for global fields, using Langlands' theory of Eisenstein series.

2. Rank two $\mathcal{O}_K$-lattices: stability & distance to cusps

2.1 Upper half space model

2.1.1 Upper half plane

The *upper half plane* $\mathcal{H}$ in complex plane $\mathbb{C}$ is defined to be $\mathcal{H} := \{z = x+iy\in\mathbb{C}, x\in\mathbb{R}, y\in\mathbb{R}_+^*\}$. On $\mathcal{H}$, the natural hyperbolic metric is given by the line element $ds^2 := \frac{dx^2+dy^2}{y^2}$ with the volume form $d\mu := \frac{dx\wedge dy}{y^2}$, and the Laplace-Beltrami operator $\Delta := y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)$.

The natural action on $\mathcal{H}$ of the group $SL(2,\mathbb{R})$ of real 2×2 metrices with determinant one is given by:

$$M\,z := \frac{az+b}{cz+d}, \qquad \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),\ \ z\in\mathcal{H}.$$

Easily, if $Mz := x^* + iy^*$ with $x^*,\ y^* \in \mathbb{R}$, then

$$x^* = \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2}, \qquad y^* = \frac{y}{(cx+d)^2+c^2y^2} > 0.$$

In particular, y^* depends only on z and the second row of M.

As said, $\mathcal{H}$ admits the real line $\mathbb{R}$ as its boundary. Consequently, to compactify it, we add on it the real projective line $\mathbb{P}^1(\mathbb{R})$ with $\infty = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Naturally, the above action of $SL(2,\mathbb{R})$ also extends to $\mathbb{P}^1(\mathbb{R})$ via $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}ax+by\\cx+dy\end{smallmatrix}\right]$.

Back to $\mathcal{H}$ itself. The stablizer of $i = (0,1) \in \mathcal{H}$ with respect to the action of $SL(2,\mathbb{R})$ on $\mathcal{H}$ is equal to $SO(2) := \{A \in O(2) : \det A = 1\}$. Since the action of $SL(2,\mathbb{R})$ on $\mathcal{H}$ is transitive, we can identify the quotient $SL(2,\mathbb{R})/SO(2)$ with $\mathcal{H}$ given by the quotient map induced from $SL(2,\mathbb{R}) \to \mathcal{H}, g\mapsto g\cdot i$.

2.1.2 Upper half Space

The *upper half space* $\mathbb{H}$ is given by

$$\begin{aligned}\mathbb{H} := \mathbb{C}\times]0,\infty[&= \{(z,r) : z = x+iy\in\mathbb{C}, r\in\mathbb{R}_+^*\}\\ &= \{(x,y,r) : x,y\in\mathbb{R}, r\in\mathbb{R}_+^*\}.\end{aligned}$$

Thinking of $\mathbb{H}$ as a subset of Hamilton's quaternions with $1,\ i,\ j,\ k$ the standard $\mathbb{R}$-basis of the quaternions, we may write points P in $\mathbb{H}$ as $P = (z,r) = (x,y,r) = z + rj$ where $z = x+iy,\ j = (0,0,1)$.

We equip $\mathbb{H}$ with the hyperbolic metric coming from the line element $ds^2 := \frac{dx^2+dy^2+dr^2}{r^2}$ with volume form $d\mu := \frac{dx\wedge dy\wedge dr}{r^3}$ and Laplace-Beltrami operator $\Delta := r^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial r^2}\Big) - r\frac{\partial}{\partial r}$.

The natural action of $SL(2,\mathbb{C})$ on $\mathbb{H}$ and on its boundary $\mathbb{P}^1(\mathbb{C})$ may be described as follows: We represent an element of $\mathbb{P}^1(\mathbb{C})$ by $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ where $x,y\in\mathbb{C}$ with $(x,y)\neq(0,0)$. Then the action of the matrix $M = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is defined to be

$$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{bmatrix}x\\y\end{bmatrix} := \begin{bmatrix}ax+by\\cx+dy\end{bmatrix}.$$

Moreover, if we represent points $P \in \mathbb{H}$ as quaternions whose fourth component equals zero, then the action of M on $\mathbb{H}$ is defined to be $P \mapsto M\,P := (aP+b)(cP+d)^{-1}$, where the inverse on the right is taken in the skew field of quaternions. Indeed, if we set $M(z+rj) = z^*+r^*j$ with $z^* \in \mathbb{C}, r^* \in \mathbb{R}$, then an obvious computation shows that

$$z^* := \frac{(az+b)(\bar{c}\bar{z}+\bar{d})+a\bar{c}r^2}{|cz+d|^2+|c|^2r^2}, \quad r^* := \frac{r}{|cz+d|^2+|c|^2r^2} = \frac{r}{\|cP+d\|^2}.$$

In particular, r^* depends only on P and the second row of M. Moreover $r^* > 0$, so $M(z+rj) \in \mathbb{H}$ as well. (Here we have set $P = z+rj$ and used $\|cP+d\|$ to denote the Euclidean norm of the vector $cP+d \in \mathbb{R}^4$, which is indeed also the square root of the norm of $cP+d$ in the quaternions.)

Furthermore, with this action, the stablizer of $j = (0,0,1) \in \mathbb{H}$ in $SL(2,\mathbb{C})$ is equal to $SU(2) := \{A \in U(2) : \det A = 1\}$. Since the action of $SL(2,\mathbb{C})$ on $\mathbb{H}$ is transitive, we obtain also a natural identification $\mathbb{H} \simeq SL(2,\mathbb{C})/SU(2)$ via the quotient map induced from $SL(2,\mathbb{C}) \to \mathbb{H},\ g \mapsto g\cdot j$.

2.1.3 Rank two $\mathcal{O}_K$-lattices: upper half space model

With above, by identifying $\mathcal{H}$ with $SL(2,\mathbb{R})/SO(2)$ and $\mathbb{H}$ with $SL(2,\mathbb{C})/SU(2)$, we conclude that

$$\mathcal{M}_{K,2;\mathfrak{a}}[N(\mathfrak{a})\cdot\Delta_K] \simeq (SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}))_{\mathrm{ss}},$$

where as before ss means the subset consisting of points corresponding to rank two semi-stable $\mathcal{O}_K$-lattices in $SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash((SL(2,\mathbb{R})/SO(2))^{r_1} \times (SL(2,\mathbb{C})/SU(2))^{r_2})$.

Put this in a more concrete term, if the metric on $\mathcal{O}_K\oplus\mathfrak{a}$ is given by matrices $g = (g_\sigma)_{\sigma\in S_\infty}$ with $g_\sigma \in SL(2,K_\sigma)$, then the corresponding points on the right hand side is $g(\mathrm{ImJ})$ with $\mathrm{ImJ} := (i^{(r_1)}, j^{(r_2)})$, i.e., the point given by $(g_\sigma\tau_\sigma)_{\sigma\in S_\infty}$ where $\tau_\sigma = i_\sigma := (0,1)$ if σ is real and $\tau_\sigma = j_\sigma := (0,0,1)$ if σ is complex. As before, $SL(\mathcal{O}_K\oplus\mathfrak{a})$ denotes elements in $GL(\mathcal{O}_K\oplus\mathfrak{a})$ with determinant 1: if $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(\mathcal{O}_K\oplus\mathfrak{a})$, then $ad-bc=1$ and $a,d \in \mathcal{O}_K, b\in\mathfrak{a}$, and $c \in \mathfrak{a}^{-1}$.

2.2 Cusps

Now the working site is the space $SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2})$ with $\mathcal{H}$ the upper half plane, $\mathbb{H}$ the upper half space, and $SL(\mathcal{O}_K\oplus\mathfrak{a})$ the special automorphism group defined by $\left\{A \in \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}\\ \mathfrak{a}^{-1} & \mathcal{O}_K\end{pmatrix} : \det A = 1\right\}$. Here the action of $SL(\mathcal{O}_K\oplus\mathfrak{a})$ is via the action of $SL(2,K)$ on $\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$. More precisely,

K^2 admits natural embeddings $K^2 \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^2 \simeq (\mathbb{R}^2)^{r_1} \times (\mathbb{C}^2)^{r_2}$ so that $\mathcal{O}_K \oplus \mathfrak{a}$ naturally embeds into $(\mathbb{R}^2)^{r_1} \times (\mathbb{C}^2)^{r_2}$ as a rank two $\mathcal{O}_K$-lattice. As such, $SL(\mathcal{O}_K \oplus \mathfrak{a})$ acts on the image of $\mathcal{O}_K \oplus \mathfrak{a}$ in $(\mathbb{R}^2)^{r_1} \times (\mathbb{C}^2)^{r_2}$ as automorphisms. Our task here is to understand the cusps of this action of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. For this, we go as follows.

First, the space $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ admits a natural boundary $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, in which the field K is imbedded via Archmidean places in S_∞: $K \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Consequently, $\mathbb{P}^1(K) \hookrightarrow \mathbb{P}^1(\mathbb{R})^{r_1} \times \mathbb{P}^1(\mathbb{C})^{r_2}$ with $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] := \infty \mapsto (\infty^{(r_1)}, \infty^{(r_2)})$. As usual, via fractional linear transformations, $SL(2,\mathbb{R})$ acts on $\mathbb{P}^1(\mathbb{R})$, and $SL(2,\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$, hence so does $SL(2,K)$ on $\mathbb{P}^1(K) \hookrightarrow \mathbb{P}^1(\mathbb{R})^{r_1} \times \mathbb{P}^1(\mathbb{C})^{r_2}$. Being a discrete subgroup of $SL(2,\mathbb{R})^{r_1} \times SL(2,\mathbb{C})^{r_2}$, for the action of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathbb{P}^1(K)$, we call the corresponding orbits (of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathbb{P}^1(K)$) the *cusps*. Very often we also call representatives cusps.

As before, we would like to study cusps by transforming $\zeta \in \mathbb{P}^1(K)$ to ∞, and hence want to assume without loss of generality that $\zeta = \infty$ in our discussion. For this becoming possible, it is then supposed that we are able to find, for $\zeta := \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \mathbb{P}^1(K)$ an element $M := M_\zeta := \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2,K)$, since then it is clear that $\left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \cdot \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$, that is, $M \cdot \infty = \zeta$. This is clearly possible, because if we set $\mathfrak{c} := \mathcal{O}_K \cdot \alpha + \mathcal{O}_K \cdot \beta$ to be the fractional ideal generated by α and β, then $1 \in \mathcal{O}_K = \mathfrak{c} \cdot \mathfrak{c}^{-1} = \alpha\mathfrak{c}^{-1} + \beta\mathfrak{c}^{-1}$. Therefore, there exist $\alpha^*,\ \beta^* \in \mathfrak{c}^{-1} \subset K$ such that $\alpha\beta^* - \alpha^*\beta = 1$.

Theorem . (Cusp and Ideal Class Correspondence) *There is a natural bijection between the ideal class group $CL(K)$ of K and the cusps $\mathcal{C}_\Gamma$ of $\Gamma = SL(\mathcal{O}_K \oplus \mathfrak{a})$ acting on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ given by $\mathcal{C}_\Gamma \to CL(K)$, $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \mapsto [\mathcal{O}_K\, \alpha + \mathfrak{a}\, \beta]$.*

This type of results are rooted back to Maaβ. But we here give a proof using a method of Siegel, while we reminder the reader that our case at hand is much more complicated.

Choose fixed integral $\mathcal{O}_K$-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ representing the ideal class group $CL(K)$. We want to show that the elements of $\mathbb{P}^1(K)$ are divided into h equivalence classes by the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $P = \left[\begin{smallmatrix} p \\ s \end{smallmatrix}\right] \in \mathbb{P}^1(K)$ defined by $\gamma \cdot P = \left[\begin{smallmatrix} ap+bs \\ cp+ds \end{smallmatrix}\right]$. Let also $P = \left[\begin{smallmatrix} p \\ s \end{smallmatrix}\right]$ be a fixed point in $\mathbb{P}^1(K)$ with $p, s \in K$. Define $\pi(P)$ to be the ideal class associated to the fractiona ideal $\mathcal{O}_K \cdot p + \mathfrak{a} \cdot s$.

Claim.

(1) $\pi : \mathbb{P}^1(K) \to CL(K)$ *is well-defined.*

(2) π *factors through the orbit space* $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash\mathbb{P}^1(K)$.

Proof.

(1) Indeed, if $P = \begin{bmatrix} p_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_2 \\ s_2 \end{bmatrix}$, then, as ideal classes,

$$[\mathcal{O}_K \cdot p_1 + \mathfrak{a} \cdot s_1] = \left[\frac{s_2}{s_1}(\mathcal{O}_K \cdot p_1 + \mathfrak{a} \cdot s_1)\right] = \left[\mathcal{O}_K \cdot s_2 \cdot \frac{p_1}{s_1} + \mathfrak{a} \cdot s_2\right]$$
$$= [\mathcal{O}_K \cdot p_2 + \mathfrak{a} \cdot s_2].$$

Here, we use [ ] to denote an ideal class.

(2) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\mathcal{O}_K \oplus \mathfrak{a})$, $\pi(\gamma \cdot P) = \pi\left(\begin{bmatrix} ap+bs \\ cp+ds \end{bmatrix}\right) = [\mathcal{O}_K \cdot (ap + bs) + \mathfrak{a} \cdot (cp + ds)]$. But, by definition, $a,\ d \in \mathcal{O}_K,\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1}$. Hence,

$$\begin{aligned} \mathcal{O}_K \cdot (ap + bs) + \mathfrak{a} \cdot (cp + ds) &= (ap) \cdot \mathcal{O}_K + (bs) \cdot \mathcal{O}_K + (cp) \cdot \mathfrak{a} \\ &\quad + (ds) \cdot \mathfrak{a} \subset p \cdot \mathcal{O}_K + s \cdot \mathfrak{a} \\ &\quad + p \cdot (\mathfrak{a}^{-1} \cdot \mathfrak{a}) + s \cdot \mathfrak{a} \\ &= p \cdot \mathcal{O}_K + s \cdot \mathfrak{a}. \end{aligned}$$

On the other hand, the inverse inclusion holds as well because the determinant of γ is one. Therefore, $\pi(\gamma \cdot P) = [\mathcal{O}_K \cdot (ap+bs) + \mathfrak{a} \cdot (cp+ds)] = [\mathcal{O}_K \cdot p + \mathfrak{a} \cdot s]$. Done.

□

Consequently, we get a well-defined map

$$\Pi: \ SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash \mathbb{P}^1(K) \to CL(K), \qquad \begin{bmatrix} p \\ s \end{bmatrix} \mapsto [\mathcal{O}_K \cdot p + \mathfrak{a} \cdot s].$$

We want to show that Π is a bijection.

Clearly the surjectity is a direct consequence of the following

Lemma. *For any two fractional $\mathcal{O}_K$-ideals $\mathfrak{a}, \mathfrak{b}$, there exist elements $\alpha, \beta \in K$ such that $\mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta = \mathfrak{b}$.*

Proof. Recall the following □

Chinese Reminder Theorem. *Let $\mathfrak{p}_j$ for $j = 1, \ldots, s$ denote distinct prime ideals of $\mathcal{O}_K$, and let e_j for $j = 1, \ldots, s$ be positive integers. Then the map given by the product of the quotient maps $f : \mathcal{O}_K \to \prod_{j=1}^s \mathcal{O}_K/\mathfrak{p}_j^{e_j}$ yields an isomorphism of rings $\mathcal{O}_K/\prod_{j=1}^s \mathfrak{p}_j^{e_j} \simeq \prod_{j=1}^s \mathcal{O}_K/\mathfrak{p}_j^{e_j}$.*

In terms of congruence, this means that given $x_j \in \mathcal{O}_K$ for $j = 1, \ldots, s$, there exists $x \in \mathcal{O}_K$ such that $x \equiv x_j \mod \mathfrak{p}_j^{e_j}$; and moreover, this uniquely determines the class of $x \mod \prod_{j=1}^s \mathfrak{p}_j^{e_j}$.

With this in mind, let us go back to the proof of the lemma. We can and hence now assume that both $\mathfrak{a}$ and $\mathfrak{b}$ are integral.(First we may assume that $\mathfrak{b}$ is an integral $\mathcal{O}_K$-ideal. Indeed, there exist $b \in K$ and an integral $\mathcal{O}_K$-ideal $\mathfrak{b}'$ such that $\mathfrak{b} = b \cdot \mathfrak{b}'$. Therefore, if there exist α', β' such that $\mathcal{O}_K \cdot \alpha' + \mathfrak{a} \cdot \beta' = \mathfrak{b}'$, then $\mathfrak{b} = b \cdot \mathfrak{b}' = b \cdot (\mathcal{O}_K \cdot \alpha' + \mathfrak{a} \cdot \beta') = \mathcal{O}_K \cdot (b\alpha') + \mathfrak{a} \cdot (b\beta')$. That is to say, $\alpha = b\alpha'$ and $\beta = b\beta'$ will do the job. Then we may further assume that $\mathfrak{a}$ is integral. Indeed, there exists an $a \in K^*$ such that $a \cdot \mathfrak{a} = \mathfrak{a}'$ is integral. Thus if there exist α', $\beta' \in K$ such that $\mathcal{O}_K \cdot \alpha' + \mathfrak{a}' \cdot \beta' = \mathfrak{b}$. Then

$$\begin{aligned}\mathfrak{b} &= \mathcal{O}_K \cdot \alpha' + \mathfrak{a}' \cdot \beta' = \mathcal{O}_K \cdot \alpha' + \mathfrak{a}' \cdot (a \cdot a^{-1})\beta' \\ &= \mathcal{O}_K \cdot \alpha' + (\mathfrak{a}' \cdot a) \cdot (a^{-1}\beta') = \mathcal{O}_K \cdot \alpha' + \mathfrak{a} \cdot (a^{-1}\beta').\end{aligned}$$

That is to say, this time, $\alpha = \alpha'$ and $\beta = a^{-1}\beta'$ do the job.)

We want to find α, $\beta \in K$ such that $\mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta = \mathfrak{b}$. (Clearly, if done, then α, β cannot be both zero at the same time, hence define a point $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \in \mathbb{P}^1(K)$. Furthermore, we get $\Pi\left(\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right) = \mathfrak{b}$ as desired.)

Choose now $\beta \in \mathfrak{a}^{-1}\mathfrak{b}\backslash\{0\}$ so that $\mathfrak{a} \cdot \beta \subset \mathfrak{a} \cdot \mathfrak{a}^{-1}\mathfrak{b} \subset \mathcal{O}_K \cdot \mathfrak{b} = \mathfrak{b}$. Therefore, by the unique factorization theorem of integral $\mathcal{O}_F$-ideals into product of prime ideals, we can assume that $\mathfrak{b} = \prod_{i=1}^l \mathfrak{p}_i^{n_i} \supset \mathfrak{a} \cdot \beta = \prod_{i=1}^l \mathfrak{p}_i^{m_i}$ with $m_i \geq n_i \geq 0$, $1 \leq i \leq l$. Now choose $b_i \in \mathfrak{p}_i^{n_i} \big/ \mathfrak{p}_i^{n_i+1}$ for all $i = 1, \ldots, l$. By the Chinese Reminder Theorem above, there exists an element $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv b_i \bmod \mathfrak{p}_i^{n_i+1}$. Since, in terms of local orders at $\mathfrak{p}_i$, $\nu_{\mathfrak{p}_i}(\alpha) = \nu_{\mathfrak{p}_i}(\mathfrak{b})$ for each i, we know that $\alpha \in \mathfrak{b}$. Thus $\mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta \subset \mathfrak{b}$.

On the other hand, if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ which does not lie in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$, then $0 = \nu_{\mathfrak{p}}(\beta) = \nu_{\mathfrak{p}}(\mathfrak{b})$. Thus we have shown that for all primes $\mathfrak{p}$ of $\mathcal{O}_K$, $\nu_{\mathfrak{p}}(\mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta) = \inf\{\nu_{\mathfrak{p}}(\mathcal{O}_K \cdot \alpha), \nu_{\mathfrak{p}}(\mathfrak{a} \cdot \beta)\} = \nu_{\mathfrak{p}}(\mathfrak{b})$. Therefore, $\mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta = \mathfrak{b}$. This completes the proof of the lemma and hence the theorem.

Remarks.

