Commun. Math. Phys. 284, 227–261 (2008)
Digital Object Identifier (DOI) 10.1007/s00220-008-0520-7

Communications in **Mathematical Physics**

The Asymptotic Behavior of the Takhtajan-Zograf Metric

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Received: 30 September 2007 / Accepted: 8 January 2008
Published online: 4 June 2008 –

Abstract: We obtain the asymptotic behavior of the Takhtajan-Zograf metric on the Teichmüller space of punctured Riemann surfaces.

0. Introduction

We consider the Teichmüller space $T_{g,n}$ and the associated Teichmüller curve $\mathcal{T}_{g,n}$ of Riemann surfaces of type (g, n) (i.e., Riemann surfaces of genus g and with $n > 0$ punctures). We will assume that $2g-2+n > 0$, so that each fiber of the holomorphic projection map $\pi: \mathcal{T}_{g,n} \to T_{g,n}$ is stable or equivalently, it admits the complete hyperbolic metric of constant sectional curvature -1. The kernel of the differential $T\mathcal{T}_{g,n} \to TT_{g,n}$ forms the so-called vertical tangent bundle over $\mathcal{T}_{g,n}$, which is denoted by $T^V\mathcal{T}_{g,n}$. The hyperbolic metrics on the fibers induce naturally a Hermitian metric on $T^V\mathcal{T}_{g,n}$.

In the study of the family of $\bar\partial_k$-operators acting on the k-differentials on Riemann surfaces (i.e., cross-sections of $\left(T^V\mathcal{T}_{g,n}\right)^{-k}|_{\pi^{-1}(s)} \to \pi^{-1}(s),\ s \in T_{g,n}$), Takhtajan and Zograf introduced in [TZ1] and [TZ2] a Kähler metric on $T_{g,n}$, which is known as the Takhtajan-Zograf metric. In [TZ2], they showed that the Takhtajan-Zograf metric is invariant under the natural action of the Teichmüller modular group $\mathrm{Mod}_{g,n}$ and it satisfies the following remarkable identity on $T_{g,n}$:

$$c_1(\lambda_k, \rho_{Q,k}) = \frac{6k^2-6k+1}{12} \cdot \frac{1}{\pi^2}\omega_{\mathrm{WP}} - \frac{1}{9}\omega_{\mathrm{TZ}}.$$

Here $\lambda_k = \det(\mathrm{ind}\,\bar\partial_k) = \wedge^{\max}\mathrm{Ker}\,\bar\partial_k \otimes (\wedge^{\max}\mathrm{Coker}\,\bar\partial_k)^{-1}$ denotes the determinant line bundle on $T_{g,n}$, $\rho_{Q,k}$ denotes the Quillen metric on λ_k, and ω_{WP} and ω_{TZ} denote the

⋆ The first author is partially supported by JSPS Grant-in-Aid for Exploratory Research 2005-2007.
⋆⋆ The second author is partially supported by the research grant R-146-000-106-112 from the National University of Singapore and the Ministry of Education.

Kähler forms of the Weil-Petersson metric and the Takhtajan-Zograf metric on $T_{g,n}$ respectively. In [We], Weng studied the Takhtajan-Zograf metric in terms of Arakelov intersection, and he expressed the class of ω_{TZ} as a rational multiple of the first Chern class of an associated Takhtajan-Zograf line bundle over the moduli space $\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$. Recently, Wolpert [Wol5] gave a natural definition of a Hermitian metric on the Takhtajan-Zograf line bundle whose first Chern form gives ω_{TZ}.

Motivated in part by these developments, we are interested in studying the boundary behavior of the Takhtajan-Zograf metric on $T_{g,n}$. Along this direction is an earlier result of Obitsu [O1], who showed that the Takhtajan-Zograf metric is incomplete. We are also inspired by Masur's beautiful paper [M], which gave the asymptotic boundary behavior of the Weil-Petersson metric on $T_g := T_{g,0}$ (see also [Wolp5] and [OW] for recent improvements of this result).

Our main result in this paper is to give the asymptotic behavior of the Takhtajan-Zograf metric near the boundary of $T_{g,n}$, which we describe heuristically as follows. Near the boundary of $T_{g,n}$, the tangent space at any point in $T_{g,n}$ can be roughly considered as the direct sum of the pinching directions and the non-pinching directions (that are 'parallel' to the boundary). Roughly speaking, our result shows that the Takhtajan-Zograf metric is smaller than the Weil-Petersson metric by an additional factor of $1/|\log|t||$ along each pinching tangential direction, i.e. it is essentially of the order of growth $1/|t|^2(\log|t|)^4$ along the pinching direction corresponding to a pinching coordinate t. Also, we show that the Takhtajan-Zograf metric extends continuously along the non-pinching tangential directions to the "nodally-depleted Takhtajan-Zograf metrics" on the boundary Teichmüller spaces, which, unlike the case of the Weil-Petersson metric, are only positive semi-definite on the boundary Teichmüller spaces. Our result also leads immediately to an alternative proof of the above mentioned result of Obitsu on the non-completeness of the Takhtajan-Zograf metric (see Theorem 1 for the precise statements of our results.)

An important ingredient in the proof of our main result is to obtain certain estimates on degenerative behavior of the Eisenstein series in the setting of holomorphic families of degenerating punctured Riemann surfaces, which seem to be of considerable independent interest. These estimates are largely obtained by geometrically constructing suitable germs of comparison functions for the Eisenstein series near the nodes and punctures. We also need to make certain adaptations from Masur's paper [M].

This paper is organized as follows. In Sect. 1, we introduce some notation and state our main results. In Sect. 2, we describe the behavior of the hyperbolic metrics on the punctured Riemann surfaces upon degenerations. In Sect. 3, we recall Masur's construction of a certain local basis of regular quadratic differentials for a degenerating family of punctured Riemann surfaces. In Sect. 4, we derive the necessary estimates of the Eisenstein series near the punctures and nodes of a degenerating family of punctured Riemann surfaces. Finally we complete the proof of our main result in Sect. 5.

1. Notation and Statement of Results

1.1. For $g \geq 0$ and $n > 0$, we denote by $T_{g,n}$ the Teichmüller space of Riemann surfaces of type (g, n). Each point of $T_{g,n}$ is a Riemann surface X of type (g, n), i.e., $X = \bar{X} \backslash \{p_1 \dots, p_n\}$, where X is a compact Riemann surface of genus g, and the punctures $p_1, \dots, p_n$ of X are n distinct points in $\bar{X}$. We will always assume that $2g-2+n > 0$, so that X admits the complete hyperbolic metric of constant sectional curvature -1. By the uniformization theorem, X can be represented as a quotient $\mathbb{H}/\Gamma$ of the upper half plane $\mathbb{H} := \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ by the natural action of Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ of the first kind. Γ is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and n parabolic transformations $P_1, \ldots, P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} P_1 P_2 \cdots P_n = \mathrm{Id}.$$

Let $z_1, \ldots, z_n \in \mathbb{R} \cup \{\infty\}$ be the fixed points of the parabolic transformations $P_1, \ldots, P_n$ respectively, which are also called cusps. The cusps $z_1, \ldots, z_n$ correspond to the punctures $p_1, \ldots, p_n$ of X under the projection $\mathbb{H} \to \mathbb{H}/\Gamma \simeq X$ respectively. For each $i = 1, 2, \ldots, n$, it is well-known that P_i generates an infinite cyclic subgroup of Γ, and we can select $\sigma_i \in \mathrm{PSL}(2, \mathbb{R})$ so that $\sigma_i(\infty) = z_i$ and $\sigma_i^{-1} P_i \sigma_i$ is the transformation $z \mapsto z + 1$ on $\mathbb{H}$. For each $i = 1, 2, \ldots, n$ and $s \in \mathbb{C}$, the Eisenstein series $E_i(z, s)$ attached to the cusp z_i is given by

$$E_i(z, s) := \sum_{\gamma \in <P_i> \backslash \Gamma} \operatorname{Im}(\sigma_i^{-1} \gamma z)^s, \quad z \in \mathbb{H}. \tag{1.1.1}$$

If $\operatorname{Re} s > 1$, then the above series is uniformly convergent on compact subsets of $\mathbb{H}$. Moreover, $E_i(z, s)$ is invariant under Γ, and thus it descends to a function on X, which we denote by the same symbol. Furthermore, it is well-known that

$$\Delta_{\mathrm{hyp}} E_j = s(s-1) E_j \quad \text{on } X, \tag{1.1.2}$$

where Δ_{hyp} denotes the hyperbolic Laplacian on X (see e.g. [Ku]).