(1) Taking $\mathfrak{a} = \mathcal{O}_K$, we in particular see that any fractional $\mathcal{O}_K$-ideal is generated by at most two elements, a simple beautiful fact that should be included in all standard textbooks on Algebraic Number Theory. The reader may find it in [8], whose proof we followed in our discussion above.

(2) We would like to reminder the reader that during this process of studying high rank zeta functions for number fields, all basic facts, not only the finiteness results on ideal class group and units, but the Chinese Reminder Theorem are used.

With this being done, we are left with the injectivity of Π. For this, we use the trick of Siegel, following the presentation of Terras [25].

Take $\zeta_1 := \left[\begin{smallmatrix} p_1 \\ s_1 \end{smallmatrix}\right]$ and $\zeta_2 := \left[\begin{smallmatrix} p_2 \\ s_2 \end{smallmatrix}\right]$ in $\mathbb{P}^1(K)$ with p_i, s_i in $\mathcal{O}_K$. Then by the discussion just above the theorem, there exist $M_1 := \left(\begin{smallmatrix} p_1 & p_1^* \\ s_1 & s_1^* \end{smallmatrix}\right)$ and $M_2 := \left(\begin{smallmatrix} p_2 & p_2^* \\ s_2 & s_2^* \end{smallmatrix}\right)$ in $SL(2,K)$ such that $M_1 \cdot \infty = \zeta_1,\ M_2 \cdot \infty = \zeta_2$. Consequently, $(M_1 \cdot M_2^{-1})\zeta_2 = \zeta_1$. In other words, $\left[\begin{smallmatrix} p_1 \\ s_1 \end{smallmatrix}\right] = \left(M_1 \cdot M_2^{-1}\right) \cdot \left[\begin{smallmatrix} p_2 \\ s_2 \end{smallmatrix}\right]$. Thus by the fact that p_i, s_i are all $\mathcal{O}_K$-integers, easily, we have $M_1 \cdot M_2^{-1} \in GL(2, \mathcal{O}_K \oplus a)$ by writing down all the entries explicitly. Clearly, by definition, $M_1 \cdot M_2^{-1} \in SL(2,K)$ as well. Hence $M_1 \cdot M_2^{-1} \in GL(2, \mathcal{O}_K \oplus a) \cap SL(2,K) = SL(2, \mathcal{O}_K \oplus a)$. This completes the proof.

In summary, what we have just established is the following bijection

$$\Pi : SL(2, \mathcal{O}_K \oplus a)\backslash \mathbb{P}^1(K) \simeq CL(K), \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mapsto [\mathcal{O}_K\alpha + \mathfrak{a}\beta := \mathfrak{b}].$$

One checks also that the inverse map Π^{-1} is given as follows: For $\mathfrak{b}$, choose $\alpha_\mathfrak{b}, \beta_\mathfrak{b} \in K$ such that $\mathcal{O}_K \cdot \alpha_\mathfrak{b} + \mathfrak{a} \cdot \beta_\mathfrak{b} = \mathfrak{b}$; With this, then $\Pi^{-1}([\mathfrak{b}])$ is simply the class of the point $\left[\begin{smallmatrix} \alpha_\mathfrak{b} \\ \beta_\mathfrak{b} \end{smallmatrix}\right]$ in $SL(2, \mathcal{O}_K \oplus a)\backslash \mathbb{P}^1(K)$. Moreover, there always exists $M_{\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]} := \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2,K)$ such that $M_{\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]} \cdot \infty = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$.

2.3 Stablizer groups of cusps

Recall that under the Cusp-Ideal Class Correspondence, there are exactly h inequivalence cusps η_i, $i = 1, 2, \dots, h$. Moreover, if we write the cusp $\eta_i = \left[\begin{smallmatrix} \alpha_i \\ \beta_i \end{smallmatrix}\right]$ for suitable α_i, $\beta_i \in K$, then the associated ideal class is exactly the one for the fractional ideal $\mathcal{O}_K\alpha_i + \mathfrak{a}\beta_i =: \mathfrak{b}_i$. Denote the stablizer group of η_i by

$$\Gamma_{\eta_i} := \{\gamma \in SL(\mathcal{O}_K \oplus \mathfrak{a}) : \gamma\eta_i = \eta_i\}, \qquad i = 1, 2, \dots, h.$$

Quite often, we use η as a running symbol for η_i.

We want to see the structure of Γ_η. As usual, we first shift η to ∞. So choose $A = \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2,K)$. Clearly $A \cdot \infty = A\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$. Consequently, $\Gamma_\eta = A \cdot \Gamma_\infty \cdot A^{-1}$.

Next, we further pin down the choice of α^* and β^* appeared in A. For this, we use a trick which according to Elstrodt roots back to Hurwitz.

Lemma. *Let α, $\beta \in K$ such that $\mathcal{O}_K\alpha + \mathfrak{a}\beta = \mathfrak{b} \neq \{0\}$. Then there exist $\alpha^*, \beta^* \in K$ such that (1) $\left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2,K)$; and (2) $\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^* = \mathfrak{b}^{-1}$.*

Proof. Note that $1 \in \mathcal{O}_K = \mathfrak{b}\cdot\mathfrak{b}^{-1} = (\mathcal{O}_K\alpha + \mathfrak{a}\beta)\cdot\mathfrak{b}^{-1} = \mathfrak{b}^{-1}\cdot\alpha + (\mathfrak{a}\mathfrak{b}^{-1})\cdot\beta$. As such, we can choose $\beta^* \in \mathfrak{b}^{-1}$, $\alpha^* \in \mathfrak{a}\mathfrak{b}^{-1}$ such that $\alpha\beta^* - \beta\alpha^* = 1$. This gives (1). As for (2), it suffices to show that $(\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^*) \cdot (\mathcal{O}_K\alpha + \mathfrak{a}\beta) = \mathcal{O}_K$. One inclusion is clear. Indeed, by our construction, $1 \in (\mathcal{O}_K\beta^* +$

$\mathfrak{a}^{-1}\alpha^*) \cdot (\mathcal{O}_K\alpha + \mathfrak{a}\beta)$, so $(\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^*) \cdot (\mathcal{O}_K\alpha + \mathfrak{a}\beta) \supset \mathcal{O}_K$. As for the inclusion in the other direction, we go as follows: Clearly,

$$(\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^*) \cdot (\mathcal{O}_K\alpha + \mathfrak{a}\beta) = \mathcal{O}_K \cdot (\beta^*\alpha) + \mathfrak{a}^{-1} \cdot (\alpha^*\alpha) + (\mathfrak{a}\mathfrak{a}^{-1}) \cdot (\alpha^*\beta) + \mathfrak{a} \cdot (\beta\beta^*).$$

But, by definition, $\mathfrak{b} = \mathcal{O}_K\alpha + \mathfrak{a}\beta$, so $\alpha \in \mathfrak{b}$, $\beta \in \mathfrak{a}^{-1}\mathfrak{b}$. This, together with $\beta^* \in \mathfrak{b}^{-1}$, $\alpha^* \in \mathfrak{a}\mathfrak{b}^{-1}$, then gives

$$(\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^*) \cdot (\mathcal{O}_K\alpha + \mathfrak{a}\beta) \subset \mathcal{O}_K \cdot (\mathfrak{b}^{-1} \cdot \mathfrak{b}) + \mathfrak{a}^{-1} \cdot ((\mathfrak{a}\mathfrak{b}^{-1}) \cdot \mathfrak{b}) + (\mathfrak{a}\mathfrak{a}^{-1}) \cdot ((\mathfrak{a}\mathfrak{b}^{-1}) \cdot (\mathfrak{a}^{-1}\mathfrak{b})) + \mathfrak{a} \cdot ((\mathfrak{a}^{-1}\mathfrak{b})\mathfrak{b}^{-1}) = \mathcal{O}_K.$$

Consequently, we have the following structure of the stablizer Γ_η; □

Corollary. $A^{-1}\Gamma_\eta A = \left\{ \begin{pmatrix} u & z \\ 0 & u^{-1} \end{pmatrix} : u \in U_K, z \in \mathfrak{a}\mathfrak{b}^{-2} \right\}$. *In particular, the associated 'lattice' for the cusp η is given by the fractional ideal $\mathfrak{a}\mathfrak{b}^{-2}$.*

Proof. All elements in $A^{-1} \cdot \Gamma_\eta \cdot A$ fix ∞, so are given by upper triangle matrices. With this observation, let us now show that $z \in \mathfrak{a}\mathfrak{b}^{-2}$. This is easy. Indeed, by definition, $A^{-1} \cdot \Gamma_\eta \cdot A$ consists of elements in the form $\begin{pmatrix} \beta^* & -\alpha^* \\ -\beta & \alpha \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{21} = 0$ and $a_{12} = (a-d)\alpha^*\beta^* - c(\alpha^*)^2 + b(\beta^*)^2$. Recall that $\alpha \in \mathfrak{b}$, $\beta \in \mathfrak{a}^{-1}\mathfrak{b}$ and $\beta^* \in \mathfrak{b}^{-1}$, $\alpha^* \in \mathfrak{a}\mathfrak{b}^{-1}$, and that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\mathcal{O}_K \oplus \mathfrak{a})$, $a,\ d \in \mathcal{O}_K,\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1}$, easily we have $z = a_{12} \subset \mathcal{O}_K \cdot ((\mathfrak{a}\mathfrak{b}^{-1}) \cdot \mathfrak{b}^{-1}) + \mathfrak{a}^{-1} \cdot (\mathfrak{a}\mathfrak{b}^{-1})^2 + \mathfrak{a} \cdot (\mathfrak{b}^{-1})^2 = \mathfrak{a}\mathfrak{b}^{-2}$ as desired.

To complete the proof, we still need to show that u is a unit. This may be done as follows. Assume, as we can, that $\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ with $\alpha,\ \beta \in \mathcal{O}_K$. Note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\mathcal{O}_K \oplus \mathfrak{a})$ such that $\gamma \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, we have $(a\alpha + b\beta) \cdot \beta = (c\alpha + d\beta) \cdot \alpha$. Here $a,\ d \in \mathcal{O}_K,\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1}$, and $\alpha \in \mathfrak{b}$, $\beta \in \mathfrak{a}^{-1}\mathfrak{b}$ with $\mathfrak{b} = \mathcal{O}_K\alpha + \mathfrak{a}\beta$. Thus note that now the ideal generated by $(a\alpha + b\beta)\beta = (c\alpha + d\beta)\alpha$ is included in $\mathfrak{a}^{-1}\mathfrak{b}^2$. Dividing by it, we have

$$\frac{(a\alpha + b\beta)}{\mathfrak{b}} = \frac{(\alpha)}{\mathfrak{b}} \qquad \text{and} \qquad \frac{(c\alpha + d\beta)}{\mathfrak{a}^{-1}\mathfrak{b}} = \frac{(\beta)}{\mathfrak{a}^{-1}\mathfrak{b}}. \tag{*}$$

But $\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \begin{pmatrix} u & z \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$, so $\begin{pmatrix} u\alpha & * \\ u\beta & * \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & * \\ c\alpha + d\beta & * \end{pmatrix}$. Therefore,

$$(u\alpha) = (a\alpha + b\beta), \qquad \text{and} \quad (u\beta) = (c\alpha + d\beta). \tag{**}$$

Clearly, from (*) and (**), as integral ideals $(u\alpha) = (\alpha), (u\beta) = (\beta)$. So $u \in U_K$ as desired. □

Set now $\Gamma_\eta' := \left\{ A\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}A^{-1} : z \in \mathfrak{a}\mathfrak{b}^{-2} \right\}$. Then $\Gamma_\eta = \Gamma_\eta' \times \left\{ A\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}A^{-1} : u \in U_K \right\}$. Note that also componentwisely, $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} z =$

$\frac{uz}{u^{-1}} = u^2 z$. So, in practice, what we really get is the following decomposition $\Gamma_\eta = \Gamma'_\eta \times U_K^2$ with

$$U_K^2 \simeq \left\{ A \cdot \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \cdot A^{-1} : u \in U_K \right\} \simeq \left\{ A \begin{pmatrix} 1 & 0 \\ 0 & u^2 \end{pmatrix} A^{-1} : u \in U_K \right\}.$$

Now we are ready to proceed a construction of a fundamental domain for the action of $\Gamma_\eta \subset SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. This is based on a construction of a fundamental domain for the action of Γ_∞ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. More precisely, with an element $A = \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2, K)$ used above, i) $A \cdot \infty = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$; and ii) The isotropy group of η in $A^{-1} SL(\mathcal{O}_K \oplus \mathfrak{a}) A$ is generated by translations $\tau \mapsto \tau + z$ with $z \in \mathfrak{a}\mathfrak{b}^{-2}$ and by dilations $\tau \mapsto u\tau$ where u runs through the group U_K^2.

Consider then the map

$$\begin{aligned} \mathrm{ImJ}: \quad & \\ \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} & \to \mathbb{R}_{>0}^{r_1+r_2}, \\ \tau := (z_1, \ldots, z_{r_1}; P_1, \ldots, P_{r_2}) & \mapsto (\Im(z_1), \ldots, \Im(z_{r_1}); J(P_1), \ldots, J(P_{r_2})), \end{aligned}$$

where if $z = x + iy \in \mathcal{H}$ resp. $P = z + rj \in \mathbb{H}$, we set $\Im(z) = y$ resp. $J(P) = r$. It induces a map $(A^{-1} \cdot \Gamma_\eta \cdot A) \backslash (\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}) \to U_K^2 \backslash \mathbb{R}_{>0}^{r_1+r_2}$, which exhibits $(A^{-1} \cdot \Gamma_\eta \cdot A) \backslash (\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$ as a torus bundle over $U_K^2 \backslash \mathbb{R}_{>0}^{r_1+r_2}$ with fiber the $n = r_1 + 2r_2$ dimensional torus $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}) / \mathfrak{a}\mathfrak{b}^{-2}$. Having factored out the action of the translations, we only have to construct a fundamental domain for the action of U_K^2 on $\mathbb{R}_{>0}^{r_1+r_2}$. This is essentially the same as in 1.4. We look first at the action of U_K^2 on the norm-one hypersurface $\mathbf{S} := \{y \in \mathbb{R}_{>0}^{r_1+r_2} : N(y) = 1\}$. By taking logarithms, it is transformed bijectively into a trace-zero hyperplane which is isomorphic to the space $\mathbb{R}^{r_1+r_2-1}$

$$\begin{aligned} \mathbf{S} \overset{\log}{\to} \mathbb{R}^{r_1+r_2-1} &:= \left\{ (a_1, \ldots a_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} : \sum a_i = 0 \right\}, \\ y \mapsto \quad & (\log y_1, \ldots, \log y_{r_1+r_2}), \end{aligned}$$

where the action of U_K^2 on $\mathbf{S}$ is carried out over an action on $\mathbb{R}^{r_1+r_2-1}$ by translations: $a_i \mapsto a_i + \log \varepsilon^{(i)}$. By Dirichlet's Unit Theorem, the logarithm transforms U_K^2 into a lattice in $\mathbb{R}^{r_1+r_2-1}$. The exponential map transforms a fundamental domain, e.g., a fundamental parallelopiped, for this action back into a fundamental domain $\mathbf{S}_{U_K^2}$ for the action of U_K^2 on $\mathbf{S}$. The cone over $\mathbf{S}_{U_K^2}$, that is, $\mathbb{R}_{>0} \cdot \mathbf{S}_{U_K^2} \subset \mathbb{R}_{>0}^{r_1+r_2}$, is a fundamental domain for the action of U_K^2 on $\mathbb{R}_{>0}^{r_1+r_2}$. If we denote by $\mathcal{T}$ a fundamental domain for the action of the translations by elements of $\mathfrak{a}\mathfrak{b}^{-2}$ on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and set

$$\mathrm{ReZ}\,(z_1, \ldots, z_{r_1}; P_1, \ldots, P_{r_2}) := (\Re(z_1), \ldots, \Re(z_{r_1}); Z(P_1), \ldots, Z(P_{r_2}))$$

with $\Re(z) := x$ resp. $Z(P) := z$ if $z = x + iy \in \mathcal{H}$ resp. $P = z + rj \in \mathbb{H}$, then what we have just said proves the following

Theorem . *The set* $\mathbf{E} := \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mathrm{ReZ}\,(\tau) \in \mathcal{T},\ \mathrm{ImJ}\,(\tau) \in \mathbb{R}_{>0} \cdot \mathbf{S}_{U_K^2}\}$ *is a fundamental domain for the action of* $A^{-1}\Gamma_\eta A$ *on* $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$.

For later use, we also set $\mathcal{F}_\eta := A_\eta^{-1} \cdot \mathbf{E}$.

2.4 Fundamental domain

Guided by Siegel's discussion on totally real fields [21], we are now ready to construct fundamental domains for general number fields.

So we are dealing with rank two $\mathcal{O}_K$-lattices whose underlying projective modules P are all given by the same $P = P_{\mathfrak{a}} := \mathcal{O}_K \oplus \mathfrak{a}$ for a fixed fractional $\mathcal{O}_K$-ideal $\mathfrak{a}$. This then leads to the space $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$.

To facilitate ensuring discussion, recall that for $\tau = (z_1, \dots, z_{r_1}; P_1, \dots, P_{r_2}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, we set $\mathrm{ImJ}(\tau) := (\Im(z_1), \dots, \Im(z_{r_1}), J(P_1), \dots, J(P_{r_2})) \in \mathbb{R}^{r_1+r_2}$ where $\Im(z) = y$ resp. $J(P) = v$ for $z = x + iy \in \mathcal{H}$ resp. $P = z + vj \in \mathbb{H}$. For our own convenience, we now set

$$N(\tau) := N(\mathrm{ImJ}(\tau)) = \prod_{i=1}^{r_1} \Im(z_i) \cdot \prod_{j=1}^{r_2} J(P_j)^2 = (y_1 \cdot \ldots \cdot y_{r_1}) \cdot (v_1 \cdot \ldots \cdot v_{r_2})^2.$$

Then by an obvious computation, we have, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, K)$,

$$N(\mathrm{ImJ}(\gamma \cdot \tau)) = \frac{N(\mathrm{ImJ}(\tau))}{\|N(c\tau + d)\|^2}. \qquad (*)$$

In particular, only the second row of γ appears.

As the first step to construct a fundamental domain, we need to have a generalization of Siegel's 'distance to cusps'. Recall that for a cusp $\eta = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \mathbb{P}^1(K)$, by the Cusp-Ideal Class Correspondence, we have a natural corresponding ideal class associated to the fractional ideal $\mathfrak{b} := \mathcal{O}_K \cdot \alpha + \mathfrak{a} \cdot \beta$. Moreover, by assuming that α, β appeared above are all contained in $\mathcal{O}_K$, as we may, we know that the corresponding stabilizer group Γ_η can be described by $A^{-1} \cdot \Gamma_\eta \cdot A = \left\{\gamma = \left(\begin{smallmatrix} u & z \\ 0 & u^{-1} \end{smallmatrix}\right) \in \Gamma : u \in U_K, z \in \mathfrak{a}\mathfrak{b}^{-2}\right\}$, where $A \in SL(2, K)$ satisfying $A\infty = \eta$ which may be further chosen in the form $A = \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2, K)$ so that $\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\alpha^* = \mathfrak{b}^{-1}$.

Now define the *reciprocal distance* $\mu(\eta, \tau)$ *from a point* $\tau = (z_1, \dots, z_{r_1}; P_1, \dots, P_{r_2})$ *in* $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ *to the cusp* $\eta = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ *in* $\mathbb{P}^1(K)$ by

$$\begin{aligned}
&\mu(\eta,\tau)\\
&:= N(\mathfrak{a}^{-1}\cdot(\mathcal{O}_K\alpha+\mathfrak{a}\beta)^2)\cdot\frac{\Im(z_1)\dots\Im(z_{r_1})\cdot J(P_1)^2\dots J(P_{r_2})^2}{\prod_{i=1}^{r_1}|(-\beta^{(i)}z_i+\alpha^{(i)})|^2 \prod_{j=1}^{r_2}\|(-\beta^{(j)}P_j+\alpha^{(j)})\|^2}\\
&=\frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\frac{N(\mathrm{ImJ}(\tau))}{\|N(-\beta\tau+\alpha)\|^2}.
\end{aligned}$$

This is well-defined: if $\eta=\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]=\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]$ in $\mathbb{P}^1(K)$, there exists $\lambda\in K^*$ such that $\alpha'=\lambda\cdot\alpha,\ \beta'=\lambda\cdot\beta$. Therefore, $\mu(\eta,\tau)$ in terms of $\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]$ is given by $\frac{1}{N(\mathfrak{a}\mathfrak{b}'^{-2})}\cdot\frac{N(\mathrm{ImJ}(\tau))}{\|N(-\beta'\tau+\alpha')\|^2}$ where $\mathfrak{b}'=\mathcal{O}_K\alpha'+\mathfrak{a}\beta'=(\lambda)\cdot\mathfrak{b}$. Hence, $\mu(\eta,\tau)$ in terms of $\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]$ becomes $\frac{N(\lambda)^2}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\frac{N(\mathrm{ImJ}(\tau))}{N(\lambda)^2\cdot\|N(-\beta\tau+\alpha)\|^2}=\frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\frac{N(\mathrm{ImJ}(\tau))}{\|N(-\beta\tau+\alpha)\|^2}$, which is nothing but $\mu(\eta,\tau)$ in terms of $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$. We are done.