The Teichmüller space $T_{g,n}$ is naturally a complex manifold of dimension $3g-3+n$. To describe its tangent and cotangent spaces at a point X, we first denote by $Q(X)$ the space of holomorphic quadratic differentials $\phi = \phi(z)\, dz^2$ on X with finite L^1 norm, i.e., $\int_X |\phi| < \infty$. Also, we denote by $B(X)$ the space of L^∞ measurable Beltrami differentials $\mu = \mu(z)\, d\bar{z}/dz$ on X (i.e., $\|\mu\|_\infty := \operatorname{ess.sup}_{z \in X} |\mu(z)| < \infty$). Let $HB(X)$ be the subspace of $B(X)$ consisting of elements of the form $\overline{\phi}/\rho$ for some $\phi \in Q(X)$. Here $\rho = \rho(z)\, dz\, d\bar{z}$ denotes the hyperbolic metric on X. Elements of $HB(X)$ are called harmonic Beltrami differentials. There is a natural Kodaira-Serre pairing $\langle\ ,\ \rangle\colon B(X) \times Q(X) \to \mathbb{C}$ given by

$$\langle \mu, \phi \rangle = \int_X \mu(z)\phi(z)\, dz d\bar{z} \tag{1.1.3}$$

for $\mu \in B(X)$ and $\phi \in Q(X)$. Let $Q(X)^\perp \subset B(X)$ be the annihilator of $Q(X)$ under the above pairing. Then one has the decomposition $B(X) = HB(X) \oplus Q(X)^\perp$. It is well-known that one has the following natural isomorphism:

$$\begin{aligned} T_X T_{g,n} &\simeq B(X)/Q(X)^\perp \simeq HB(X), \quad \text{and} \\ T_X^* T_{g,n} &\simeq Q(X) \end{aligned} \tag{1.1.4}$$

with the duality between $T_X T_{g,n}$ and $T_X^* T_{g,n}$ given by (1.1.3). It should be remarked that Bers was responsible for many of the concepts described above (see [Be]).

The Weil-Petersson metric g^{WP} and the Takhtajan-Zograf metric g^{TZ} on $T_{g,n}$ (the latter being introduced in [TZ1] and [TZ2]) are defined as follows (see e.g. [IT, Wolp2] and the references therein for background materials on g^{WP}): for $X \in T_{g,n}$ and $\mu, \nu \in HB(X)$, one has

$$
\begin{aligned}
g^{\mathrm{WP}}(\mu,\nu) &= \int_X \mu\bar{\nu}\rho, \\
g^{\mathrm{TZ}}(\mu,\nu) &= \sum_{i=1}^{n} g^{(i)}(\mu,\nu), \quad \text{where} \\
g^{(i)}(\mu,\nu) &= \int_X E_i(\cdot,2)\mu\bar{\nu}\rho, \quad i = 1,2,\ldots,n
\end{aligned} \tag{1.1.5}
$$

(see (1.1.1)). It follows from results in [A2,Ch,Wolp1,TZ2,O1] that the metrics g^{WP}, $g^{(i)}$, g^{TZ} are all Kählerian and non-complete. Note that g^{TZ} is well-defined only when $n > 0$. Moreover, each $g^{(i)}$ is intrinsic to the corresponding cusp p_i in the sense that if an element γ in the Teichmüller modular group $\mathrm{Mod}_{g,n}$ carries the cusp p_i to another cusp p_j, then γ also carries $g^{(i)}$ to $g^{(j)}$. To facilitate subsequent discussion, we will call $g^{(i)}$ the Takhtajan-Zograf cuspidal metric on $T_{g,n}$ associated to the cusp z_i (or the puncture p_i).

The moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of type (g,n) is obtained as the quotient of $T_{g,n}$ by the Teichmüller modular group $\mathrm{Mod}_{g,n}$, i.e., $\mathcal{M}_{g,n} \simeq T_{g,n}/\mathrm{Mod}_{g,n}$ (see e.g. [N]). As such, $\mathcal{M}_{g,n}$ is naturally endowed with the structure of a complex V-manifold ([Ba]). The metrics g^{WP} and g^{TZ} (but not each individual $g^{(i)}$ unless $n = 1$) are invariant under $\mathrm{Mod}_{g,n}$ and thus they descend to Kähler metrics on (the smooth points of) $\mathcal{M}_{g,n}$, which we denote by the same names and symbols.

1.2. To facilitate the ensuing discussion, we consider some related pseudo-metrics on the associated boundary Teichmüller spaces of $T_{g,n}$.

As in [M] (in the case of $T_{g,0}$), we denote by $\delta_{\gamma_1,\ldots,\gamma_m}T_{g,n}$ the boundary Teichmüller space of $T_{g,n}$ arising from pinching m distinct points. Take a point $X_0 \in \delta_{\gamma_1,\ldots,\gamma_m}T_{g,n}$. Then X_0 is a Riemann surface with n punctures $p_1,\ldots,p_n$ and m nodes $q_1,\ldots,q_m$. Observe that $X_0^o := X\backslash\{q_1,\ldots,q_m\}$ is a non-singular Riemann surface with $n+2m$ punctures. Each node q_i corresponds to two punctures on X_0^o (other than $p_1,\ldots,p_n$). Denote the components of X_0^o by S_α, $\alpha = 1,2,\ldots,d$. Each S_α is a Riemann surface of genus g_α and with n_α punctures, i.e., S_α is of type (g_α, n_α). It will be clear in Sect. 1.3 that we will only need to consider the case where $2g_\alpha - 2 + n_\alpha > 0$ for each α, so that each S_α also admits the complete hyperbolic metric of constant sectional curvature -1. It is easy to see that $\sum_{\alpha=1}^{d}(3g_\alpha - 3 + n_\alpha) + m = 3g - 3 + n$. With respect to the disjoint union $X_0^o = \cup_{\alpha=1}^{d} S_\alpha$, one easily sees that $\delta_{\gamma_1,\ldots,\gamma_m}T_{g,n}$ is a product of lower dimensional Teichmüller spaces given by

$$
\delta_{\gamma_1,\ldots,\gamma_m}T_{g,n} = T_{g_1,n_1} \times T_{g_2,n_2} \times \cdots \times T_{g_d,n_d} \tag{1.2.1}
$$

with each $S_\alpha \in T_{g_\alpha,n_\alpha}$, $\alpha = 1,2,\ldots,d$. Recall that the punctures of S_α arise from either the punctures or the nodes of X_0, and for simplicity, they will be called old cusps and new cusps of S_α respectively. Denote the number of old cusps (resp. new cusps) of S_α by n'_α (resp. n''_α), so that $n_\alpha = n'_\alpha + n''_\alpha$. We index the punctures of S_α such that $\{p_{\alpha,i}\}_{1\le i\le n'_\alpha}$ denotes the set of old cusps, and $\{p_{\alpha,i}\}_{n'_\alpha+1\le i\le n_\alpha}$ denotes the set of new cusps. For each α and i, we denote by $g^{(\alpha,i)}$ the Takhtajan-Zograf cuspidal metric on

T_{g_α,n_α} with respect to the puncture $p_{\alpha,i}$ (cf. (1.1.5)). Now we define a pseudo-metric $\hat{g}^{\mathrm{TZ},\alpha}$ on T_{g_α,n_α} by summing the $g^{(\alpha,i)}$'s over the old cusps, i.e.,

$$\hat{g}^{\mathrm{TZ},\alpha} := \sum_{1\le i\le n'_\alpha} g^{(\alpha,i)}. \tag{1.2.2}$$

If none of the punctures of S_α are old cusps, then $\hat{g}^{\mathrm{TZ},\alpha}$ is simply defined to be zero identically. As such, $\hat{g}^{\mathrm{TZ},\alpha}$ is positive definite precisely when S_α possesses at least one old cusp. Note that by contrast, the Takhtajan-Zograf metric $g^{\mathrm{TZ},\alpha}$ on T_{g_α,n_α} is given by $g^{\mathrm{TZ},\alpha} := \sum_{1\le i\le n_\alpha} g^{(\alpha,i)}$, and $g^{\mathrm{TZ},\alpha}$ is always positive definite.

Definition 1.2.1. *The* **nodally depleted Takhtajan-Zograf pseudo-metric** $\hat{g}^{TZ,(\gamma_1,\dots,\gamma_m)}$ *on* $\delta_{\gamma_1,\dots,\gamma_m}T_{g,n}$ *is defined to be the product pseudo-metric of the* $\hat{g}^{TZ,\alpha}$*'s on the* T_{g_α,n_α}*'s, i.e.,*

$$\left(\delta_{\gamma_1,\dots,\gamma_m}T_{g,n}, \hat{g}^{TZ,(\gamma_1,\dots,\gamma_n)}\right) = \prod_{i=1}^{d}\left(T_{g_\alpha,n_\alpha}, \hat{g}^{TZ,\alpha}\right). \tag{1.2.3}$$

1.3. Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of type (g,n) as in (1.1), and let $\overline{\mathcal{M}}_{g,n}$ denote the Knudsen-Deligne-Mumford stable curve compactification of $\mathcal{M}_{g,n}$ ([DM][KM,Kn]). Like $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$ admits a V-manifold structure, which we describe as follows. Similar description for $\overline{\mathcal{M}}_g$ (i.e., when $n=0$) can be found in [M] or [Wolp3].