As such, our definition is clearly a generalization and more importantly a normalization of Siegel's distance to cusps. In particular, this definition is enviromentally free.

Lemma 1. *μ is invariant under the action of $SL(\mathcal{O}_K\oplus\mathfrak{a})$.*

Proof. By the well-defined argument above, we may simply assume that for a cusp η, α,β are fixed. Then the proof is based on the following observation. For the cusp $\eta=\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\in\mathbb{P}^1(K)$, we may choose $A_\eta=\left(\begin{smallmatrix}\alpha&\alpha^*\\ \beta&\beta^*\end{smallmatrix}\right)\in SL(2,K)$ such that $A\infty=\eta$. (Surely, A_η is not unique, however this does not matter.) Clearly, $A_\eta^{-1}=\left(\begin{smallmatrix}\beta^*&-\alpha^*\\ -\beta&\alpha\end{smallmatrix}\right)$. Therefore, by defintion,

$$\mu(\eta,\tau)=\frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot N(\mathrm{ImJ}(A_\eta^{-1}(\tau))). \tag{**}$$

Note that even A_η is not unique, as said above, with a fixed τ, from (*), $N(\mathrm{ImJ}(A_\eta^{-1}(\tau)))$ depends only on the second row of A_η^{-1}, which is simply $(-\beta,\alpha)$, uniquely determined by the cusp η.

With (**), the proof may be completed easily as follows. First, let us consider the factor $N(\mathfrak{a}\mathfrak{b}^{-2})$. Clearly, with the change from η to $\gamma\eta$ for $\gamma\in SL(\mathcal{O}_K\oplus\mathfrak{a})$, the fractional ideal $\mathfrak{a}\mathfrak{b}^{-2}$ does not really change, so this factor remains unchanged. Therefore, it suffices to consider the second factor $N(\mathrm{ImJ}(A_\eta^{-1}(\tau)))$. By an easy calculation, $A_{\gamma\eta}=\gamma A_\eta$. Consequently, $A_{\gamma\eta}^{-1}(\gamma\tau)=(\gamma A_\eta)^{-1}(\gamma\tau)=A_\eta^{-1}\gamma^{-1}(\gamma\eta)=A_\eta^{-1}(\gamma^{-1}\gamma\eta)=A_\eta^{-1}(\eta)$. □

Lemma 2. *There exists a positive constant C depending only on K and $\mathfrak{a}$ such that if $\mu(\eta,\tau)>C$ and $\mu(\eta',\tau)>C$ for $\tau\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$ and $\eta,\ \eta'\in\mathbb{P}^1(K)$, then $\eta=\eta'$.*

Proof. Set $\mu(\eta,\tau)=\frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\frac{1}{\Delta(\eta,\tau)}$. Since $N(\mathfrak{a}^{-1}\mathfrak{b})\geq N(\mathfrak{a}^{-1})$, it suffices to show that *there exists a positive constant c depending only on K such that if $\Delta(\eta,\tau)<c$ and $\Delta(\eta',\tau)<c$ for $\tau\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$ and $\eta,\ \eta'\in\mathbb{P}^1(K)$, then $\eta=\eta'$.*

By the Cusp-Ideal Class correspondence and the invariance property just proved, we can write $\eta=\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right],\ \eta'=\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right]$ with $\mathcal{O}_K$-integers $\alpha,\beta,\alpha',\beta'$ such that $\mathfrak{b}:=\mathcal{O}_K\alpha+\mathfrak{a}\beta$ and $\mathfrak{b}':=\mathcal{O}_K\alpha'+\mathfrak{a}\beta'$ have norm less than a constant C depending only on K. For every (r_1+r_2)-tuple $(t_1,\dots,t_{r_1+r_2})$ of non-zero real numbers, by Dirichlet's Unit Theorem, there exists a unit $\varepsilon\in K$ such that $|t_i\varepsilon^{(i)}|\leq c\cdot|N(t)|^{\frac{1}{r_1+r_2}}$ where $N(t):=\prod_{i=1}^{r_1}t_i\cdot\prod_{j=r_1+1}^{r_1+r_2}t_j^2$ with c a constant depending only on K. Hence, after multiplying α and β by a suitable uint, we have

$$\begin{aligned}&\max\{\,\Im(z_i)^{-1}|-\beta^{(i)}z_i+\alpha^{(i)}|,\ \ J(P_j)^{-2}\|-\beta^{(j)}P_j+\alpha^{(j)}\|^2\}\\ &\quad\leq c\cdot\Delta(\eta,\tau)^{-\frac{1}{r_1+r_2}}\cdot C^{\frac{2}{r_1+r_2}}\leq c\cdot T^{-\frac{1}{r_1+r_2}}\cdot C^{\frac{2}{r_1+r_2}}.\end{aligned}$$

This gives

$$\begin{aligned}&\max\{\,|-\beta^{(i)}\Re(z_i)+\alpha^{(i)}|\cdot\Im(z_i)^{-1/2},\ \ \|-\beta^{(j)}Z(P_j)+\alpha^{(j)}\|\cdot J(P_j)^{-1}\}\\ &\quad\leq c^{1/2}\cdot T^{-\frac{1}{2(r_1+r_2)}}\cdot C^{\frac{1}{r_1+r_2}}\end{aligned}$$

and $\max\{\,|\beta^{(i)}|\cdot\Im(z_i)^{1/2},\ \|\beta^{(j)}\|\cdot J(P_j)\}\leq c^{1/2}\cdot T^{-\frac{1}{2(r_1+r_2)}}\cdot C^{\frac{1}{r_1+r_2}}$. For α' and β', we obtain similar inequalities. But now, for real places

$$\begin{aligned}\alpha^{(i)}(\beta')^{(i)}-\beta^{(i)}(\alpha')^{(i)}&=(-\beta^{(i)}\Re(z_i)+\alpha^{(i)})\Im(z_i)^{-1/2}\cdot(\beta')^{(i)}\Im(z_i)^{1/2}\\ &\quad-(-(\beta')^{(i)}\Re(z_i)+(\alpha)^{(i)})\Im(z_i)^{-1/2}\\ &\quad\times\cdot\beta^{(i)}\Im(z_i)^{1/2},\end{aligned}$$

while for complex places,

$$\begin{aligned}\alpha^{(j)}(\beta')^{(j)}-\beta^{(j)}(\alpha')^{(j)}&=(-\beta^{(j)}Z(P_j)+\alpha^{(j)})J(P_j)\cdot(\beta')^{(j)}J(P_j)\\ &\quad-(-(\beta')^{(j)}Z(P_j)+(\alpha')^{(j)})J(P_j)\cdot\beta^{(j)}J(P_j).\end{aligned}$$

Consequently $N(\alpha\beta'-\beta\alpha')\leq(2c)^{r_1+r_2}\cdot T^{-1}\cdot C^2$. So if $T>(2c)^{r_1+r_2}\cdot C^2$, the norm of the algebraic integer $\alpha\beta'-\beta\alpha'$ has absolute value less than 1, that is, $\alpha\beta'-\beta\alpha'=0$. This implies that $\eta=\eta'$ as desired. □

More correctly, we should consider $\frac{1}{\mu(\eta,\tau)^{1/2}}$ as the 'distance' of τ to the cusp η. For example, if $\eta=\infty$, the distance is just $\frac{1}{N(\tau)^{1/2}}\cdot\frac{1}{N(\mathfrak{a})^{1/2}}$, since by definition, $\mu(\infty,\tau)=\frac{N(\mathcal{O}_K\cdot 1+\mathfrak{a}\cdot 0)^2N(\tau)}{|N(-0\tau+1)|^2}=N(\tau)$.

Lemma 3. *There exists a positive real number $T := T(K)$ depending only on K such that for $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, there exists a cusp η such that $\mu(\eta, \tau) > T$.*

Proof. Since $N(\mathfrak{a}^{-1}\mathfrak{b}^2) \geq N(\mathfrak{a}^{-1})$, and there are finitely many inequivalent cusps, it is sufficient to find a solution of α, β in $\mathcal{O}_K$ satisfying the inequality

$$|N(-\beta\tau + \alpha)|^2 \cdot N(\mathrm{ImJ}(\tau))^{-1} \leq T^{-1}.$$

Consider the inequalities

$$|-\beta^{(i)}\Re(z_i)+\alpha^{(i)}| \cdot \Im(z_i)^{-1/2} \leq c_i, \;\; |\beta^{(i)}| \cdot \Im(z_i)^{1/2} \leq d_i, \;\; i = 1, \dots, r_1$$

$$\|-\beta^{(j)}Z(P_j)+\alpha^{(j)}\| \cdot J(P_j)^{-1} \leq c_j, \;\; \|\beta^{(j)}\| \cdot J(P_j) \leq d_j, \;\; j = 1, \dots, r_2,$$

which we may write, using a $\mathbb{Z}$-basis $\omega_1, \dots, \omega_{r_1+r_2}$ of $\mathcal{O}_K$ as a system of $r_1 + 2r_2$ linear inequalities (by changing the last r_2 to the $2r_2$ inequalities involving only real numbers with respect to complex conjugations). According to a theorem of Minkowski, we can find a solution $\alpha = \sum a_i\omega_i, \; \beta = \sum b_i\omega_i$ with $a_i, \, b_i \in \mathbb{Z}$ provided that $(\prod c_i \cdot \prod d_j^2)$ is no less than the absolute of the determinant of this system. Clearly, this absolute value is simply $|\omega_i^{(k)}|^2 = \Delta_K$, the (absolute value of) discriminant of K. So we can take $c_i = d_j = \Delta_K^{\frac{1}{r_1+2r_2}}$, and hence $T = 2^{r_2} \cdot \Delta_K$. This completes the proof. □

Now for the cusp $\eta = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \in \mathbb{P}^1(K)$, we define the 'sphere of influence' of η by

$$F_\eta := \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mu(\eta, \tau) \geq \mu(\eta', \tau), \forall \eta' \in \mathbb{P}^1(K)\}.$$

Lemma 4. *The action of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ in the interior F_η^0 of F_η reduces to that of the isotropy group Γ_η of η, i.e., if τ and $\gamma\tau$ both belong to F_η^0, then $\gamma\tau = \tau$.*

Proof. We have $\begin{matrix} \mu(\gamma^{-1}\eta, \tau) & \leq & \mu(\eta, \tau) \\ \| & & \| \\ \mu(\eta, \gamma\tau) & \geq & \mu(\gamma\eta, \gamma\tau) \end{matrix}$ for $\tau, \, \gamma\tau \in F_\eta^o$, and the inequalities are strict if $\gamma\eta \neq \eta$.

Consequently, the boundary of F_η consists of pieces of 'generalized isometric circles' given by equalities $\mu(\eta, \tau) = \mu(\eta', \tau)$ with $\eta' \neq \eta$. □

Using above discussion, we arrive at decomposing the orbit space $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$ into h pieces glued in some way along pants of their boundary.

Theorem . *Let* $i_\eta : \Gamma_\eta \backslash F_\eta \hookrightarrow SL(\mathcal{O}_K \oplus \mathfrak{a}) \backslash (\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$ *be the natural map. Then*

$$SL(\mathcal{O}_K \oplus \mathfrak{a}) \backslash (\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}) = \cup_\eta i_\eta(\Gamma_\eta \backslash F_\eta),$$

where the union is taken over a set of h cusps representing the ideal classes of K. Each piece corresponds to an ideal class of K.

Note that the action of Γ_η on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is free. Consequently, all fixed points of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ lie on the boundaries of F_η.

Further, we may give a more precise description of the fundamental domain, based on our understanding of the fundamental domains for stabilizer groups of cusps. To state it, denote by $\eta_1, \ldots, \eta_h$ inequivalent cusps for the action of $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. Choose $A_{\eta_i} \in SL(2, K)$ such that $A_{\eta_i}\infty = \eta_i$, $i = 1, 2, \ldots, h$. Write $\mathbf{S}$ for the norm-one hypersurface $\mathbf{S} := \{y \in \mathbb{R}_{>0}^{r_1+r_2} : N(y) = 1\}$, and $\mathbf{S}_{U_K^2}$ for the action of U_K^2 on $\mathbf{S}$. Denote by $\mathcal{T}$ a fundamental domain for the action of the translations by elements of $\mathfrak{a}\mathfrak{b}^{-2}$ on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and $\mathbf{E} := \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mathrm{ReZ}\,(\tau) \in \mathcal{T},\ \mathrm{ImJ}\,(\tau) \in \mathbb{R}_{>0} \cdot \mathbf{S}_{U_K^2}\}$ for a fundamental domain for the action of $A_\eta^{-1}\Gamma_\eta A_\eta$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. Easily, we know that the intersections of $\mathbf{E}$ with $i_\eta(F_\eta)$ are connected. Consequently, we have the following

Theorem′.

(1) *The set $A_\eta^{-1}\mathbf{E} \cap F_\eta$ is a fundamental domain for the action of Γ_η on F_η which we call D_η;*
(2) *There exist $\alpha_1, \ldots, \alpha_h \in SL(\mathcal{O}_K \oplus \mathfrak{a})$ such that $\cup_{i=1}^h \alpha(D_{\eta_i})$ is connected and hence a fundamental domain for $SL(\mathcal{O}_K \oplus \mathfrak{a})$.*

We may present this concrete discussion on fundamental domains in a more theoretical manner. For this, we first introduce a natural geometric truncation for the fundamental domain. So define a compact manifold with boundary

$$S_T := SL(\mathcal{O}_K \oplus \mathfrak{a}) \backslash \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mu(\eta, \tau) \leq T\ \forall \eta \in \mathcal{C}_{SL(\mathcal{O}_K \oplus \mathfrak{a})}\},$$

where $C_{SL(\mathcal{O}_K \oplus \mathfrak{a})}$ denotes the collections of cusps, and T is so large that for all cusps η, $W(\eta, T) := \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : \mu(\eta, \tau) \leq T\}$ is contained in F_η, so disjoint for different classes η and η'. Clearly, then the boundary ∂S_T consists of h component manifold $i_\eta(\Gamma_\eta \backslash \partial W(\eta, T))$ of dimension $2r_1 + 3r_2 - 1$. Moreover, let $\Sigma := \big\{(t_1, \ldots, t_{r_1}; s_1, \ldots, s_{r_2}) \in \mathbb{R}_{>0}^{r_1+r_2} : \prod_{i=1}^{r_1} t_i \prod_{i=1}^{r_2} s_j^2 = 1\big\}$ act on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ by component-wise multiplication. The semi-direct product $\mathcal{E} = (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}) \times \Sigma$ acts on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ by

$$((u_i, v_j), (t_i, s_j)) \cdot (\tau = (z_i; P_j)) := (\lambda_i z_i + u_i; s_j P_j + v_j).$$

The boundary $\partial W(\infty, T)$ is a partial homogeneous space for this semi-direct product. We view $A_\eta^{-1}\Gamma_\eta A_\eta\backslash\partial W(\infty, Y)$ as the quotient of $\mathcal{E}$ by the discrete subgroup $A_\eta^{-1}\Gamma_\eta A_\eta$. It is a r_1+2r_2-torus bundle over $U_K^2\backslash\Sigma$ with fiber $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ modulo the translations in $A_\eta^{-1}\Gamma_\eta A_\eta$. The manifold with boundary S_T is homotopically equivalent to $SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash\Big(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}\Big)$. (See e.g. [**?**].) Consequently, we have

$$SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}) = S_T\cup_{\partial S_T}(\partial S_T\times[0,\infty)),$$

i.e., $SL(\mathcal{O}_K\oplus\mathfrak{a})\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2})$ is topologically a manifold with h 'ends' of the form $T^{r_1+2r_2}$-bundle over $T^{r_1+r_2-a}\times[0,\infty)$.

With all this, we may end our long discussion on the fundamental domain for the action of $SL(\mathcal{O}_K\oplus\mathfrak{a})$ on $\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$. The essentials are, of course, that a fundamental domain may be given as $S_Y\cup\mathcal{F}_1(Y_1)\cup\ldots\cup\mathcal{F}_h(Y_h)$ with $\mathcal{F}_i(Y_i)=A_i\cdot\widetilde{\mathcal{F}}_i(Y_i)$ and

$$\widetilde{\mathcal{F}}_i(Y_i) := \{\tau\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2} : \mathrm{ReZ}(\tau)\in\Sigma,\ \mathrm{ImJ}(\tau)\in\mathbb{R}_{>T}\cdot\mathbf{S}_{U_K^2}\}.$$

Moreover, all $\mathcal{F}_i(Y_i)$'s are disjoint from each other when Y_i are sufficiently large.

2.5 Stability

2.5.1 Upper half plane

So we are working with rank two $\mathbb{Z}$-lattice of volume 1. The space, i.e., the moduli space of all such lattices, is simply $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$, or better, $SL(2,\mathbb{Z})\backslash\mathcal{H}$. For it, we have a well-known fundamental domain $\mathcal{D}$ whose closure is given by $\overline{\mathcal{D}} := \big\{z\in\mathcal{H} : |z|\geq 1, |x|\leq\frac{1}{2}\big\}$. Our question then is:

What are the points in $\mathcal{D}$ corresponding to isometric classes of rank 2 semi-stable lattices of volume 1?

The answer is given by classical reduction theory. For any rank two $\mathbb{Z}$-lattice Λ of volume 1 in $\mathbb{R}^2$ (equipped with the standard Euclideal metric), fix $\mathbf{x}\in\Lambda\backslash\{0\}$ such that its length gives the first Minkowski successive minimum $\lambda_1=\lambda_1(\Lambda)$ of Λ. Then via rotation when necessary, we may assume that $\mathbf{x}=(\lambda_1,0)$. Furthermore, classical reduction theory tells us that $\frac{1}{\lambda_1}\Lambda$ is simply the lattice of the volume $\lambda_1^{-2}=: y_0$ generated by the vectors $(1,0)$ and $\omega := x_0+iy_0\in\overline{\mathcal{D}}$. In particular, with one generator $(1,0)$ being fixed, all lattices are parametrized by only one vector, i.e., the (other) generator $\omega=x_0+iy_0\in\overline{\mathcal{D}}$. Consequently, our problem now becomes:

What are the points $\omega \in \overline{\mathcal{D}}$ *whose corresponding lattices, i.e., those generated by* $(1,0)$ *and* ω*, are semi-stable*?

To answer this, set $\mathcal{D}_T := \{z \in \overline{\mathcal{D}} : y = \Im(z) \leq T\}$. Then by the above discussion, up to points on the boundary, the points in $\mathcal{D}_T$ are in one-to-one corresponding with rank two $\mathbb{Z}$-lattices (in $\mathbb{R}^2$) of volume one whose first Minkowski successive minimums λ_1 satisfying $\lambda_1^{-2} \leq T$, since $\lambda_1^{-2} = y_0 \leq T$. Write this condition in a better form, we have $\lambda_1(\Lambda) \geq T^{-1/2}$, or equivalently, $\deg(\Lambda) \leq \frac{1}{2}\log T$. Then what we have just siad may be restated in a more theoretical form as the following

Fact (VI)$_{\mathbb{Q}}$ (Grometric Truncation=Algebraic Truncation) *Up to a subset of measure zero, there is a natural one-to-one and onto morphism* $\mathcal{M}_{\mathbb{Q},2}^{\leq\frac{1}{2}\log T}[1] \simeq \mathcal{D}_T$, *where* $\mathcal{M}_{\mathbb{Q},2}^{\leq\frac{1}{2}\log T}[1]$ *denotes the moduli space of rank two* $\mathbb{Z}$*-lattices* Λ *of volume 1 whose sublattices of rank 1 all have degrees* $\leq \frac{1}{2}\log T$. *In particular,*

$$\mathcal{M}_{\mathbb{Q},2}^{\leq 0}[1] = \mathcal{M}_{\mathbb{Q},2}[1] \simeq \mathcal{D}_1.$$

2.5.2 Rank two $\mathcal{O}_K$-lattices: level two

We start with our discussion by citing a result of Tsukasa Hayashi [11].