Take a point $X_0 \in \overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$. Then X_0 is a stable Riemann surface with n punctures $p_1,\dots,p_n$ and m nodes $q_1,\dots,q_m$ for some $m>0$. Thus we may regard X_0 as a point in $\delta_{\gamma_1,\dots,\gamma_m}T_{g,n}$ (cf. (1.2)). Write $X_0\backslash\{q_1,\dots,q_m\} = \cup_{1\le\alpha\le d}S_\alpha$ and write $\delta_{\gamma_1,\dots,\gamma_m}T_{g,n} = \prod_{\alpha=1}^{d} T_{g_\alpha,n_\alpha}$ with each component $S_\alpha \in T_{g_\alpha,n_\alpha}$ as in Sect. 1.2. Note that since X_0 is stable, each S_α admits the complete hyperbolic metric of constant sectional curvature -1. Also, for some $0<r<1$, each node q_j in X_0 admits an open neighborhood

$$N_j = \{(z_j,w_j)\in\mathbb{C}^2\colon |z_j|,\ |w_j|<r,\ z_j\cdot w_j = 0\} \tag{1.3.1}$$

so that $N_j = N_j^1\cup N_j^2$, where $N_j^1 = \{(z_j,0)\in\mathbb{C}^2\colon |z_j|<r\}$ and $N_j^2 = \{(0,w_j)\in\mathbb{C}^2\colon |w_j|<r\}$ are the coordinate discs in $\mathbb{C}^2$. Without loss of generality, we will assume that r is independent of j, upon shrinking r if necessary. For each α, we choose $3g_\alpha - 3+n_\alpha$ linearly independent Beltrami differentials $\nu_i^{(\alpha)}$, $1\le i\le 3g_\alpha-3+n_\alpha$, which are supported on $S_\alpha\backslash\cup_{j=1}^{n}N_j$, so that their harmonic projections form a basis of $T_{S_\alpha}T_{g_\alpha,n_\alpha}$ (cf. (1.1.4)). For simplicity, we rewrite $\{v_i^{(\alpha)}\}_{1\le\alpha\le d,1\le i\le 3g_\alpha-3+n_\alpha}$ as $\{v_i\}_{1\le i\le 3g-3+n-m}$. Then one has an associated local coordinate neighborhood V of X_0 in $\delta_{\gamma_1,\dots,\gamma_m}T_{g,n}$ with holomorphic coordinates $\tau = (\tau_1,\dots,\tau_{3g-3+n-m})$ such that X_0 corresponds to 0. Shrinking and reparametrizing V if necessary, we may assume $V\simeq\Delta^{3g-3+n-m}$, where $\Delta = \{z\in\mathbb{C}\colon |z|<1\}$ denotes the unit disc in $\mathbb{C}$. For a point $\tau\in V$, one has the associated Beltrami differential $\mu(\tau) = \sum_{i=1}^{3g-3+n-m}\tau_i\, v_i$ and a quasi-conformal homeomorphism $w^{\mu(\tau)}\colon X_0\to X_\tau$ onto a Riemann surface X_τ satisfying

$$\frac{\partial w^{\mu(\tau)}}{\partial\bar{z}} = \mu(z)\frac{\partial w^{\mu(\tau)}}{\partial z}. \tag{1.3.2}$$

The map $w^{\mu(\tau)}$ is conformal on each N_j, $j = 1, \ldots, m$, so that we may regard $N_j \subset X_\tau$ for each j. Then for each $t = (t_1, \ldots, t_m)$ with each $|t_j| < r$, we obtain a new Riemann surface $X_{t,\tau}$ for X_τ by removing the disks $\{z_j \in N_j^1 : |z_j| < |t_j|\}$ and $\{w_j \in N_j^2 : |w_j| < |t_j|\}$ and identifying $z_j \in N_j^1$ with $w_j = t_j/z_j \in N_j^2$, $j = 1, \ldots, m$. Then one obtains a holomorphic family of noded Riemann surfaces $\{X_{t,\tau}\}$ parametrized by the coordinates $(t, \tau) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m})$ of $\Delta^m(r) \times V \simeq \Delta^m(r) \times \Delta^{3g-3+n-m}$, where $\Delta^m(r)$ denotes the m-fold Cartesian product of the disc $\Delta(r) = \{z \in \mathbb{C} : |z| < r\}$ in $\mathbb{C}$. Moreover, the Riemann surfaces $X_{t,\tau}$ with $(t, \tau) \in (\Delta^*(r))^m \times V$ are of type (g, n), where $\Delta^*(r) = \Delta(r)\backslash\{0\}$. The coordinates $t = (t_1, \ldots, t_m)$ will be called pinching coordinates, and $\tau = (t_1, \ldots, t_{3g-3+n-m})$ will be called boundary coordinates. For $1 \le j \le m$, let α_j denote the simple closed curve $|z_j| = |w_j| = |t_j|^{\frac{1}{2}}$ on $X_{t,\tau}$. Shrinking $\Delta^m(r)$ and V if necessary, it is known that the universal cover of $(\Delta^*(r))^m \times V$ is naturally a domain in $T_{g,n}$ and the corresponding covering transformations are generated by a Dehn twist about the α_j's. Since Dehn twists are elements of $\mathrm{Mod}_{g,n}$, the $\mathrm{Mod}_{g,n}$-invariant metrics g^{WP} and g^{TZ} descend to metrics on $(\Delta^*(r))^m \times V$, which we denote by the same symbols and names. It is well-known that each $X_0 \in \overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$ admits an open neighborhood $\hat{U}$ in $\overline{\mathcal{M}}_{g,n}$ together with a local uniformizing chart $\chi : U \simeq \Delta^m(r) \times V \to \hat{U}$ for some $\Delta^m(r) \times V$ as described above, where χ is a finite ramified cover. Obviously the metrics g^{WP} and g^{TZ} on $(\Delta^*(r))^m \times V \subset U$ may also be regarded as extensions of the pull-back of the corresponding metrics on the smooth points of $\hat{U} \cap \mathcal{M}_{g,n}$ via the map χ.