Let Λ be a rank two $\mathcal{O}_K$-lattice of volume $N(\mathfrak{a}) \cdot \Delta_K$ with underlying projective module $\mathcal{O}_K \oplus \mathfrak{a}$. Recall that, by definition, Λ is semi-stable if for any rank one $\mathcal{O}_K$-sublattice Λ_1 of Λ, equipped with the induced metric, $\mathrm{Vol}(\Lambda_1)^2 \geq N(\mathfrak{a})\Delta_K$. To understand this condition, let us first understand the structure of rank one $\mathcal{O}_K$-sublattices Λ_1 of Λ.

By the discussion in §1.1, any rank one $\mathcal{O}_K$-submodule of Λ has the form $\mathfrak{c} \cdot \binom{x}{y}$ where $\mathfrak{c}$ is a fractional $\mathcal{O}_K$-ideal and $\mathfrak{c} \cdot \binom{x}{y} \in \mathcal{O}_K \oplus \mathfrak{a}$. Set $\mathfrak{b} = \mathcal{O}_K x + \mathfrak{a}^{-1} y$. Since $\mathfrak{c} \cdot x \in \mathcal{O}_K,\ \mathfrak{c}\, y \in \mathfrak{a}$, we have

$$\mathfrak{b} \cdot \mathfrak{c} \subset (\mathcal{O}_K x + \mathfrak{a}^{-1} y) \cdot \mathfrak{c} = \mathfrak{c} \cdot x + \mathfrak{a}^{-1}(\mathfrak{c} \cdot y) \subset \mathcal{O}_K + \mathfrak{a}^{-1} \cdot \mathfrak{a} = \mathcal{O}_K.$$

Therefore, $\mathfrak{c} \subset \mathfrak{b}^{-1}$. This then proves (1) of the following

Proposition. ([11]).

(1) *Any rank one sublattice of* $\Lambda = (\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda)$ *is contained in* $\mathfrak{b}^{-1}\binom{x}{y} \cap \Lambda$ *where* $\binom{x}{y} \in K^2 \backslash \{\binom{0}{0}\}$ *and* $\mathfrak{b} = \mathcal{O}_K x + \mathfrak{a}^{-1} y$;

(2) Λ *is semi-stable if and only if*

$$\prod_{\sigma \in S_\infty} \left\| \begin{pmatrix} x_\sigma \\ y_\sigma \end{pmatrix} \right\|_{\Lambda_\sigma}^2 \geq N(\mathfrak{a}\mathfrak{b}^2) = N(\mathcal{O}_K x + \mathfrak{a}^{-1} y) \cdot N(\mathcal{O}_K y + \mathfrak{a} x),$$

$$\times \forall \begin{pmatrix} x \\ y \end{pmatrix} \in K^2 \Big\backslash \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Proof. From (1), it suffices to check the semi-stable condition for all rank one sublattices Λ_1 induced from the submodules $\mathfrak{b}^{-1}\binom{x}{y}$, where $\binom{x}{y} \in K^2\backslash\{\binom{0}{0}$ with $\mathfrak{b} := \mathcal{O}_K x + \mathfrak{a}^{-1}y$. Now, by Arakelov–Riemann–Roch, $\mathrm{Vol}(\Lambda_1) = N(\mathfrak{c}) \cdot \Delta_K^{1/2} \cdot \prod_\sigma \left\|\binom{x_\sigma}{y_\sigma}\right\|_{\Lambda_\sigma}$. Therefore, the semi-stable condition becomes $\left((N(\mathfrak{b}^{-1})\Delta_K^{1/2}) \cdot \prod_{\sigma\in S_\infty} \left\|\binom{x_\sigma}{y_\sigma}\right\|\right)^2 \geq N(\mathfrak{a}) \cdot \Delta_K$. That is to say,

$$\prod_{\sigma\in S_\infty} \left\|\begin{pmatrix} x_\sigma \\ y_\sigma \end{pmatrix}\right\|_{\Lambda_\sigma}^2 \geq N(\mathfrak{a}\mathfrak{b}^2) = N(\mathfrak{a}(\mathcal{O}_K x + \mathfrak{a}^{-1}y) \cdot \mathfrak{b})$$

$$= N((\mathcal{O}_K y + \mathfrak{a}x)\mathfrak{b})$$

$$= N(\mathfrak{a}x + \mathcal{O}_K y) \cdot N(\mathcal{O}_K x + \mathfrak{a}^{-1}y).$$

□

2.5.3 Stability and distance to cusps

In this subsection, we expose an intrinsic relation in Geometric Arithmetic, which connects stability and distance to cusps in a very beautiful way.

Assume that $\Lambda = (\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda)$ is semi-stable. Then for any non-zero element $(x, y) \in K \oplus K$, set $\mathfrak{b}_0 := \mathcal{O}_K x + \mathfrak{a}^{-1}y$ so that $x \in \mathfrak{b}_0,\ y = \mathfrak{a}\mathfrak{b}_0$. Thus $\mathfrak{b}_0^{-1}x \subset \mathcal{O}_K$ and $\mathfrak{b}_0^{-1}y \subset \mathfrak{a}$ and $\mathfrak{b}_0^{-1}(x, y) \subset (\mathfrak{b}_0^{-1}x, \mathfrak{b}_0^{-1}y) \subset \mathcal{O}_K \oplus \mathfrak{a}$. Moreover, if P_1 is a projective $\mathcal{O}_K$-submodule of rank 1 in $\mathcal{O}_K \oplus \mathfrak{a}$, then $P_1 = \mathfrak{c}(x, y)$ with $\mathfrak{c}$ a fractional ideal and $(x, y) \in K \oplus K\backslash\{(0, 0)\}$. Since $\mathfrak{c}x \subset \mathcal{O}_K$ and $\mathfrak{c}y \subset \mathfrak{a}$, we have

$$\mathfrak{c} \cdot \mathfrak{b}_0 = \mathcal{O}_K \cdot \mathfrak{c}x + \mathfrak{a}^{-1} \cdot \mathfrak{c}y \subset \mathcal{O}_K \cdot \mathcal{O}_K + \mathfrak{a}^{-1} \cdot \mathfrak{a} = \mathcal{O}_K.$$

Hence $\mathfrak{c} \subset \mathfrak{b}_0^{-1}$. Consequently, $P_1 = \mathfrak{c}(x, y) \subset \mathfrak{b}_0^{-1}(x, y)$. Therefore,

(i) $\mathfrak{b}_0^{-1}(x, y)$ *is a projective* $\mathcal{O}_K$*-submodule of rank 1 in* $\mathcal{O}_K \oplus \mathfrak{a}$; and
(ii) *Any projective* $\mathcal{O}_K$*-submodule of rank 1 in* $\mathcal{O}_K \oplus \mathfrak{a}$ *is contained in* $\mathfrak{b}_0^{-1}(x, y)$.

Thus, the semi-stability condition becomes $(\mathrm{Vol}(\mathfrak{b}_0^{-1}(x, y), \rho_\Lambda))^2 \geq \mathrm{Vol}(\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda)$. That is, $\left(N(\mathfrak{b}_0)^{-2} \cdot (\Delta_K^{\frac{1}{2}})^2\right) \cdot \|(x, y)\|_{\rho_\Lambda}^2 \geq N(\mathfrak{a}) \cdot \Delta_K^{2\times\frac{1}{2}}$ or better

$$\|(x, y)\|_{\rho_\Lambda}^2 \geq N(\mathfrak{a}\mathfrak{b}_0^2). \tag{*}$$

On the other hand, for $g_\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\rho_\Lambda = \rho(g_\Lambda)$,

$$\|(x, y)\|_{\rho_\Lambda}^2 = \|(x, y)g_\Lambda\|^2 = \prod_{\sigma\in S_\infty} ||a_\sigma x_\sigma + c_\sigma y_\sigma|^2 + |b_\sigma x_\sigma + d_\sigma y_\sigma|^2|^{N_\sigma}$$

$$= N\left(\begin{pmatrix} * & * \\ x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathrm{ImJ}\right)^{-1}, \tag{**}$$

where $N_\sigma = 1$ resp. 2 if σ is real resp. complex, and $\mathrm{ImJ} := (i, \dots, i,\ j, \dots, j) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ with $i = \sqrt{-1} \in \mathcal{H}$ and $j = (0, 0, 1) \in \mathbb{H}$. (Recall that we have set $N(\tau) := N(\mathrm{ImJ}(\tau))$.) Indeed,

$$\begin{aligned}
\begin{pmatrix} * & * \\ x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{ImJ} &= \begin{pmatrix} * & * \\ ax + cy & bx + dy \end{pmatrix} \mathrm{ImJ} \\
&= \Bigg(\begin{pmatrix} * & * \\ a_{\sigma_1} x_{\sigma_1} + c_{\sigma_1} y_{\sigma_1} & b_{\sigma_1} x_{\sigma_1} + d_{\sigma_1} y_{\sigma_1} \end{pmatrix} i, \dots, \\
&\quad \times \begin{pmatrix} * & * \\ a_{\sigma_{r_1}} x_{\sigma_{r_1}} + c_{\sigma_{r_1}} y_{\sigma_{r_1}} & b_{\sigma_{r_1}} x_{\sigma_{r_1}} + d_{\sigma_{r_1}} y_{\sigma_{r_1}} \end{pmatrix} i, \\
&\quad \times \begin{pmatrix} * & * \\ a_{\tau_1} x_{\tau_1} + c_{\tau_1} y_{\tau_1} & b_{\tau_1} x_{\tau_1} + d_{\tau_1} y_{\tau_1} \end{pmatrix} j, \dots, \\
&\quad \times \begin{pmatrix} * & * \\ a_{\tau_{r_2}} x_{\tau_{r_2}} + c_{\tau_{r_2}} y_{\tau_{r_2}} & b_{\tau_{r_2}} x_{\tau_{r_2}} + d_{\tau_{r_2}} y_{\tau_{r_2}} \end{pmatrix} j \Bigg) \\
&\quad \times \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2},
\end{aligned}$$

where $\sigma_1, \dots, \sigma_{r_1}$ (resp. $\tau_1, \dots, \tau_{r_2}$) denote real places (resp. complex places) in S_∞. From here, to get (**), we use the following obvious calculations:

(a) For reals, if $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, for $z = X + Yi \in \mathcal{H}$ with $X, Y \in \mathbb{R}$, set $M(X + iY) = X^* + Y^* i$ with $X^*,\ Y^* \in \mathbb{R}$. Then $Y^* := \frac{Y}{(CX+D)^2 + C^2 Y^2}$. In particular, when applied to the local factor for real σ in (**), we have $C = a_\sigma x_\sigma + c_\sigma y_\sigma$, $D = b_\sigma x_\sigma + d_\sigma y_\sigma$ and $X = 0, Y = 1$. Therefore, the corresponding Y^* is simply

$$\begin{aligned}
&\frac{1}{((a_\sigma x_\sigma + b_\sigma y_\sigma) \cdot 0 + (b_\sigma x_\sigma + d_\sigma y_\sigma))^2 + (a_\sigma x_\sigma + c_\sigma y_\sigma)^2 \cdot 1^2} \\
&\quad = \frac{1}{(b_\sigma x_\sigma + d_\sigma y_\sigma)^2 + (a_\sigma x_\sigma + c_\sigma y_\sigma)^2};
\end{aligned}$$

(b) For complexes, if $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2, \mathbb{C})$, for $P = Z + Vj \in \mathbb{H}$ with $Z \in \mathbb{C}, V \in \mathbb{R}$, set $M(Z + Vj) = Z^* + V^* i$ with $Z^* \in \mathbb{C},\ V^* \in \mathbb{R}$. Then

$$V^* := \frac{V}{|CZ + D|^2 + |C|^2 V^2} = \frac{V}{\|CP + D\|^2}.$$

In particular, when applied to the local factor for complex τ in (**), we have $C = a_\tau x_\tau + c_\tau y_\tau$, $D = b_\tau x_\tau + d_\tau y_\tau$ and $Z = 0,\ V = 1$. Therefore, the corresponding $(V^*)^2$ is simply

$$\Big(\frac{1}{|(a_\tau x_\tau + c_\tau y_\tau)\cdot 0 + (b_\tau x_\tau + d_\tau y_\tau)|^2 + |a_\tau x_\tau + c_\tau y_\tau|^2 \cdot 1^2}\Big)^2$$
$$= \quad \Big(\frac{1}{|b_\tau x_\tau + d_\tau y_\tau|^2 + |a_\tau x_\tau + c_\tau y_\tau|^2}\Big)^2.$$

Consequently, the relation (**), together with (*), implies

(iii) *The lattice* $\Lambda = (\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda(g))$ *with* $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is semi-stable if and only if for any non-zero* $(x,y) \in K \oplus K$, $N\big(\left(\begin{smallmatrix} * & * \\ x & y \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\mathrm{ImJ}\big) \cdot N(\mathfrak{a}\mathfrak{b}_0^2) \leq 1$, *where* $\mathfrak{b}_0 := \mathcal{O}_K x + \mathfrak{a}^{-1} y$.

But, by definition, for the lattice $\Lambda = (\mathcal{O}_K \oplus \mathfrak{a}, \rho(g_\Lambda))$, the corresponding point $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is given by $g_\Lambda(\mathrm{ImJ})$. Hence, we have the following equivalent

(iii′) *The lattice* $\Lambda = \Big(\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda(g)\Big)$ *is semi-stable if and only if for any non-zero* $(x,y) \in K \oplus K$, $N\Big(\left(\begin{smallmatrix} * & * \\ x & y \end{smallmatrix}\right)\tau_\Lambda\Big) \cdot N(\mathfrak{a}\mathfrak{b}_0^2) \leq 1$, *where* $\mathfrak{b}_0 := \mathcal{O}_K x + \mathfrak{a}^{-1} y$.

Set now $x = -\beta$ and $y = \alpha$. Then $\mathfrak{b}_0 = \mathcal{O}_K \beta + \mathfrak{a}^{-1}\alpha$. In particular, $\beta \in \mathfrak{b}_0$ and $\alpha \in \mathfrak{a}\mathfrak{b}_0$. So if we define $\mathfrak{b} := \mathfrak{a}\mathfrak{b}_0$. Then $\alpha \in \mathfrak{b},\ \beta \in \mathfrak{a}^{-1}\mathfrak{b}$, and

$$\mathcal{O}_K\alpha + \mathfrak{a}\beta \subset \mathfrak{b} = \mathfrak{a}\mathfrak{b}_0 = \mathfrak{a}\cdot(\mathcal{O}_K\beta + \mathfrak{a}^{-1}\alpha) \subset \mathcal{O}_K \cdot \mathfrak{a}\beta + \mathfrak{a}^{-1}\cdot\mathfrak{a}\alpha$$
$$= \mathfrak{a}\beta + \mathcal{O}_K\alpha.$$

Therefore, $\mathfrak{b} = \mathcal{O}_K\alpha + \mathfrak{a}\beta$, and $\mathfrak{a}\mathfrak{b}_0^2 = \mathfrak{a}\cdot(\mathfrak{a}^{-1}\mathfrak{b})^2 = \mathfrak{a}^{-1}\mathfrak{b}^2$. Consequently, the semi-stability condition (iii′) becomes for any cusp $\eta = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \mathbb{P}^1(K)$,

$$\mu(\eta, \tau_\Lambda) = N\left(\begin{pmatrix} * & * \\ -\beta & \alpha \end{pmatrix}\tau_\Lambda\right) \cdot N(\mathfrak{a}^{-1}\mathfrak{b}^2) \leq 1.$$

Or better, in terms of *distance to cusp*,

$$d(\eta, \tau_\Lambda) := \frac{1}{\mu(\eta, \tau_\Lambda)} \geq 1.$$

In this way, we arrive at the following fundamental result, which exposes a beautiful intrinsic relation between stability and the distance to cusps.

Fact (VII) *The lattice* Λ *is semi-stable if and only if the distances of corresponding point* $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ *to all cusps are all bigger or equal to 1.*

Remark. One can never overestimate the importance of this relation. Being stable, lattices should be away from cusps. More generally, while the stability condition is defined in terms of sublattices, the relation above transforms these volumes inequalities in terms of distances to cusps. In a more theoretical term for higher rank lattices, the essence of this fact is that, sublattices and cusps, as two different aspects of parabolic subgroups, are naturally corresponding to each other: stability conditions for various sublattices are naturally related with these for generalized distances to all cusps.

2.5.4 Moduli space of rank two semi-stable $\mathcal{O}_K$-lattices

For a rank two $\mathcal{O}_K$-lattice Λ, denote by $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ the corresponding point. Then, from the Fact in the previous subsection, Λ is semi-stable if and only if for all cusps η, $d(\eta, \tau_\Lambda) := \frac{1}{\mu(\eta,\tau_\Lambda)}$ are bigger than or equal to 1. This then leads to the consideration of the following truncation of the fundamental domain $\mathcal{D}$ of $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$: For $T \geq 1$, denote by $\mathcal{D}_T := \{\tau \in \mathcal{D} : d(\eta, \tau_\Lambda) \geq T^{-1},\ \forall \text{cusp}\ \eta\}$.

$\mathcal{D}_T$ may be precisely described in terms of $\mathcal{D}$ and certain neighborhood of cusps.

Lemma. *For a cusp η, denote by $X_\eta(T) := \{\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2} : d(\eta, \tau) < T^{-1}\}$. Then for $T \geq 1$, $X_{\eta_1}(T) \cap X_{\eta_2}(T) \neq \emptyset \Leftrightarrow \eta_1 = \eta_2$.*

Proof. One direction is clear. Hence, it suffices to show that if $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ satisfies $d(\tau, \eta_1) < 1$ and $d(\tau, \eta_2) < 1$, then $\eta_1 = \eta_2$. For this, let $\eta_1 = \left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix}\right]$, $\eta_2 = \left[\begin{smallmatrix} \alpha_2 \\ \beta_2 \end{smallmatrix}\right]$ and $\mathfrak{b}_1 = \mathcal{O}_K\alpha_1 + \mathfrak{a}\beta_1,\ \mathfrak{b}_2 = \mathcal{O}_K\alpha_2 + \mathfrak{a}\beta_2$. Clearly,

$$
\begin{aligned}
&N\left(\begin{pmatrix} * & * \\ -\beta_1 & \alpha_1 \end{pmatrix}\tau\right) \cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) \\
&\quad = N\left(\begin{pmatrix} * & * \\ -\beta_1 & \alpha_1 \end{pmatrix} \cdot \left(\begin{pmatrix} \alpha_2 & \alpha_2^* \\ \beta_2 & \beta_2^* \end{pmatrix} \cdot \begin{pmatrix} \beta_2^* & -\alpha_2^* \\ -\beta_2 & \alpha_2 \end{pmatrix}\right) \cdot \tau\right) \cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) \\
&\quad = N\left(\left(\begin{pmatrix} * & * \\ -\beta_1 & \alpha_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & \alpha_2^* \\ \beta_2 & \beta_2^* \end{pmatrix}\right) \cdot \begin{pmatrix} * & * \\ -\beta_2 & \alpha_2 \end{pmatrix}\tau\right) \cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) \\
&\quad = N\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \cdot \begin{pmatrix} * & * \\ -\beta_2 & \alpha_2 \end{pmatrix}\tau\right) \cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) \\
&\quad = \frac{N\left(\begin{pmatrix} * & * \\ -\beta_2 & \alpha_2 \end{pmatrix}\tau\right)}{\left\| c \cdot \begin{pmatrix} * & * \\ -\beta_2 & \alpha_2 \end{pmatrix}\tau + d \right\|^2} \cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2),
\end{aligned}
$$

where $c = \alpha_1\beta_2 - \beta_1\alpha_2$.