1.4. Before we state our main result, we first need to make the following definition.

Definition 1.4.1. *Let X_0 be a Riemann surface with n punctures $p_1, \ldots, p_n$ and m nodes $q_1, \ldots, q_m$. A node q_i is said to be* **adjacent to punctures (resp. a puncture** p_j**)** *if the component of $X_0\backslash\{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m\}$ containing q_i also contains at least one of the p_j's (resp. the puncture p_j). Otherwise, it is said to be* **non-adjacent to punctures (resp. the puncture** p_j**)**.

Now we are ready to state our main result in the following

Theorem 1. *For $g \ge 0$ and $n > 0$, let $X_0 \in \overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$ be a stable Riemann surface with n punctures $p_1, \ldots, p_n$ and m nodes $q_1, \ldots, q_m$ arranged in such a way that q_i is adjacent (resp. non-adjacent) to punctures for $1 \le i \le m'$ (resp. $m' + 1 \le i \le m$). Let $\hat{U}$ be an open neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}$, together with a local uniformizing chart $\psi : U \simeq \Delta^m(r) \times V \to \hat{U}$, where $V \simeq \Delta^{3g-3+n-m}$ is a domain in the boundary Teichmüller space $\delta_{\gamma_1,\ldots,\gamma_m} T_{g,n}$ corresponding to X_0 and with each γ_i corresponding to q_i. Let $(s_1, \ldots, s_{3g-3+n}) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m}) = (t, \tau)$ be the pinching and boundary coordinates of U, and let the components of the Takhtajan-Zograf metric g^{TZ} be given by*

$$g^{TZ}_{i\bar{j}} = g^{TZ}\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right), \quad 1 \le i, j \le 3g - 3 + n, \tag{1.4.1}$$

on $U^* := (\Delta^*(r))^m \times V \subset U$. Then the following statements hold:

(*i*) *For each $1 \le j \le m$ and any $\varepsilon > 0$, one has*

$$\limsup_{(t,\tau)\in U^* \to (0,0)} |t_j|^2(-\log|t_j|)^{4-\varepsilon} g^{TZ}_{j\bar{j}}(t, \tau) = 0. \tag{1.4.2}$$

(*ii*) *For each* $1 \leq j \leq m'$ *and any* $\varepsilon > 0$*, one has*

$$\liminf_{(t,\tau)\in U^*\to(0,0)} |t_j|^2(-\log|t_j|)^{4+\varepsilon} g^{TZ}_{j\bar{j}}(t,\tau) = +\infty. \tag{1.4.3}$$

(*iii*) *For each* $1 \leq j,\, k \leq m$ *with* $j \neq k$*, one has*

$$\left|g^{TZ}_{j\bar{k}}(t,\tau)\right| = O\left(\frac{1}{|t_j|\,|t_k|\,(\log|t_j|)^3(\log|t_k|)^3}\right) \quad as\ (t,\tau)\in U^*\to(0,0). \tag{1.4.4}$$

(*iv*) *For each* $j\,,k \geq m+1$*, one has*

$$\lim_{(t,\tau)\in U^*\to(0,0)} g^{TZ}_{j\bar{k}}(t,\tau) = \hat{g}^{TZ,(\gamma_1,\ldots,\gamma_m)}_{j\bar{k}}(0,0). \tag{1.4.5}$$