We want to show that $c = 0$, since then $\eta_1 = \eta_2$. Thus to continue, let us recall that we have the following conditions ready to use:

$$N\left(\begin{pmatrix} * & * \\ -\beta_1 & \alpha_1 \end{pmatrix}\tau\right)\cdot N(a^{-1}\mathfrak{b}_1^2) > 1,\ N\left(\begin{pmatrix} * & * \\ -\beta_2 & \alpha_2 \end{pmatrix}\tau\right)\cdot N(a^{-1}\mathfrak{b}_2^2) > 1.$$

As such, set $\tau' = \left(\begin{smallmatrix} * & * \\ -\beta_2 & \alpha_2 \end{smallmatrix}\right)\tau$, then what we need to show becomes the following □

Lemma′. *Assume that (i)* $N(\tau')\cdot N(\mathfrak{a}^{-1}\mathfrak{b}_2^2) > 1$, *(ii)* $N(\tau')\cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) > \|c\tau' + d\|^2$, *and (iii)* $c = \alpha_1\beta_2 - \beta_1\alpha_2$ *with* $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2 \in \mathcal{O}_K$. *Then* $c = 0$.

Proof. First note that $\alpha_1 \in \mathfrak{b}_1,\ \beta_1 \in \mathfrak{a}^{-1}\mathfrak{b}_1$ and $\alpha_2 \in \mathfrak{b}_2,\ \beta_2 \in \mathfrak{a}^{-1}\mathfrak{b}_2$, we have $c \in \mathfrak{a}^{-1}\mathfrak{b}_1\mathfrak{b}_2$. Thus

$$N(c) \geq N(\mathfrak{a}^{-1}\mathfrak{b}_1\mathfrak{b}_2). \tag{*}$$

□

Sublemma'. $\|c\tau' + d\|^2 \geq N(c)^2 \cdot N(\tau')^2$.

Proof. Indeed, if suffices to prove this inequality locally. This is however an obvious calculation. Say for real σ, by definition, $\left\|c_\sigma z_\sigma + d_\sigma\right\|^2 = (c_\sigma x_\sigma + d_\sigma)^2 + c_\sigma^2 y_\sigma^2 \geq c_\sigma^2 y_\sigma^2$, done. (We leave the complex case to the reader.)

Thus by (ii), we have $N(\tau')\cdot N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) > N(c)^2\cdot N(\tau')^2$. That is to say, $N(\mathfrak{a}^{-1}\mathfrak{b}_1^2) > N(c)^2\cdot N(\tau')$. Consequently, by (i), we have $N(\mathfrak{a}^{-1}\mathfrak{b}_1^2)\cdot N(\mathfrak{a}^{-1}\mathfrak{b}_2^2) > N(c)^2$, or better $N(\mathfrak{a}^{-1}\mathfrak{b}_1\mathfrak{b}_2) > N(c)$, contradicting to (*). □

All in all, then we have exposed the following

Fact $(\mathrm{VI})_K$. *There is a natural identification between*

(a) *the moduli space of rank two semi-stable* $\mathcal{O}_K$*-lattices of volume* $N(\mathfrak{a})\Delta_K$ *with underlying projective module* $\mathcal{O}_K \oplus \mathfrak{a}$*; and*

(b) *the truncated compact domain* $\mathcal{D}_1$ *consisting of points in the fundamental domain* $\mathcal{D}$ *whose distances to all cusps are bigger than 1.*

In other words, the truncated compact domain $\mathcal{D}_1$ is obtained from the fundamental domain $\mathcal{D}$ of $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash\Big(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}\Big)$ by delecting the disjoint open neighborhoods $\cup \cup_{i=1}^h \mathcal{F}_i(1)$ associated to inequivalent cusps $\eta_1, \eta_2, \ldots, \eta_h$, where $\mathcal{F}_i(T)$ denotes the neighborhood of η_i consisting of $\tau \in \mathcal{D}$ whose distance to η_i is strictly less than T^{-1}.

For later use, we set also $\mathcal{D}_T := \mathcal{D}\backslash \cup \cup_{i=1}^h \mathcal{F}_i(T), T \geq 1$.

3. Epstein zeta function and its Fourier expansion

3.1 Epstein zeta function and Eisenstein series

We start with a relation between Epstein zeta function and Eisenstein series on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$.

Motivated by our study on high rank zeta functions for number fields in Chapter 1, for a fixed integer $r \geq 1$ and a fractional ideal $\mathfrak{a}$ of a number field K, let us define *the Epstein type zeta function* $\widehat{E}_{r,\mathfrak{a};\Lambda}(s)$ associated to an $\mathcal{O}_K$-lattice Λ with underlying projective module $P_{\mathfrak{a}} = \mathcal{O}_K^{(r-1)} \oplus \mathfrak{a}$ to be

$$\widehat{E}_{r,\mathfrak{a};\Lambda}(s) := \left(\pi^{-\frac{rs}{2}}\Gamma\left(\frac{rs}{2}\right)\right)^{r_1} (2\pi^{-rs}\Gamma(rs))^{r_2} \cdot (N(\mathfrak{a})\Delta_K^{\frac{r}{2}})^s \cdot \sum_{\mathbf{x}\in\mathcal{O}_K^{(r-1)}\oplus\mathfrak{a}/U_{K,r}^+,\mathbf{x}\neq(0,\ldots,0)} \frac{1}{\|\mathbf{x}\|_\Lambda^{rs}}$$

where $U_{K,r}^+ := \{\varepsilon^r : \varepsilon \in U_K,\ \varepsilon^r \in U_K^+\} = U_K^+ \cap U_K^r$. For example, note that in the case $r = 2$, $U_{K,2}^+ = U_K^2$, we have

$$\widehat{E}_{2,\mathfrak{a};\Lambda}(s) := (\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2} \cdot (N(\mathfrak{a})\Delta_K)^s \cdot \sum_{\mathbf{x}\in\mathcal{O}_K\oplus\mathfrak{a}/U_K^2,\mathbf{x}\neq(0,0)} \frac{1}{\|\mathbf{x}\|_\Lambda^{2s}}.$$

From now on, we will concentrate on this rank 2 case.

We want to relate the rank 2 Epstein zeta function defined in terms of lattices to an Eisenstein series defined over $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. This is based on the following simple but key observation, which serves as a bridge between lattices model and the upper half space model. (See also our discussion on stability and distance to cusps in Ch 2 above.)

Recall that, for any non-zero vector $(x, y) \in \mathcal{O}_K \oplus \mathfrak{a}$, the lattice norm of (x, y) associated with the lattice $\Lambda = (\mathcal{O}_K \oplus \mathfrak{a}, \rho_\Lambda(g))$ where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is given by

$$\begin{aligned}\|(x,y)\|_\Lambda^2 &= \left\|(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|^2 \\ &= \prod_{\sigma:\mathbb{R}}((a_\sigma x_\sigma + c_\sigma y_\sigma)^2 + (b_\sigma x_\sigma + d_\sigma y_\sigma)^2) \\ &\quad \cdot \prod_{\tau:\mathbb{C}}(|a_\tau x_\tau + c_\tau y_\tau|^2 + |b_\tau x_\tau + d_\tau y_\tau|^2)^2 \\ &= \left(\frac{N(g_\Lambda(\mathrm{ImJ}))}{\|x \cdot g_\Lambda(\mathrm{ImJ}) + y\|^2}\right)^{-1},\end{aligned}$$

where, by ImJ, we mean the point $\mathrm{ImJ} := (\overbrace{i, \ldots, i}^{r_1-\mathrm{times}}, \overbrace{j, \ldots, j}^{r_2-\mathrm{times}}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$. (Recall that we have set $N(\tau) := N(\mathrm{ImJ}(\tau))$.) Here for $X \in K$, we set $\|X\| := N(X) := \prod_{\sigma:\mathbb{R}} |X_\sigma| \cdot \prod_{\tau:\mathbb{C}} |X_\tau|^2$. Also change the action of units to the one induced from the diagonal action. Then,

$$\widehat{E}_{2,\mathfrak{a};\Lambda}(s) := (\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2} \cdot (N(\mathfrak{a})\Delta_K)^s \cdot \sum_{(x,y)\in\mathcal{O}_K\oplus\mathfrak{a}/U_K,(x,y)\neq(0,0)} \left(\frac{N(\mathrm{ImJ}(\tau_\Lambda))}{\|x\cdot\tau_\Lambda + y\|^2}\right)^s.$$

Set then for $\Re(s) > 1$,

$$\widehat{E}_{2,\mathfrak{a}}(\tau, s) :=(\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2} \cdot (N(\mathfrak{a})\Delta_K)^s \cdot \sum_{(x,y)\in\mathcal{O}_K\oplus\mathfrak{a}/U_K,(x,y)\neq(0,0)} \left(\frac{N(\mathrm{ImJ}(\tau_\Lambda))}{\|x\cdot\tau + y\|^2}\right)^s.$$

Then we have just completed the proof of the following

Lemma. *For a rank two $\mathcal{O}_K$-lattice $\Lambda = (\mathcal{O}_K \oplus a, \rho_\Lambda)$, denote by τ_Λ the corresponding point in the moduli space $SL(\mathcal{O}_K \oplus a)\backslash(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$. Then $\widehat{E}_{2,\mathfrak{a};\Lambda}(s) = \widehat{E}_{2,\mathfrak{a}}(\tau_\Lambda, s)$.*

3.2 Fourier expansions

For simplicity, introduce the standard Eisenstein series by setting

$$E_{2,\mathfrak{a}}(\tau, s) := \sum_{(x,y)\in\mathcal{O}_K\oplus\mathfrak{a}/U_K,(x,y)\neq(0,0)} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|x\cdot\tau + y\|^2}\right)^s, \qquad \Re(s) > 1.$$

Then the completed one becomes

$$\widehat{E}_{2,\mathfrak{a}}(\tau, s) = (\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2} \cdot (N(\mathfrak{a})\Delta_K)^s \cdot E_{2,\mathfrak{a}}(\tau, s). \quad (*)$$

Following the classics, see e.g., [7], with suitable generalizations, we in this subsection give an explicit expression of Fourier expansion for the Eisenstein series.

As before, for the cusp $\eta = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, choose a (normalized) matrix $A = \left(\begin{smallmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{smallmatrix}\right) \in SL(2, F)$ such that if $\mathfrak{b} = \mathcal{O}_K\alpha + \mathfrak{a}\beta$, then $\mathcal{O}_K\beta^* + \mathfrak{a}\alpha^* = \mathfrak{b}^{-1}$. Clearly, $A\infty = \eta$, and moreover, $A^{-1}\Gamma'_\eta A = \left\{\left(\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right) : \omega \in \mathfrak{a}\mathfrak{b}^{-2}\right\}$. Since $\widehat{E}_{2,\mathfrak{a}}(\tau, s)$, and hence $E_{2,\mathfrak{a}}(\tau, s)$, is $SL(\mathcal{O}_K \oplus \mathfrak{a})$-invariant, $E_{2,\mathfrak{a}}(\tau, s)$ is $\Gamma'_\eta \subset SL(\mathcal{O}_K \oplus \mathfrak{a})$-invariant. Therfeore $E_{2,\mathfrak{a}}(A\tau, s)$ is $\mathfrak{a}\mathfrak{b}^{-2}$-invariant, that is, $E_{2,\mathfrak{a}}(A\tau, s)$ is invariant under parallel transforms by elements of $\mathfrak{a}\mathfrak{b}^{-2}$.

As a direct consequence, we have the Fourier expansion $E_{2,\mathfrak{a}}(A\tau,s) = \sum_{\omega'\in(\mathfrak{a}\mathfrak{b}^{-2})^\vee} a_{\omega'}\ (\mathrm{ImJ}(\tau),s)\cdot e^{2\pi i\langle\omega',\mathrm{ReZ}(\tau)\rangle}$, where $(\mathfrak{a}\mathfrak{b}^{-2})^\vee$ denotes the dual lattice of $\mathfrak{a}\mathfrak{b}^{-2}$. Thus, if we use Q to denote a fundamental parallolgram of $\mathfrak{a}\mathfrak{b}^{-2}$ in $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$, then

$$a_{\omega'}(\mathrm{ImJ}(\tau),s) := \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})}\sum_{(c,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K,(c,d)\neq(0,0)} \times\int_Q\left(\frac{N(\mathrm{ImJ}(\tau))}{\|c\tau+d\|^2}\right)^s cdote^{-2\pi i\langle\omega',\mathrm{ReZ}(\tau)\rangle}\prod_{\sigma:\mathbb{R}}dx_\sigma \cdot\prod_{\tau:\mathbb{C}}dx_\tau dy_\tau.$$

(As such, we are using in fact the standard Lebesgue measure, rather than the canonical one. So the notation may cause a bit confusion. However, since the canonical metric and the Lebesgue one differ by a constant factor depending only on the field K, we, up to such a constant factor, may ignore the actual difference: As can be seen in §4 below, even this makes results in this section not as explicit as possible, it serves our purpose of understanding rank two zetas quite well.)

Now, let us compute the Fourier coefficients in more details. For this, we break the summation about $a_{\omega'}$ into two cases according to whether $c=0$ or not.

1) *Case when* $c=0$. Then the contribution becomes

$$\frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})}\sum_{(0,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K,d\neq0}\int_Q\left(\frac{N(\mathrm{ImJ}(\tau))}{\|d\|^2}\right)^s \cdot e^{-2\pi i\langle\omega',\mathrm{ReZ}(\tau)\rangle}\prod_{\sigma:\mathbb{R}}dx_\sigma\cdot\prod_{\tau:\mathbb{C}}dx_\tau dy_\tau$$
$$=\frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})}\sum_{(0,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K,d\neq0}\left(\frac{N(\mathrm{ImJ}(\tau))}{\|d\|^2}\right)^s \times\int_Q e^{-2\pi i\langle\omega',\mathrm{ReZ}(\tau)\rangle}\prod_{\sigma:\mathbb{R}}dx_\sigma\cdot\prod_{\tau:\mathbb{C}}dx_\tau dy_\tau.$$

So according to whether $\omega'=0$ or not, this case may further be classified into two subcases.

1. a) *Subcase when* $\omega'\neq0$. Then, $\int_Q e^{-2\pi i\langle\omega',\mathrm{ReZ}(\tau)\rangle}\prod_{\sigma:\mathbb{R}}dx_\sigma\cdot\prod_{\tau:\mathbb{C}}dx_\tau dy_\tau=0$.

1. b) *Subcase when* $\omega' = 0$. Then $\int_Q \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau = \mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})$. Accordingly,

$$a_0(\mathrm{ImJ}(\tau), s) = \sum_{(0,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K, d\neq 0} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|d\|^2}\right)^s$$

$$= \left(\sum_{(0,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K, d\neq 0} N(d)^{-2s}\right) \cdot N(\mathrm{ImJ}(\tau))^s.$$

To go further, let us look at the summation $\sum_{(0,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K, d\neq 0} N(d)^{-2s}$ more carefully.

By definition, $(\mathcal{O}_K \oplus \mathfrak{a})A = (\mathcal{O}_K \oplus \mathfrak{a})\left(\begin{smallmatrix}\alpha & \alpha^* \\ \beta & \beta^*\end{smallmatrix}\right)$ with $\left(\begin{smallmatrix}\alpha & \alpha^* \\ \beta & \beta^*\end{smallmatrix}\right) \in SL(2,F)$ such that if $\mathfrak{b} = \mathcal{O}_K\alpha + \mathfrak{a}\beta$, then $\mathcal{O}_K\beta^* + \mathfrak{a}\alpha^* = \mathfrak{b}^{-1}$.

Claim. $(\mathcal{O}_K \oplus \mathfrak{a})A/U_K = (\mathcal{O}_K\alpha + \mathfrak{a}\beta, \mathcal{O}_K\alpha^* + \alpha\beta^*)/U_K = (\mathfrak{b} \oplus \mathfrak{a}\mathfrak{b}^{-1})/U_K$.

Indeed, by defintion $\mathcal{O}_K\alpha + \mathfrak{a}\beta = \mathfrak{b}$. So it suffices to prove that $\mathcal{O}_K\alpha^* + \mathfrak{a}\beta^* = \mathfrak{a}\mathfrak{b}^{-1}$. Clearly $\alpha^* \in \mathfrak{a}\mathfrak{b}^{-1}$, $\beta^* \in \mathfrak{b}^{-1}$ (as already used several times), so $\mathcal{O}_K\alpha^* + \alpha\beta^* \subset \mathfrak{a}\mathfrak{b}^{-1}$. On the other hand, as showed before $\mathfrak{b}^{-1} = \mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\beta^*$ so $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{a} \cdot (\mathcal{O}_K\beta^* + \mathfrak{a}^{-1}\beta^*) \subset \mathfrak{a}\beta^* + \mathcal{O}_K\alpha^*$.

As such, then the corresponding summation in the coeffcient a_0 becomes the one over $(\mathfrak{a}\mathfrak{b}^{-1}\backslash\{0\})/U_K$. Now we use the following

Lemma. *For a fractional* $\mathcal{O}_K$ *ideal* $\mathfrak{a}$, *denote by* $\mathfrak{R}$ *the ideal class associated with* $\mathfrak{a}^{-1}$.

(1) *There is a natural bijection*

$$\begin{array}{ccc}(\mathfrak{a}\backslash\{0\})/U_K & \rightarrow & \{\mathfrak{b} \in [\mathfrak{a}^{-1}] = \mathfrak{R} : \mathfrak{b} \textit{ integral } \mathcal{O}_K - \mathrm{ideal}\} \\ \overline{a} & \mapsto & \mathfrak{b} := a\mathfrak{a}^{-1}\end{array}.$$

(2) $\zeta(\mathfrak{R}, s) := \sum_{\mathfrak{b}\in\mathfrak{R}:\mathfrak{b} \textit{ integral } \mathcal{O}_K-\mathrm{ideal}} N(\mathfrak{b})^{-s} = N(\mathfrak{a})^s \cdot \Big(\sum_{a\in(\mathfrak{a}\backslash\{0\})/U_K} N(a)^{-s}\Big)$.

Proof. All are standard. For example, (1) may be found in [19], while (2) is a direct consequence of (1).

Therefore, we arrive at the following □

Proposition. *For the subcases at hand, the corresponding Fourier coefficient is given by* $a_0(\mathrm{ImJ}(\tau), s) = (N(\mathfrak{a}^{-1}\mathfrak{b})^{2s} \cdot \zeta([\mathfrak{a}^{-1}\mathfrak{b}], 2s)) \cdot N(\mathrm{ImJ}(\tau))^s$.

2) *Case when $c \neq 0$*. In this case,

$$a_{\omega'}(\mathrm{ImJ}(\tau), s) := \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{(c,d)\in(\mathcal{O}_K\oplus\mathfrak{a})A/U_K, c\neq 0} \times \int_Q \left(\frac{N(\mathrm{ImJ}(\tau))}{\|c\tau+d\|^2}\right)^s \cdot e^{-2\pi i\langle\omega', \mathrm{ReZ}(\tau)\rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau.$$

To compute this, consider the coset of $A^{-1}\Gamma'_\eta A$ among $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$.

Claim. *For $(c, d) \in (\mathcal{O}_K \oplus \mathfrak{a})A/U_K$, $c \neq 0$ and $\omega \in \mathfrak{a}\mathfrak{b}^{-2}$, we have $(c, c\omega + d) \in (\mathcal{O}_K \oplus \mathfrak{a})A/U_K$, $c \neq 0$.*

It suffices to deal the component of d and $c\omega + d$. Note that $A = \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in SL(2, F)$ with $\alpha \in \mathfrak{b}$, $\beta \in \mathfrak{a}^{-1}\mathfrak{b}$, $\alpha^* \in \mathfrak{a}\mathfrak{b}^{-1}$, $\beta^* \in \mathfrak{b}^{-1}$, we have

$$c \in \mathcal{O}_K \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{a}^{-1}\mathfrak{b} = \mathfrak{b} \qquad \text{and} \qquad d \in \mathcal{O}_K \cdot \mathfrak{a}\mathfrak{b}^{-1} + \mathfrak{a} \cdot \mathfrak{b}^{-1} = \mathfrak{a}\,\mathfrak{b}^{-1}.$$

So, we should show that with $c \in \mathfrak{b}$, $d \in \mathfrak{a}\,\mathfrak{b}^{-1}$ and $\omega \in \mathfrak{a}\mathfrak{b}^{-2}$, we have $c\omega + d \in \mathfrak{a}\,\mathfrak{b}^{-1}$. But this is clear since $c\omega + d \in \mathfrak{b} \cdot \mathfrak{a}\mathfrak{b}^{-2} + \mathfrak{a}\,\mathfrak{b}^{-1} = \mathfrak{a}\,\mathfrak{b}^{-1}$. Done.

Now since $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ c & c\omega+d \end{smallmatrix}\right)$ with $(c, d) \in (\mathcal{O}_K \oplus \mathfrak{a})A/U_K$, $c \neq 0$ and $\omega \in \mathfrak{a}\mathfrak{b}^{-2}$. Consequently, if we let $\mathcal{R}$ to be a system of representatives of $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ modulo the right action of $A^{-1}\Gamma'_\eta A$, or better, to be a system of representatives of (c, d) modulo the relation $(c, d) \sim (c, c\omega+d)$ with $(c, d) \in (\mathcal{O}_K \oplus \mathfrak{a})A/U_K$, $c \neq 0$ and $\omega \in \mathfrak{a}\mathfrak{b}^{-2}$, then for the case at hand, note that for $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, $\mathrm{ImJ}(\tau + \omega) = \mathrm{ImJ}(\tau)$, we have

$$\begin{aligned} & a_{\omega'}(\mathrm{ImJ}(\tau), s) \\ &= \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\mathcal{R}} \int_{\omega\in\mathfrak{a}\mathfrak{b}^{-2}} \int_Q \left(\frac{N(\mathrm{ImJ}(\tau))}{\|c(\tau+\omega)+d\|^2}\right)^s \\ &\quad \cdot e^{-2\pi i\langle\omega', \mathrm{ReZ}(\tau)\rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau \\ &= \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\mathcal{R}} \int_{\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|c\tau+d\|^2}\right)^s \\ &\quad \cdot e^{-2\pi i\langle\omega', \mathrm{ReZ}(\tau)\rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau \end{aligned}$$

$$= \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathcal{R}} \frac{1}{N(c)^{2s}} \int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|\tau + \frac{d}{c}\|^2} \right)^s$$

$$\cdot e^{-2\pi i \langle \omega', \mathrm{ReZ}(\tau) + \frac{d}{c} - \frac{d}{c} \rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau$$

$$= \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathcal{R}} \frac{e^{2\pi i \langle \omega', \frac{d}{c} \rangle}}{N(c)^{2s}} \int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|\tau\|^2} \right)^s$$

$$\cdot e^{-2\pi i \langle \omega', \mathrm{ReZ}(\tau) \rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau$$

2.a) *Subcase when* $\omega' = 0$. Then

$$a_0(\mathrm{ImJ}(\tau), s) = \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathcal{R}} \frac{1}{N(c)^{2s}}$$

$$\cdot \int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|\tau\|^2} \right)^s \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau.$$

So according to whether $\sigma : \mathbb{R}$ or $\tau : \mathbb{C}$, we have to compute the following integrations:

2.a.i) *For reals*,

$$\int_{\mathbb{R}} \left(\frac{y}{x^2 + y^2} \right)^s dx = \frac{1}{y^s} \int_{\mathbb{R}} \left(\frac{1}{\left(\frac{x}{y}\right)^2 + 1} \right)^s d\frac{x}{y} \cdot y = y^{1-s} \int_{\mathbb{R}} \frac{dt}{(1+t^2)^s}$$

$$= y^{1-s} \cdot \pi^{\frac{1}{2}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)};$$

2.a.ii) *For complexes*,

$$\int_{\mathbb{C}} \left(\frac{r}{|z|^2 + r^2} \right)^{2s} dx\, dy = r^{2-2s} \int_{\mathbb{C}} \frac{dx\, dy}{(1 + |z|^2)^{2s}} = r^{2-2s} \cdot \frac{\pi}{2s - 1};$$

2.b) *Subcase when* $\omega' \neq 0$. Then

$$a_{\omega'}(\mathrm{ImJ}(\tau), s) = \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathcal{R}} \frac{e^{-2\pi i \langle \omega', \frac{d}{c} \rangle}}{N(c)^{2s}}$$
$$\times \int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \left(\frac{N(\mathrm{ImJ}(\tau))}{\|\tau\|^2} \right)^s$$
$$\cdot\, e^{-2\pi i \langle \omega', \mathrm{ReZ}(\tau) \rangle} \prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dx_\tau dy_\tau.$$

So according to whether $\sigma : \mathbb{R}$ or $\tau : \mathbb{C}$, we have to compute the following integrations:
2.b.i) *For reals*,

$$\int_{\mathbb{R}} \left(\frac{y}{x^2 + y^2} \right)^s e^{-2\pi i |\omega'| \cdot x} dx = \frac{1}{y^s} \int_{\mathbb{R}} \left(\frac{1}{(\frac{x}{y})^2 + 1} \right)^s e^{-2\pi i |\omega'| \cdot \frac{x}{y} y} d\frac{x}{y} \cdot y$$
$$= y^{1-s} \int_{\mathbb{R}} \frac{1}{(1+t^2)^s} e^{-2\pi i |\omega'| y t} dt = y^{1-s} \cdot$$
$$\times \left(2\pi^s |\omega'|^{s-\frac{1}{2}} \cdot y^{s-\frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(2\pi|\omega'| y) \right)$$
$$= 2\pi^s |\omega'|^{s-\frac{1}{2}} \cdot y^{\frac{1}{2}} \cdot \frac{1}{\Gamma(s)} \cdot K_{s-\frac{1}{2}}(2\pi|\omega'| y);$$

2.b.ii) *For complexes*,

$$\int_{\mathbb{C}} \left(\frac{r}{|z|^2 + r^2} \right)^{2s} e^{-2\pi i |\omega'| \cdot x} dx\, dy$$
$$= \int_{\mathbb{R}^2} \left(\frac{r}{x^2 + y^2 + r^2} \right)^{2s} e^{-2\pi i |\omega'| \cdot x} dx\, dy$$
$$= \int_{\mathbb{R}^2} \left(\frac{1}{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + 1} \right)^{2s} r^{-2s} e^{-2\pi i |\omega'| \cdot \frac{x}{r} \cdot r} d\frac{x}{r}\, d\frac{y}{r} \cdot r^2$$
$$= r^{2-2s} \int_{\mathbb{R}^2} \frac{e^{-2\pi i |\omega'| \cdot x \cdot r}}{(x^2 + y^2 + 1)^{2s}} dx\, dy$$
$$= r^{2-2s} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{dy}{(y^2 + x^2 + 1)^{2s}} \cdot e^{-2\pi i |\omega'| r x} dy\, dx$$

$$= r^{2-2s}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{\left(\left(\frac{y}{\sqrt{x^2+1}}\right)^2+1\right)^{2s}}d\frac{y}{\sqrt{x^2+1}}\cdot\sqrt{x^2+1}\cdot\frac{1}{(x^2+1)^{2s}}\right)$$

$$\cdot e^{-2\pi i|\omega'|rx}dx$$

$$= r^{2-2s}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{dt}{(1+t^2)^{2s}}\right)\cdot\frac{e^{-2\pi i|\omega'|rx}}{(x^2+1)^{2s-\frac{1}{2}}}dx$$

$$= r^{2-2s}\cdot\left(\pi^{\frac{1}{2}}\cdot\frac{\Gamma\left(2s-\frac{1}{2}\right)}{\Gamma(2s)}\right)\cdot\int_{\mathbb{R}}\frac{e^{-2\pi i|\omega'|rx}}{(x^2+1)^{2s-\frac{1}{2}}}dx$$

$$= r^{2-2s}\cdot\left(\pi^{\frac{1}{2}}\cdot\frac{\Gamma\left(2s-\frac{1}{2}\right)}{\Gamma(2s)}\right)$$

$$\cdot\left(2\pi^{2s-\frac{1}{2}}|\omega'|^{2s-1}r^{(2s-1)}\frac{1}{\Gamma\left(2s-\frac{1}{2}\right)}K_{2s-1}(2\pi|\omega'|r)\right)$$

$$= \frac{2\pi^{2s}|\omega'|^{2s-1}}{\Gamma(2s)}rK_{2s-1}(2\pi|\omega'|r)$$

by using the calculation for reals. Or more directly,

$$\int_{\mathbb{C}}\left(\frac{r}{|z|^2+r^2}\right)^{2s}e^{-2\pi i|\omega'|\cdot x}dx\,dy$$

$$= r^{-2s}\int_{\mathbb{C}}\left(\frac{1}{\left(\frac{|z|}{r}\right)^2+1}\right)^{2s}de^{-2\pi i|\omega'|\cdot\frac{x}{r}\cdot r}\frac{x}{r}\cdot rd\frac{y}{r}\cdot r$$

$$= r^{2-2s}\int_{\mathbb{C}}\frac{1}{(1+|z|^2)^{2s}}e^{-2\pi i|\omega'|\cdot x\cdot r}\cdot dx\,dy$$

$$= r^{2-2s}\cdot\frac{2\pi^{2s}|\omega'|^{2s-1}}{\Gamma(2s)}r^{2s-1}K_{2s-1}(2\pi|\omega'|r)$$

$$= \frac{2\pi^{2s}|\omega'|^{2s-1}}{\Gamma(2s)}rK_{2s-1}(2\pi|\omega'|r).$$

All in all, we have then obtain the following

Theorem . *With the same notation as above, we have the following Fourier expansion for the Eisenstein series*

$$\begin{aligned}
E_{2,\mathfrak{a}}(A\tau,s) = {} & \zeta([\mathfrak{a}^{-1}\mathfrak{b}],2s)\cdot N(\mathfrak{a}\mathfrak{b}^{-1})^{-2s}\cdot N(\mathrm{ImJ}(\tau))^s \\
& + \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & c\omega+d \end{smallmatrix}\right)\in\mathcal{R}} \frac{1}{N(c)^{2s}} \\
& \cdot \left(\pi^{\frac{1}{2}}\right)^{r_1}\cdot\left(\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{r_1}\cdot\left(\frac{\pi}{2s-1}\right)^{r_2}\cdot N(\mathrm{ImJ}(\tau))^{1-s} \\
& + \frac{1}{\mathrm{Vol}(\mathfrak{a}\mathfrak{b}^{-2})} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\in\mathcal{R}} \frac{e^{2\pi i\langle\omega',\frac{d}{c}\rangle}}{N(c)^{2s}}\cdot N(\mathrm{ImJ}(\tau))^{\frac{1}{2}}\cdot N(\omega')^{s-\frac{1}{2}} \\
& \times\left(\frac{2\pi^s}{\Gamma(s)}\right)^{r_1}\prod_{\sigma:\mathbb{R}} K_{s-\frac{1}{2}}(2\pi|\omega'|_\sigma y_\sigma)\cdot\left(\frac{2\pi^{2s}|\omega'|^{2s-1}}{\Gamma(2s)}\right)^{r_2} \\
& \cdot\prod_{\tau:\mathbb{C}} K_{2s-1}(2\pi|\omega'|_\tau r_\tau).
\end{aligned}$$

4. Explicit formula for rank two zetas: Rankin-Selberg & Zagier method

4.1 The rank two zeta function for $\mathbb{Q}$

Recall that if we set $\mathcal{D}_T := \{x\in\mathcal{D} : y=\Im(z)\leq T\}$, the points in $\mathcal{D}_T$ are in one-to-one corresponding to rank two $\mathbb{Z}$-lattices (in $\mathbb{R}^2$) of volume one whose first Minkowski successive minimums λ_1 satisfying $\lambda_1(\Lambda)\geq T^{-1/2}$. Thus if we set $\mathcal{M}_{\mathbb{Q},2}^{\leq\frac{1}{2}\log T}[1]$ be the moduli space of rank two $\mathbb{Z}$-lattices Λ of volume 1 (over $\mathbb{Q}$) whose sublattices of rank one have degree $\leq\frac{1}{2}\log T$, then up to a measure zero subset, *there is a natural one-to-one and onto morphism* $\mathcal{M}_{\mathbb{Q},2}^{\leq\frac{1}{2}\log T}[1]\simeq\mathcal{D}_T$. In particular, the corresponding moduli space of semi-stable lattices is given by $\mathcal{M}_{\mathbb{Q},2}^{\leq 0}[1]=\mathcal{M}_{\mathbb{Q},2}[1]\simeq\mathcal{D}_1$. Moreover, motivated by our definition of zeta functions, we introduce a (generalized) rank two zeta function $\xi_{\mathbb{Q},2}^T(s)$ by setting

$$\xi_{\mathbb{Q},2}^T(s) := \int_{\mathcal{D}_T}\widehat{E}(z,s)\,\frac{dx\wedge dy}{y^2},\qquad \Re(s)>1.$$

Fact (VIII)$_{\mathbb{Q}}$ *For the generalized zeta function $\xi_{\mathbb{Q},2}^T(s)$,*

$$\xi_{\mathbb{Q},2}^T(s) = \frac{\xi(2s)}{s-1} \cdot T^{s-1} - \frac{\xi(2s-1)}{s} \cdot T^{-s}.$$

In particular, the rank two zeta function $\xi_{\mathbb{Q},2}(s)$ is given by

$$\xi_{\mathbb{Q},2}(s) = \frac{\xi(2s)}{s-1} - \frac{\xi(2s-1)}{s}, \qquad \Re(s) > 1.$$

Proof. This is a direct consequence of [32]. Indeed, it is well known that the Fourier expansion of $\widehat{E}(z,s)$ is given by

$$\widehat{E}(z,s) = \xi(2s)y^s + \xi(2s-1)y^{1-s} + \mathrm{non-constant\ term}.$$

Hence we have

$$\begin{aligned}\xi_{\mathbb{Q},2}^T(s) &= \int_0^T (\xi(2s)y^s)\frac{dy}{y^2} - \int_T^\infty (\xi(2s-1)y^{1-s})\frac{dy}{y^2} \\ &= \frac{\xi(2s)}{s-1} \cdot T^{s-1} - \frac{\xi(2s-1)}{s} \cdot T^{-s}.\end{aligned}$$

□

Remarks.

(1) Even though originally $T \geq 1$, we may extend it as a function of complex variable T in terms of the right hand side. Denote this resulting function also by $\xi_{\mathbb{Q},2}^T(s)$: for $\Re(s) > 1$, if T is real and $T \geq 1$, then $\xi_{\mathbb{Q},2}^T(s)$ is the integration of $\widehat{E}(z,s)$ over the domain $\mathcal{D}_T$. Based on this, even when T is real and $0 < T \leq 1$, we have a geometric interpretation for $\xi_{\mathbb{Q},2}^T(s)$: it is simply $(\int_{D_{1,T}} - \int_{D_{-1,T}})\widehat{E}(z,s) \cdot \frac{dx\wedge dy}{y^2}$, where $D_{1,T} := \mathcal{D} \cap \{z = x + iy : y \leq T, |x| \leq \frac{1}{2}\}$ and $D_{-1,T} := \{z \in \mathcal{H} : |z| \leq 1\} \cap \{z = x + iy : y \geq T, |x| \leq \frac{1}{2}\}$;

(2) By taking the residue at $s = 1$, $(\mathrm{Res}_{s=1}\widehat{E}(z,s)) \cdot \mathrm{Vol}(\mathcal{D}_1) = \xi(2) - \mathrm{Res}_{s=1}\xi(2s-1)$;

(3) We see, in particular, for half positive integers $n \geq \frac{3}{2}$,

$$((n-1)n) \cdot \xi_{\mathbb{Q},2}(n) = n \cdot \xi(2n) - (n-1) \cdot \xi(2n-1).$$

So special values of the Riemann zeta at two successive integers are related naturally via the special values of rank two zeta function. This clearly is a fact which should be taken seriously. In particular, in view of

Remark (1) above, we suggest the reader to see what happens for small n's by writting out rank two zeta in terms of the integrations for the terms defining Eisenstein series. With this, it is very likely that the reader is convinced that, when talking about specail values of $\xi(s)$ at odd integers, it is better to distinguish the values at $4\mathbb{Z}_{>0}-1$ from these at $4\mathbb{Z}_{>0}+1$.

4.2 Rankin-Selberg & Zagier method

In this section, $T \geq 1$ is assumed to be a positive real number.

Now let us compute the integration $\iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau,s)d\mu(\tau)$. Here $\mathcal{D}_T$ is the compact part obtained from the fundamental domain $\mathcal{D}$ for $SL(\mathcal{O}_K \oplus \mathfrak{a})\backslash(\mathcal{H}^{r_1} \times \mathbb{H}^{r_2})$ by cutting off the cusp neighbouhoods defined by the conditions that the distance to cusps is less than T^{-1}. (Recall that, as such, $\mathcal{D}_1$ is simply the part corresponding to semi-stable lattices.) We use the Rankin-Selberg & Zagier method, but in its simplest form as a generalization of the one stated in the previous section.

For doing so, let us first consider the integration $\iiint_{\mathcal{D}_T}(\Delta_K\, \widehat{E}_{2,\mathfrak{a}}(\tau,s))\, d\mu(\tau)$ where $\Delta_K := \sum_{\sigma:\mathbb{R}} \Delta_\sigma + \sum_{\tau:\mathbb{C}} \Delta_\tau$ with $\Delta_\sigma := y_\sigma^2\big(\frac{\partial^2}{\partial x_\sigma^2} + \frac{\partial^2}{\partial y_\sigma^2}\big)$ and $\Delta_\tau := r_\tau^2\big(\frac{\partial^2}{\partial x_\tau^2} + \frac{\partial^2}{\partial y_\tau^2} + \frac{\partial^2}{\partial r_\tau^2}\big) - r\frac{\partial}{\partial r_\tau}$. (For the time being, by an abuse of notation, we use Δ_K to denote the hyperbolic Laplace operator for the space $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, not the absolute value of the discriminant of K which accordingly is changed to D_K.)

Note that $\Delta_\sigma(y_\sigma^s) = s(s-1)\cdot y_\sigma^s$, while $\Delta_\tau(r_\tau^{2s}) = 2s(2s-2)\cdot r_\tau^{2s}$ by the SL-invariance of the metrics, we conclude hence that

$$\begin{aligned}\Delta_K(\widehat{E}_{2,\mathfrak{a}}(\tau,s)) = (r_1\cdot(s(s-1)) + r_2\cdot(2s(2s-2)))\\ \cdot\,\widehat{E}_{2,\mathfrak{a}}(\tau,s), \qquad \Re(s)>1.\end{aligned}$$

Hence $\iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau,s)\, d\mu(\tau) = \frac{r_1+4r_2}{s(s-1)} \iiint_{\mathcal{D}_T} \Delta_K \widehat{E}_{2,\mathfrak{a}}(\tau,s)\, d\mu(\tau)$.

On the other hand, using Stokes' Formula, we have

$$\begin{aligned}&\iiint_{\mathcal{D}_T} \Delta_K\widehat{E}_{2,\mathfrak{a}}(\tau,s)d\mu(\tau)\\ &= \iiint_{\mathcal{D}_T} (\Delta_K\widehat{E}_{2,\mathfrak{a}}(\tau,s))\cdot 1 d\mu(\tau) - \iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau,s)\cdot(\Delta_K 1)\, d\mu(\tau)\\ &= \iiint_{\mathcal{D}_T} ((\Delta_K\widehat{E}_{2,\mathfrak{a}}(\tau,s))\cdot 1 - \widehat{E}_{2,\mathfrak{a}}(\tau,s)\cdot(\Delta_K 1))\, d\mu(\tau)\end{aligned}$$

$$= \iint_{\partial D(T)} \left(\frac{\partial \widehat{E}_{2,\mathfrak{a}}(\tau, s)}{\partial \nu} \cdot 1 - \widehat{E}_{2,\mathfrak{a}}(\tau, s) \cdot \frac{\partial 1}{\partial \nu} \right) d\mu$$

$$= \iint_{\partial D(T)} \frac{\partial \widehat{E}_{2,\mathfrak{a}}(\tau, s)}{\partial \nu} d\mu$$

with $\frac{\partial}{\partial \nu}$ the outer normal derivative and $d\mu$ the volume element of the boundary $\partial \mathcal{D}_T$.

To calculate this latest integration, we start with a trick initially used by Siegel to make the following change of variables at cusps. (See e.g., [6].)

Two directions have to be studied: the ReZ direction for x_σ resp. $z_\tau = x_\tau + iy_\tau$, and the ImJ directions for y_σ resp. for $r_\tau =: v_\tau$ when σ is real resp. τ is complex. As used in the discussion for fundamental domains, the change with respect to the ReZ direction is simpler, while the change with respect to the ImJ direction is a bit complicated.