(*v*) *For each* $j \leq m$ *and* $k \geq m+1$*, one has*

$$\left|g^{TZ}_{j\bar{k}}(t,\tau)\right| = O\left(\frac{1}{|t_j|(-\log|t_j|)^3}\right) \quad as\ (t,\tau)\in U^*\to(0,0). \tag{1.4.6}$$

Here in (1.4.5)*,* $\hat{g}^{TZ,(\gamma_1,\ldots,\gamma_m)}_{j\bar{k}}$ *denotes the* $(j,k)^{\text{th}}$ *component of the nodally depleted Takhtajan-Zograf pseudo-metric on* $\delta_{\gamma_1,\ldots,\gamma_m}T_{g,n}$ *(cf. Definition* 1.2.1*).*

Remark 1.4.2. (i) Theorem 1(i) is equivalent to the following statement: For each $1 \leq j \leq m$ and any $\varepsilon > 0$, there exists a constant $C_{1,\varepsilon} > 0$ (depending on ϵ) such that

$$g^{\mathrm{TZ}}_{j\bar{j}}(t,\tau) \leq \frac{C_{1,\varepsilon}}{|t_j|^2(-\log|t_j|)^{4-\varepsilon}} \quad \text{for all } (t,\tau)\in U^*. \tag{1.4.7}$$

Similarly, Theorem 1(ii) is equivalent to the following statement: For each $1 \leq j \leq m'$ and any $\varepsilon > 0$, there exists a constant $C_{2,\varepsilon} > 0$ (depending on ϵ) such that

$$g^{\mathrm{TZ}}_{j\bar{j}}(t,\tau) \geq \frac{C_{2,\varepsilon}}{|t_j|^2(-\log|t_j|)^{4+\varepsilon}} \quad \text{for all } (t,\tau)\in U^*. \tag{1.4.8}$$

(ii) In view of Theorem 1(i) and (ii), it is natural to ask the following question: Does the stronger estimate

$$g^{\mathrm{TZ}}_{j\bar{j}}(t,\tau) \sim \frac{1}{|t_j|^2(-\log|t_j|)^4} \text{ hold for } 1 \leq j \leq m' \text{ and } (t,\tau)\in U^*? \tag{1.4.9}$$

The methods of this paper does not seem to generalize easily to answer this question.

2. The Hyperbolic Metric

2.1. In Sect. 2, we are going to give some uniform estimates for the family of hyperbolic metrics near the punctures and nodes of degenerating Riemann surfaces. For a degenerating family of compact Riemann surfaces (i.e. $n = 0$), Wolpert [Wolp3] has developed from the prescribed curvature equation results which are stronger than what is described in this section. Since the estimates in the form that we need in our ensuing discussion were discussed explicitly only in the case when $n = 0$ in [M] and [Wolp3], we include here the modifications arising from the punctures for the convenience of the reader. Throughout this article, hyperbolic metrics will always be normalized to be of constant sectional curvature -1. First we have

Lemma 2.1.1. *Let S be a hyperbolic punctured Riemann surface with hyperbolic metric ρ. Let $\Delta^*(r) = \{z \in \mathbb{C}: |z| < r\}$ (with $r > 0$) be a punctured coordinate neighborhood of a puncture p of S with the origin 0 corresponding to p. Write $\rho = \rho(z)\, dz \otimes d\bar{z}$ on $\Delta^*(r)$. Then one has*

$$\lim_{z\to 0} |z|^2 (\log |z|)^2 \rho(z) = 1.$$

Proof. First recall from (1.1) that we may write $S = \mathbb{H}/\Gamma$, where $\mathbb{H} = \{Z = X + iY \in \mathbb{C}: Y > 0\}$ and $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$. Moreover, upon conjugation by an element in $\mathrm{PSL}(2, \mathbb{R})$ if necessary, we may assume that the puncture p corresponds to the cusp ∞ and the subgroup Γ_∞ of Γ fixing ∞ is generated by the transformation $Z \mapsto Z + 1$. It is well-known that for some $R > 0$, one has $\gamma A \cap A = \emptyset$ for some $\gamma \in \Gamma_\infty\backslash\Gamma$, where $A = \{Z = X + iY \in \mathbb{H}: Y > R\}$ (cf. e.g. [FK, p. 216] or Remark-Definition 2.1.2(ii) below). It follows that the function

$$w(Z) = e^{2\pi i Z} \tag{2.1.1}$$

on A descends to the coordinate function on the punctured coordinate neighborhood $\Delta^*(r_0) = \{w \in \mathbb{C}: 0 < |w| < r_0\}$ of p in S with p corresponding to the origin 0, where $0 < r_0 = e^{-2\pi R} < 1$. Being descended from the hyperbolic metric $dZ \otimes d\bar{Z}/Y^2$ on $\mathbb{H}$, one easily sees that

$$\rho = \frac{dw \otimes d\bar{w}}{|w|^2(\log |w|)^2} \quad \text{on} \quad \Delta^*(r_0). \tag{2.1.2}$$