Indeed, for the ImJ direction, recall that here all components are positive. In particular, $(y_{\sigma_1}, \dots, y_{\sigma_{r_1}}, v_{\tau_1}, \dots, v_{\tau_{r_2}}) \in \mathbb{R}_+^{r_1+r_2}$ resulting from the ImJ direction of a point $(z_1, \dots, z_{r_1}, P_1, \dots, P_{r_2})$ in $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ admits a natural norm

$$\begin{aligned} N(y_{\sigma_1}, \dots, y_{\sigma_{r_1}}, v_{\tau_1}, \dots, v_{\tau_{r_2}}) &= (y_{\sigma_1} \cdot \dots \cdot y_{\sigma_{r_1}}) \cdot (v_{\tau_1} \cdot \dots \cdot v_{\tau_{r_2}})^2 \\ &= \prod_{\sigma:\,\mathbb{R}} y_\sigma \cdot \prod_{\tau:\,\mathbb{C}} v_\tau^2. \end{aligned}$$

As such, the key here is that we need to find a variable change for the ImJ directions so that

(a) the outer normal direction will be seen more clearly; and
(b) the fundamental domain for the stablizer group of cusps can be written in a very simple way.

Our generalized version of Siegel's change of variables in the discussion of fundamental domains does exactly this. It is carried out by replacing the original variables $y_{\sigma_1}, \dots, y_{\sigma_{r_1}}, v_{\tau_1}, \dots, v_{\tau_{r_2}}$ with the new variables $Y_0, Y_1, \dots, Y_{r_1+r_2-1}$. To give a precise definition, let $\varepsilon_1, \dots, \varepsilon_{r_1+r_2-1}$ be a generator of the unit group U_K (modulo the torsion). Then by Dirichlet's Unit Theorem, the matrix $\begin{pmatrix} 1 & \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_{r_1+r_2-1}^{(1)}| \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \log|\varepsilon_1^{(r_1+r_2)}| & \cdots & \log|\varepsilon_{r_1+r_2-1}^{(r_1+r_2)}| \end{pmatrix}$ is invertible. Set $(e_j^{(i)})_{i=0,j=1}^{r_1+r_2-1,r_1+r_2}$ be its inverse. Then by definition and an obvious calculation,

i) the entries of the first row is given by $e_j^{(0)} = \frac{1}{r_1+r_2}$, $j = 1, 2, \dots, r_1+r_2$;

ii) $\sum_{j=1}^{r_1+r_2} e_j^{(i)} = 0, i = 1, \dots, r_1 + r_2 - 1$; and

iii) $\sum_{j=1}^{r_1+r_2} e_j^{(i)} \log |\varepsilon_k^{(j)}| = \delta_{ik}, i, k = 1, \dots, r_1 + r_2 - 1$.

In particular, $(e_j^{(i)}) = \begin{pmatrix} \frac{1}{r_1+r_2} & \frac{1}{r_1+r_2} & \cdots & \frac{1}{r_1+r_2} \\ e_1^{(1)} & e_2^{(1)} & \cdots & e_{r_1+r_2}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ e_1^{(r_1+r_2-1)} & e_2^{(r_1+r_2-1)} & \cdots & e_{r_1+r_2}^{(r_1+r_2-1)} \end{pmatrix}$.

With this, make a change of variables by

$$Y_0 := N(y_{\sigma_1}, \dots, y_{\sigma_{r_1}}, v_{\tau_1}, \dots, v_{\tau_{r_2}}) = \prod_{\sigma:\mathbb{R}} y_\sigma \cdot \prod_{\tau:\mathbb{C}} v_\tau^2,$$

$$Y_1 := \frac{1}{2}\left(\sum_{i=1}^{r_1} e_i^{(1)} \log y_{\sigma_i} + \sum_{j=1}^{r_2} e_{r_1+j}^{(1)} \log v_{\tau_j}^2\right)$$

$$\cdots\ \cdots\ \cdots\ \cdots$$

$$Y_{r_1+r_2-1} := \frac{1}{2}\left(\sum_{i=1}^{r_1} e_i^{(r_1+r_2-1)} \log y_{\sigma_i} + \sum_{j=1}^{r_2} e_{r_1+j}^{(r_1+r_2-1)} \log 0 v_{\tau_j}^2\right)$$

Consequently, by inverting these relations, we have

$$y_{\sigma_i} = Y_0^{\frac{1}{r_1+r_2}} \prod_{q=1}^{r_1+r_2-1} |\varepsilon_q^{(i)}|^{2Y_q}, \qquad i = 1, \dots, r_1,$$

$$v_{\tau_j}^2 = Y_0^{\frac{1}{r_1+r_2}} \prod_{q=1}^{r_1+r_2-1} (|\varepsilon_q^{(r_1+j)}|^{2Y_q})^2, \qquad j = 1, \dots, r_2.$$

Further, by taking the fact that $N_\tau = 2$ for complex places τ, for later use, we set

$$t_j := v_j^2 = Y_0^{\frac{1}{r_1+r_2}} \prod_{q=1}^{r_1+r_2-1} (|\varepsilon_q^{(r_1+j)}|^2)^{2Y_q}, \qquad j = 1, \dots, r_2.$$

After this change of variables, from the precise construction of the fundamental domain for the action of Γ_η in $SL(\mathcal{O}_K \oplus \mathfrak{a})$ on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ in 2.4, it now becomes clear that this fundamental domain for the action of Γ_η on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is simply given by

$$0 < Y_0 < \infty, \qquad -\frac{1}{2} \leq Y_1, \dots, Y_{r_1+r_2-1} \leq \frac{1}{2},$$

$$(x_{\sigma_1}, \dots, x_{\sigma_{r_1}}; z_{\tau_1}, \dots, z_{\tau_{r_2}}) \in \mathcal{F}_\eta(\mathfrak{a}\mathfrak{b}^{-2}),$$

with $\mathcal{F}_\eta(\mathfrak{a}\mathfrak{b}^{-2})$ a fundamental parallelopiped associated with $\mathfrak{a}\mathfrak{b}^{-2}$ in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

To go further, we need to know the precise change of the volume forms in accordance with the above change of variables. So we must compute some of the Riemannian geometric invariants in terms of these coordinates at the cusps. Clearly the hyperbolic metric on $\mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ is given by $g = \begin{pmatrix} g_{\rm ImJ} & 0 \\ 0 & g_{\rm ReZ} \end{pmatrix}$ with the matrics for the ImJ and ReZ directions being given by

$$g_{\rm ImJ} = g_{\rm ReZ} = \mathrm{diag}\left(\frac{1}{y_1^2}, \dots, \frac{1}{y_{r_1}^2}, \left(\frac{1}{v_1^2}\right)^2, \dots \left(\frac{1}{v_{r_2}^2}\right)^2\right).$$

(Here we remind the reader that the twist resulting from $N_\tau = 2$ for complex places τ is entered in the discussion: In the above matrix we used $\left(\frac{1}{v_j^2}\right)^2$ instead of a simple $\frac{1}{v_j^2}$.)

First for the ReZ directions, there is no changes here. Hence the ReZ part of the matrix for the metric remains the same. As for the ImJ directions, from above, the metric in the ImJ directions is given by

$$(g_{ij}) := g_{\rm ImJ} = \mathrm{diag}\left(\frac{1}{y_1^2}, \dots, \frac{1}{y_{r_1}^2}, \left(\frac{1}{v_1^2}\right)^2, \dots \left(\frac{1}{v_{r_2}^2}\right)^2\right).$$

Recall that, in general, to find the matrix $(\tilde{g}_{ij})$ obtained from (g_{ij}) by the change of variables, we need to calculate the partial derivatives so as to get $\tilde{g}_{ij}$ from $\tilde{g}_{ij} = \sum_{\alpha,\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^i} \frac{\partial x^\beta}{\partial \tilde{x}^j} g_{\alpha\beta}$. (Here, in terms of g_{ij} and $\tilde{g}_{ij}$, the variables are assumed to be renumbered as $x^1, x^2, \dots, x^{r_1+r_2}$ and $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^{r_1+r_2}$ respectively.)

More precisely, by an obvious calculation,

$$\frac{\partial y_{\sigma_i}}{\partial Y_0} = \frac{1}{(r_1+r_2)Y_0} y_{\sigma_i}, \qquad i = 1, \dots, r_1,$$

$$\frac{\partial y_{\sigma_i}}{\partial Y_q} = 2\log|\varepsilon_q^{(i)}| \cdot y_{\sigma_i}, \qquad i = 1, \dots, r_1,\ q = 1, \dots, r_1+r_2-1$$

$$\frac{\partial t_{\tau_j}}{\partial Y_0} = \frac{1}{(r_1+r_2)Y_0} t_{\tau_j}, \qquad j = 1, \dots, r_2,$$

$$\frac{\partial t_{\tau_j}}{\partial Y_q} = 2\log|\varepsilon_q^{(j)}|^2 \cdot t_{\tau_j}, \qquad j = 1, \dots, r_2,\ q = 1, \dots, r_1+r_2-1.$$

Thus by the formula $\tilde{g}_{ij} = \sum_{\alpha,\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^i} \frac{\partial x^\beta}{\partial \tilde{x}^j} g_{\alpha\beta}$ for the change of variables and the symmetry of the matric matrix, we are led to calculate the following three types of products of matrices:

$$X_0 \cdot \mathrm{diag}\left(\frac{1}{y_1^2}, \ldots, \frac{1}{y_{r_1}^2}, \left(\frac{1}{v_1^2}\right)^2, \ldots \left(\frac{1}{v_{r_2}^2}\right)^2\right) \cdot X_0^t,$$

$$X_0 \cdot \mathrm{diag}\left(\frac{1}{y_1^2}, \ldots, \frac{1}{y_{r_1}^2}, \left(\frac{1}{v_1^2}\right)^2, \ldots \left(\frac{1}{v_{r_2}^2}\right)^2\right) \cdot X_q^t,$$

$$X_p \cdot \mathrm{diag}\left(\frac{1}{y_1^2}, \ldots, \frac{1}{y_{r_1}^2}, \left(\frac{1}{v_1^2}\right)^2, \ldots \left(\frac{1}{v_{r_2}^2}\right)^2\right) \cdot X_q^t$$

where $X_0 := \left(\frac{y_{\sigma_1}}{(r_1+r_2)Y_0}, \ldots, \frac{y_{\sigma_{r_1}}}{(r_1+r_2)Y_0}, \frac{t_{\tau_1}}{(r_1+r_2)Y_0}, \ldots, \frac{t_{\tau_{r_2}}}{(r_1+r_2)Y_0}\right)$ and

$$\begin{aligned} X_p = (&2\log|\varepsilon_p^{(1)}| \cdot y_{\sigma_1}, \ldots, 2\log|\varepsilon_p^{(r_1)}| \cdot y_{\sigma_{r_1}}, 2\log|\varepsilon_p^{(r_1+1)}|^2 \\ &\cdot v_{\tau_1}^2, \ldots, 2\log|\varepsilon_p^{(r_1+r_2)}|^2 \cdot v_{\tau_{r_2}}^2), \end{aligned}$$

for $p,\ q = 1, 2, \ldots, r_1 + r_2 - 1$.

Hence, $\tilde g_{11} = \frac{1}{(r_1+r_2)Y_0^2}$, $\tilde g_{1j} = \tilde g_{j1} = 0$, $j = 2, \ldots, r_1 + r_2$, since $\sum_{i=1}^{r_1} \log|\varepsilon_p^{(i)}| + \sum_{j=1}^{r_2} \log|\varepsilon_p^{(r_1+j)}|^2 = 0$; while, for $i, j = 1, \ldots, r_1+r_2-1$,

$$\tilde g_{(i+1)(j+1)} = 4\sum_{p=1}^{r_1} \log|\varepsilon_i^{(p)}| \log|\varepsilon_j^{(p)}| + 4\sum_{p=r_1+1}^{r_1+r_2} \log|\varepsilon_i^{(p)}|^2 \log|\varepsilon_j^{(p)}|^2.$$

As for the new volume element, we use $\det(\tilde g_{ij}) = \frac{4^{r_1+r_2-1}}{(r_1+r_2)Y_0^2} R^2$, where R is the regulator of K, i.e., $R := \det\left(\log\|\varepsilon_q^{(p)}\|\right)_{p,q=1}^{r_1+r_2-1}$. (See e.g. [19].) Thus, by taking also account of ReZ direction, we have

$$\begin{aligned} d\omega &= \left(\sqrt{\det(\tilde g_{ij}) \cdot \frac{1}{Y_0^2}}\right) dY_0\, dY_1 \cdots dY_{r_1+r_2-1} \prod_{\sigma:\mathbb{R}} dx_\sigma \prod_{\tau:\mathbb{C}} dz_\tau \\ &= \frac{2^{r_1+r_2-1}}{\sqrt{r_1+r_2}} R \frac{dY_0}{Y_0^2}\, dY_1 \cdots dY_{r_1+r_2-1} \prod_{\sigma:\mathbb{R}} dx_\sigma \prod_{\tau:\mathbb{C}} dz_\tau. \end{aligned}$$

Clearly, the boundary $\partial\mathcal{D}_T$ of $\mathcal{D}_T$ consists of

1) (the corresponding parts of) the boundary of the fundamental domain of $\mathcal{D}$; and
2) the hyperplane of $\mathcal{D}$ defined by the condition $Y_0 = T' := N(\mathfrak{a}\mathfrak{b}^{-2}) \cdot T$.
   (Please note that the factor $N(\mathfrak{a}\mathfrak{b}^{-2})$ is added in 2). This is because in the definition of $\mathcal{D}_T$, what we used is the distance to cusp, not simply Y_0.)

Consequently, if we set $d\mu$ to be the volume element of this hypersurface, then

$$\begin{aligned} d\mu &= \frac{1}{\sqrt{\tilde{g}_{11}}} d\omega\Big|_{Y_0=T'} \\ &= \frac{\sqrt{r_1+r_2}}{T'} 2^{r_1+r_2-1} R \cdot dY_1 \cdots dY_{r_1+r_2-1} \prod_{\sigma:\mathbb{R}} dx_\sigma \prod_{\tau:\mathbb{C}} dz_\tau. \end{aligned}$$

Also if ν is the unit normal to the hypesurface, since $\left\langle \frac{\partial}{\partial Y_0}, \frac{\partial}{\partial Y_0} \right\rangle = \tilde{g}^{11}\big|_{Y_0=T'} = (r_1+r_2)T'^2$, we have $\nu = \left(\frac{1}{\sqrt{r_1+r_2}T'}, 0, \ldots, 0\right)$. Thus the outer normal derivative of a function f is given by $\frac{\partial f}{\partial \nu} = \left(\frac{1}{\sqrt{r_1+r_2}T'}, 0, \ldots, 0\right) \cdot \operatorname{grad} f = \sqrt{r_1+r_2} \cdot T' \frac{\partial f}{\partial Y_0}$.

Now by (the fact that the group $SL(\mathcal{O}_K \oplus \mathfrak{a})$ is finitely generated and) the concrete construction of our fundamental domain, we see that the boundary $\partial\mathcal{D}_T$ consists of finitely many of surfaces which are either parts of horospheres or parts $X_i(T)$ of planes cut out by $Y_0 = T_i'$, where $T_i' = N(\mathfrak{a}\mathfrak{b}_i^{-2}) \cdot T$ (with $\mathfrak{b}_i$ the fractional ideal associated to the cusp η_i). Moreover, besides the hyperplanes associated with $Y_0 = T'$, the set of horospheres appeared on the boundary is divided into the sets of equivalent pairs for which the integral of the outer normal derivative along one surface in a pair is equal to the integral of the inner normal derivative along the other surface in the pair. (Say, in terms of $Y_{p\geq 1}$, they are given by $Y_p = \pm\frac{1}{2}$ in pairs.) As such, we further conclude that

$$\begin{aligned} \iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau,s) d\mu(\tau) &= \frac{1}{r_1+4r_2} \frac{1}{s(s-1)} \iiint_{\mathcal{D}_T} \Delta_K \widehat{E}_{2,\mathfrak{a}}(\tau,s) d\mu(\tau) \\ &= \iint_{\partial\mathcal{D}_T} \frac{\partial \widehat{E}_{2,\mathfrak{a}}(\tau,s)}{\partial\nu} d\mu \\ &= \frac{1}{r_1+4r_2} \frac{1}{s(s-1)} \sum_{i=1}^h \iint_{X_i(T)} \frac{\partial \widehat{E}_{2,\mathfrak{a}}(\tau,s)}{\partial\nu} ds, \end{aligned}$$

where $X_i(T)$ denotes the part of the boundary of $\mathcal{D}_T$ coming from the pull back of the intersection of the hypersurface $Y_0 = T_i'$ with $F_{\eta_i}, i = 1, 2, \ldots, h$. (Here we used the fact that for $T \geq 1$, $X_i(T)$ are disjoint from each other. See e.g., the Lemma in 2.5.4.)

Thus far, we are ready to use Fourier expansion to do the final calculation. Note that the average for $e^{2\pi it}$ (together with its derivative) over an interval of length 1 is zero. Hence, in the above integration for $\widehat{E}$ over $\mathcal{D}_T$, we are in fact left with only the constant terms of the Fourier expansion for $\widehat{E}(s)$.

Consequently, with $T_i' = N(\mathfrak{a}\mathfrak{b}_i^{-2})\cdot T$, then, up to constant factors depending only on K,

$$\iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau,s)d\mu(\tau)$$

$$= \frac{1}{r_1+4r_2}\frac{1}{s(s-1)}\iiint_{\mathcal{D}_T} \Delta_K \widehat{E}_{2,\mathfrak{a}}(\tau,s)d\mu(\tau)$$

$$= \frac{1}{r_1+4r_2}\frac{1}{s(s-1)}\sum_{i=1}^h \iint_{X_i(T)} \frac{\partial}{\partial\nu}(A_{0i}Y_0^s + B_{0i}Y_0^{1-s})d\mu$$

$$= \frac{1}{r_1+4r_2}\frac{1}{s(s-1)}\sum_{i=1}^h \iint_{X_i(T)} \sqrt{r_1+r_2}$$

$$\cdot T_i' \frac{\partial}{\partial Y_0}(A_{0i}Y_0^s + B_{0i}Y_0^{1-s})$$

$$\cdot \frac{\sqrt{r_1+r_2}}{T_i'} 2^{r_1+r_2-1}R\cdot dY_1\dots dY_{r_1+r_2-1}\cdot\prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dz_\tau$$

$$= \frac{r_1+r_2}{r_1+4r_2}\frac{1}{s(s-1)}\cdot 2^{r_1+r_2-1}R$$

$$\cdot \sum_{i=1}^h \iint_{X_i(T)} (s\,A_{0i}{Y_0}^{s-1} - (s-1)\,B_{0i}{Y_0}^{-s})$$

$$\times\, dY_1\dots dY_{r_1+r_2-1}\cdot\prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dz_\tau$$

$$= \frac{r_1+r_2}{r_1+4r_2}\frac{1}{s(s-1)} 2^{r_1+r_2-1}R\cdot \sum_{i=1}^h (s\,A_{0i}{T_i'}^{s-1} - (s-1)\,B_{0i}{T_i'}^{-s})$$

$$\cdot \iint_{X_i(T)} dY_1\dots dY_{r_1+r_2-1}\cdot\prod_{\sigma:\mathbb{R}} dx_\sigma \cdot \prod_{\tau:\mathbb{C}} dz_\tau$$

$$= \frac{r_1+r_2}{r_1+4r_2} 2^{r_1+r_2-1}R\cdot D_K^{\frac{1}{2}}\sum_{i=1}^h N(\mathfrak{a}\mathfrak{b}_i^{-2})$$

$$\cdot \left(\frac{A_{0i}}{s-1}\cdot {T_i'}^{s-1} - \frac{B_{0i}}{s}{T_i'}^{-s}\right),$$

due to the fact that the lattice corresponding to the cusp $\eta_i = \frac{\alpha_i}{\beta_i}$ is given by $\mathfrak{a}\mathfrak{b}_i^{-2}$ with $\mathfrak{b}_i = \mathcal{O}_K\alpha_i + \mathfrak{a}\beta_i$ and $Y_p \in \big[-\frac{1}{2}, \frac{1}{2}\big]$. Thus with the precisely formula we have for $A_{0i}(s)$ and the functional equation with the change $s \leftrightarrow 1-s$, we finally obtain the following

Theorem . *Up to a constant factor depending only on K,*

$$\iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau, s)d\mu(\tau) = \frac{\xi_K(2s)}{s-1}\cdot T^{s-1} - \frac{\xi_K(2-2s)}{s}\cdot T^{-s}.$$

Proof. By functional equation, we only need to calculate the coefficient of $\frac{T^{s-1}}{s-1}$. Note that, by Theorem in 3.3.2, the partial constant term $A_{0,i}$ for the completed $\widehat{E}_{2,\mathfrak{a}}(\tau, s)$ is given by $((\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2}(N(\mathfrak{a})\Delta_K)^s)\cdot \zeta([\mathfrak{a}^{-1}\mathfrak{b}_i], 2s)\cdot N(\mathfrak{a}\mathfrak{b}_i^{-1})^{-2s}$. Hence, up to a constant factor depending only on K, the coefficient of $\frac{T^{s-1}}{s-1}$ in the integration $\iiint_{\mathcal{D}_T} \widehat{E}_{2,\mathfrak{a}}(\tau, s)d\mu(\tau)$ is simply the summation $\sum_{i=1}^h$ of $N(\mathfrak{a}\mathfrak{b}_i^{-2})A_{0,i}$ timing with the factor $N(\mathfrak{a}\mathfrak{b}_i^{-2})^{s-1}$ resulting from the discrepency between T and T_i'. That is to say, up to a constant factor depending only on K, the coefficient of $\frac{T^{s-1}}{s-1}$ is nothing but

$$\begin{aligned} &\sum_{i=1}^h (\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2}(N(\mathfrak{a})\Delta_K)^s \\ &\qquad \cdot \zeta([\mathfrak{a}^{-1}\mathfrak{b}_i], 2s)N(\mathfrak{a}\mathfrak{b}_i^{-1})^{-2s}\cdot N(\mathfrak{a}\mathfrak{b}_i^{-2})\cdot N(\mathfrak{a}\mathfrak{b}_i^{-2})^{s-1} \\ &= \Delta_K^s\cdot(\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2}\sum_{i=1}^h \zeta([\mathfrak{a}^{-1}\mathfrak{b}_i], 2s) \\ &= \Delta_K^s\cdot(\pi^{-s}\Gamma(s))^{r_1}(2\pi^{-2s}\Gamma(2s))^{r_2}\zeta_K(2s) = \xi_K(2s), \end{aligned}$$

since $\sum_{i=1}^h \zeta([\mathfrak{a}^{-1}\mathfrak{b}_i], 2s) = \zeta_K(2s)$, resulting from the facts that

(i) the h ideal classes $[\mathfrak{a}^{-1}\mathfrak{b}_i]$ for fixed $\mathfrak{a}$ run over all elements of the class group of K;

(ii) the total Dedekind zeta function decomposes into a summation of partial zeta functions associated to ideal classes.

□

Consequently, we have the following

Fact (VIII) *Up to a constant factor depending only on K, the rank two non-abelian zeta function $\xi_{K,2}(s)$ is given by*

$$\xi_{K,2}(s) = \frac{\xi_K(2s)}{s-1} - \frac{\xi_K(2s-1)}{s} \qquad \Re(s) > 1.$$

Proof. This is because, by Fact VI in 2.5, we have the moduli space of rank two semi-stable lattices of volume $N(\mathfrak{a})\Delta_K$ with underlying projective module $\mathcal{O}_K \oplus \mathfrak{a}$ is given by $\mathcal{D}_1$. But from the Theorem above, up to a constant factor depending only on K,

$$\iiint_{\mathcal{D}_1} \widehat{E}_{2,\mathfrak{a}}(\tau, s)d\mu(\tau) = \frac{\xi_K(2s)}{s-1} - \frac{\xi_K(2-2s)}{s}.$$

That is, up to a constant factor depending only on K, $\xi_{K,2;\mathfrak{a}}(s) = \frac{\xi_K(2s)}{s-1} - \frac{\xi_K(2-2s)}{s}$. Therefore, by Fact IV, up to a constant factor depending only on K,

$$\xi_{K,2}(s) = \frac{\xi_K(2s)}{s-1} - \frac{\xi_K(2-2s)}{s}. \qquad \square$$

5. Zeros of rank two zetas for number fields

5.1 Zeros of rank two zeta of $\mathbb{Q}$

Following what was happened in history, let me first start with Masatoshi Suzuki's weak result [23] and then give Jeff Lagarias' unconditional result [Lag] and/or [15]. Meanwhile, for an independent parallel work, please go to Haseo Ki's paper [14].

Theorem . *If the Riemann Hypothesis for the Riemann zeta function holds, then all zeros of $\xi_{\mathbb{Q},2}(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

5.1.1 Proof

Let $F(z) = -Z\left(\frac{1}{2} + 2iz\right)$ with $Z(s) = s(1-s)\xi(s)$.

Proposition. (Suzuki)

(1) $F\left(z + \frac{i}{4}\right) - F\left(z - \frac{i}{4}\right) = iz(1+4z^2)\,\xi_{\mathbb{Q},2}\left(\frac{1}{2} + iz\right)$.

(2) *Assume the RH, then all zeros of $F\left(z + \frac{i}{4}\right) - F\left(z - \frac{i}{4}\right)$ are real.*

In particular, then the RH implies that $\xi_{\mathbb{Q},2}\left(\frac{1}{2} + zi\right)$ admits only real zeros.

Proof.

(1) Simple calculation. Indeed,

$$\begin{aligned} F\left(z + \frac{i}{4}\right) &= -Z\left(\frac{1}{2} + 2i\left(\frac{i}{4} + z\right)\right) \\ &= -Z\left(\frac{1}{2} - \frac{1}{2} + 2iz\right) = -Z(2iz). \end{aligned}$$

So $F\big(z-\frac{i}{4}\big) = -Z\big(1+2iz\big)$ and

$$\begin{aligned}
&F\left(z+\frac{i}{4}\right) - F\left(z-\frac{i}{4}\right)\\
&\quad = (1+2iz)(-2iz)\xi(1+2iz) - 2iz(1-2iz)\xi(2iz)\\
&\quad = 2iz(1-2iz)(1+2iz)\cdot\left(\frac{\xi(1+2iz)}{2iz-1} - \frac{\xi(2iz)}{1+2iz}\right)\\
&\quad = iz(1+4z^2)\cdot\left(\frac{\xi\left(2\left(\frac{1}{2}+iz\right)\right)}{\left(\frac{1}{2}+iz\right)-1} - \frac{\xi\left(2\left(\frac{1}{2}+iz\right)-1\right)}{\frac{1}{2}+iz}\right)\\
&\quad = iz(1+4z^2)\,\xi_{\mathbb{Q},2}\left(\frac{1}{2}+iz\right).
\end{aligned}$$

(2) Clearly, $F(z)$ is an entire function of order 1, so there are constants A, B such that

$$F(z) = e^{A+Bz}\cdot\prod_{\rho:F(\rho)=0}\left(1-\frac{z}{\rho}\right)\cdot\exp\left(\frac{z}{\rho}\right).$$

Note that essentially, ρ are zeros of the completed Riemann zeta but transformed from z to $\frac{1}{2}+2iz$. Hence, *by the RH*, all ρ are real.

□

Moreover, since $F(z) = -Z\big(\frac{1}{2}+2iz\big)$ with $Z(s) = s(1-s)\,\xi(s)$, we have for $x\in\mathbb{R}$,

$$\overline{F(x)} = \overline{-Z(1/2+2ix)} = -Z(\overline{1/2+2ix}) = -Z\left(\frac{1}{2}-2ix\right)$$

which by functional equation is simply $-Z\big(1-\big(\frac{1}{2}-2ix\big)\big) = -Z\big(\frac{1}{2}+2ix\big) = F(x)$. That is to say, for $x\in\mathbb{R}$, $F(x)$ takes only real values. Hence, A and B are both real.

Now let $z_0 = x_0+iy_0$ be a zero of $F\big(z+\frac{i}{4}\big) - F\big(z-\frac{i}{4}\big) = iz(1+4z^2)\,\xi_{\mathbb{Q},2}\big(\frac{1}{2}+iz\big)$. Then $z_0 = 0$ and/or z_0 is a zero of $\xi_{\mathbb{Q},2}\big(\frac{1}{2}+iz\big)$ since $\xi_{\mathbb{Q},2}\big(\frac{1}{2}+iz\big)$ admits simple poles at $z=\pm\frac{1}{2}i$.

In any case, $F\big(z_0+\frac{i}{4}\big) = F\big(z_0-\frac{i}{4}\big)$. By taking absolute values on both sides,

$$\begin{aligned}
&\left|e^{A+B\left(z_0+\frac{i}{4}\right)}\cdot\prod\left(1-\frac{z_0+\frac{i}{4}}{\rho}\right)\cdot\exp\left(\frac{z_0+\frac{i}{4}}{\rho}\right)\right|\\
&\qquad = \left|e^{A+B\left(z_0-\frac{i}{4}\right)}\cdot\prod\left(1-\frac{z_0-\frac{i}{4}}{\rho}\right)\cdot\exp\left(\frac{z_0-\frac{i}{4}}{\rho}\right)\right|.
\end{aligned}$$

Since $B \in \mathbb{R}$ and $\rho_n \in \mathbb{R}$ (which is obtained by the RH as said above), we hence get $1 = \prod_{n=1}^{\infty} \frac{(x_0-\rho_n)^2+\left(y_0-\frac{1}{4}\right)^2}{(x_0-\rho_n)^2+\left(y_0+\frac{1}{4}\right)^2}$. Thus if $y_0 > 0$, then the right hand side is < 1, while if $y_0 < 0$, then the right hand side is > 1. Contradiction. This leads then $y_0 = 0$, hence completes the proof.

With this in mind, note that in the proof above, the RH was used to ensure that ρ are real, which have the effect that then in the calculation for the exponential factor $\exp\left(\frac{z_0+\frac{i}{4}}{\rho}\right)$, the ratio $\frac{\left|\exp\left(\frac{z_0+\frac{i}{4}}{\rho}\right)\right|}{\left|\exp\left(\frac{z_0-\frac{i}{4}}{\rho}\right)\right|}$ gives us the exact value 1. That is to say, this factor of ratio of exp's does not contribute.

However, one does not need such an argument from the very beginning to eliminate the factors $\exp\left(\frac{z_0\pm\frac{i}{4}}{\rho}\right)$. In fact, this is the improvement of Lagarias, who gets his own unconditional result totally independently, as a part of his understanding of de Branges's work. The trick is very simple: Use the functional equation. So instead of working on individual ρ_n in the product, we may equally use the functional equation to pair ρ and $1-\rho$ for the zeros of the completed Riemann zeta function, or even to group ρ, $1-\rho$, $\overline{\rho}$ and $1-\overline{\rho}$ together. Consequently, the exponential factor appeared inside the infinite product may be totally omitted. That is to say, from the very beginning, we may simply assume that the Hadamard product involved takes the form $F(z) = e^{A+Bz} \cdot \prod'_{\rho:F(\rho)=0} \left(1-\frac{z}{\rho}\right)$ where $\prod'$ means that ρ's are paired or grouped as above. Form here, it is an easy exercise to deduce the following result of (Suzuki and) Lagarias.

Fact $(\mathrm{IX})_{\mathbb{Q}}$ *All zeros of* $\xi_{\mathbb{Q},2}(s)$ *lie on the line* $\Re(s) = \frac{1}{2}$.

Proof. Alternatively, as above, we have

$$\left|e^{A+B\left(z_0+\frac{i}{4}\right)} \cdot \prod \left(1-\frac{z_0+\frac{i}{4}}{\rho}\right) \cdot \exp\left(\frac{z_0+\frac{i}{4}}{\rho}\right)\right|$$
$$= \left|e^{A+B\left(z_0-\frac{i}{4}\right)} \cdot \prod \left(1-\frac{z_0-\frac{i}{4}}{\rho}\right) \cdot \exp\left(\frac{z_0-\frac{i}{4}}{\rho}\right)\right|.$$

Since $B \in \mathbb{R}$, so if we can take care of the factors $\exp\left(\frac{z_0+\frac{i}{4}}{\rho}\right)$ and $\exp\left(\frac{z_0-\frac{i}{4}}{\rho}\right)$ in a nice way, we are done. For this, as said above, let us group ρ, $\bar{\rho}$, $1-\rho$, $1-\bar{\rho}$ together, we see that $\frac{1}{\rho}+\frac{1}{\bar{\rho}} = \frac{2\,\Re(\rho)}{|\rho|^2}$ and $\frac{1}{1-\rho}+\frac{1}{1-\bar{\rho}} = \frac{2-2\,\Re(\rho)}{|1-\rho|^2}$ are all reals, hence, the same prove as above works. $\square$

5.1.2 A simple generalization

The above method works for the functions $\xi_{\mathbb{Q},2}^T(s)$ as well, provided that $T \geq 1$. Indeed, first, recall that we have the precise relation $\xi_{\mathbb{Q},2}^T(s) = \frac{\xi(2s)}{s-1} \cdot T^{s-1} - \frac{\xi(2s-1)}{s} \cdot T^{-s}$. Consequently, $F\big(z + \frac{i}{4}\big) \cdot T^{-\frac{1}{2}-iz} - F\big(z - \frac{i}{4}\big) \cdot T^{-\frac{1}{2}+iz} = iz(1+4z^2)\,\xi_{\mathbb{Q},2}^T\big(\frac{1}{2} + zi\big)$. Therefore, using the same proof, we arrive at the relation

$$\left| e^{A+B\left(z_0+\frac{i}{4}\right)} \cdot \prod \left(1 - \frac{z_0 + \frac{i}{4}}{\rho}\right) \cdot \exp\left(\frac{z_0 + \frac{i}{4}}{\rho}\right) \right| \cdot \left|T^{-iz-\frac{1}{2}}\right|$$
$$= \left| e^{A+B\left(z_0-\frac{i}{4}\right)} \cdot \prod \left(1 - \frac{z_0 - \frac{i}{4}}{\rho}\right) \cdot \exp\left(\frac{z_0 - \frac{i}{4}}{\rho}\right) \right| \cdot \left|T^{iz-\frac{1}{2}}\right|.$$

That is to say, $1 = \prod_{n=1}^{\infty} \frac{(x_0-\rho_n)^2+(y_0-\frac{1}{4})^2}{(x_0-\rho_n)^2+(y_0+\frac{1}{4})^2} \cdot \frac{T^{-y_0}}{T^{y_0}}$. Or equivalently, $T^{2y_0} = \prod_{n=1}^{\infty} \frac{(x_0-\rho_n)^2+(y_0-\frac{1}{4})^2}{(x_0-\rho_n)^2+(y_0+\frac{1}{4})^2}$.

Thus with $T \geq 1$, we have

(i) if $y_0 > 0$, the left hand side is > 1, while the right hand side is < 1, contradiction; while
(ii) if $y_0 < 0$, the left hand side is < 1, while the right hand side is > 1, contradiction. That is to say, we obtain the following

Fact $(\mathrm{IX}')_{\mathbb{Q}}$ *For $T \geq 1$, all zeros of $\xi_{\mathbb{Q},2}^T(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

5.2 Zeros of rank two zetas for number fields

Fact (IX) *All zeros of rank two zeta functions for number fields are on the critical line $\Re(s) = \frac{1}{2}$.*

Proof. This is a direct consequence of the following two facts:

First, we know that, by the Rankin-Selberg & Zagier method, up to a constant factor depending only on K, $\xi_{K,2}(s) = \frac{\xi_K(2s)}{s-1} - \frac{\xi_K(2s-1)}{s}$. Secondly, $s(s-1) \cdot \xi_K(s)$ is an entire function of order one [16]. Consequently, the proof in the previous section on the zeros works here as well, by a simple change from the Riemann ξ for the field of rationals $\mathbb{Q}$ to the Dedekind ξ_K for the number field K. □

Acknowledgement

I would like to thank C. Deninger, J. Elstrodt, and D. Zagier for their enlightenment; I. Fensenko, E. Freitag, and H. Kim for their keen interests; and K. Murty for his kind invitation of the submission.[1]

[1] The project is partially supported by JSPS.

References

[1] A. Borel, Some finiteness properties of adele groups over number fields, *Publ. Math.*, IHES, **16** (1963) 5–30.

[2] A. Borel, *Introduction aux groupes arithmetictiques*, Hermann, (1969).

[3] A. Borisov, Convolution structures and arithmetic cohomology, *Compositio Math.*, **136** (2003), No. 3, 237–254.

[4] C. Chevalley, *Introduction to the theory of algebraic functions of one variables*, AMS (1951).

[5] H. M. Edwards, *Riemann's Zeta Function*, Dover Publications, INC., (1974).

[6] I. Y. Efrat, *The Selberg Trace Formula for* $PSL_2(\mathbb{R})^n$, Memoirs of AMS, No. 359, (1987).

[7] J. Elstrodt, F. Grunewald and J. L. Mennicke, *Groups Acting on Hyperbolic Space: Harmonic Analysis and Number Theory*, Springer Verlag (1997).

[8] A. Fröhlich and M. J.Taylor, *Algebraic Number Theory*, Cambridge studies in advanced mathematics **27**, Cambridge Univ. Press (1991).

[9] G. van der Geer and R. Schoof, Effectivity of Arakelov Divisors and the Theta Divisor of a Number Field, *Sel. Math., New ser.*, **6** (2000), 377–398.

[10] D. R. Grayson, Reduction theory using semistability. *Comment. Math. Helv.*, **59** (1984), No. 4, 600–634.

[11] Hayashi, Tsukasa: The degeneration of new non-abelian zeta function of rank 2, priprint, 2003.

[12] K. Iwasawa, Letter to Dieudonné, April 8, 1952, in *Zeta Functions in Geometry*, Advanced Studies in Pure Math. **21** (1992), 445–450.

[13] K. Iwasawa, *Lectures notes on Riemann(-Artin) Hypothesis*, noted by T. Kimura, Princeton Univ. (1975).

[14] H. Ki, All but finitely many non-trivial zeros of the approximations of the Epstein zeta function are simple and on the critical line, *Proceedings of London Math. Society*, (3) **90** (2005), No. 2, 321–344.

[15] J. Lagarias and M. Suzuki, The Riemann hypothesis for certain integrals of Eisenstein series, *J. Number Theory*, **118** (2006), No. 1, 98–122.

[16] S. Lang, *Algebraic Number Theory*, Springer-Verlag, (1986).

[17] S. Lang, *Introduction to Arekelov Theory*, Springer-Verlag, (1988).

[18] C. Moreno, *Algebraic curves over finite fields*. Cambridge Tracts in Mathematics, **97**, Cambridge University Press, (1991).

[19] J. Neukirch, *Algebraic Number Theory*, Grundlehren der Math. Wissenschaften, Vol. **322**, Springer-Verlag, (1999).

[20] J.-P. Serre, *Algebraic Groups and Class Fields*, GTM **117**, Springer-Verlag (1988).

[21] C. L. Siegel, *Lectures on advanced analytic number theory*. Notes by S. Raghavan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 23 (1965).

[22] U. Stuhler, Eine Bemerkung zur Reduktionstheorie quadratischer Formen. *Arch. Math.*, **27** (1976), No. 6, 604–610.

[23] Suzuki, Masatoshi: priviate communication, (2003).

[24] J. Tate, Fourier analysis in number fields and Hecke's zeta functions, Thesis, Princeton University, (1950).

[25] A. Terras, *Harmonic analysis on symmetric spaces and applications II*, Springer-Verlag, (1988).

[26] E. C. Titchmarsh, *The Theory of Riemann Zeta-Function*, Oxford Univ. Press, (1951).

[27] A. Weil, *Basic Number Theory*, Springer-Verlag, (1973).

[28] L. Weng, Geometric Arithmetic: A Program, *Arithmetic Geometry and Number Theory*, World Scientific, (2006) 211–400.

[29] L. Weng, Non-Abelian L-Functions for Number Fields, to appear.

[30] L. Weng, Non-Abelian Zeta Functions for Function Fields, *Amer J Math.*, **127** (2005), No. 5, 937–1017.

[31] L. Weng, Refined Brill-Noether locus and non-abelian Zeta functions for elliptic curves, *Algebraic geometry in East Asia* (Kyoto, 2001), 245–262, World Sci. (2002).
[32] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay. J. Fac. Sci. Univ. Tokyo Sect. *IA Math.*, 28(3), 415–437. (1982).