Now, if z is any coordinate function of S near p with $z(p) = 0$. Then w can be regarded as a holomorphic function of z near p with $w(0) = 0$ and $C := w'(0) \neq 0$. By Taylor's theorem, we have

$$w = Cz + O(z^2) \quad \text{and} \quad w'(z) = C + O(z)$$

as $z \to 0$. Together with (2.1.2), it follows that ρ is given in terms of z near p by

$$\rho = \frac{|w'(z)|^2 dz \otimes d\bar{z}}{|w(z)|^2(\log |w(z)|)^2} = \frac{|C + O(z)|^2 dz \otimes d\bar{z}}{|z|^2|C + O(z)|^2(\log |z| + \log |C + O(z)|)^2},$$

and upon letting $z \to 0$, Lemma 2.1.1 follows immediately. □

Remark-Definition 2.1.2. (i) For simplicity, a local holomorphic coordinate function w of a hyperbolic Riemann surface S defined near a puncture p with $w(p) = 0$ will be said to be *standard* if it is descended from the Euclidean coordinate function on $\mathbb{H}$ via (2.1.1) (so that the hyperbolic metric ρ of S satisfies (2.1.2) near p). If S has nodes, such a definition will also be applied to the punctures of $S \setminus \{\text{nodes}\}$ (instead of S). As seen above, such standard coordinate functions always exist near the punctures of S.
(ii) It follows from the collar lemma for non-compact surfaces (cf. e.g. [Bu, Theorem 4.4.6, p.111-112] that we may always take $R = \frac{1}{2}$ (and thus $r_o = e^{-\pi}$) in the proof of Lemma 2.1.1.

Notation as in §1. Let $X_0 \in \overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$ be a Riemann surface with n punctures $p_1, \ldots, p_n$ and m nodes $q_1, \ldots, q_m$, and let $\hat{U}$ be an open neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}$ together with a local uniformizing chart $\chi: U \to \hat{U}$, where $U \simeq \Delta^m(r) \times V = \{(t, \tau) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m}): t \in \Delta^m(r), \tau \in V\}$ and $V \simeq \Delta^{3g-3+n-m}$ is an open coordinate neighborhood of X_0 in $\delta_{\gamma_1,\ldots,\gamma_m} T_{g,n}$ as in (1.3). Let $\mathcal{X} := \{X_{t,\tau}\}_{(t,\tau)\in U}$ be the corresponding family of Riemann surfaces parametrized by U with $X_0 = X_{(0,0)}$, and let $\pi: \mathcal{X} \to U$ denote the holomorphic projection map. Fix a puncture p_i of X_0.

Shrinking U if necessary, it is easy to see that there exists an open coordinate subset $W_i = \Delta^*(R) \times U$ of $\mathcal{X}$ such that $\pi|_{W_i}$ is given by the projection onto the second factor, and each point $(0, (t, \tau))$ corresponds to the puncture on $X_{t,\tau}$ associated to p_i (in particular, $(0, (0, 0))$ corresponds to p_i itself). Shrinking R and V if necessary, we will assume without loss of generality that R is independent of i, and each $W_i \subset\subset W_i'$ for some similarly defined open coordinate subset $\mathcal{X}$ of the form

$$W_i' = \Delta^*(R') \times U' \quad \text{with} \quad U' = \Delta^m(r') \times V', \quad V' \simeq \Delta^{3g-3+n-m}(\delta) \tag{2.1.3}$$

for some $0 < R < R' < 1$, $0 < r < r' < 1$ and $\delta > 1$. For each $(t, \tau) \in U$, we denote the hyperbolic metric on $X_{t,\tau}$ by $\rho_{t,\tau}$, and we denote $W_{i,t,\tau} := W_i \cap X_{t,\tau} \simeq \Delta^*(R)$ and $W'_{i,t,\tau} := W_i' \cap X_{t,\tau} \simeq \Delta^*(R')$. We also write

$$\rho_{t,\tau} = \rho_{t,\tau}(z_i) dz_i \otimes d\bar{z}_i = \rho(z_i, t, \tau) dz_i \otimes d\bar{z}_i \quad \text{on } W'_{i,t,\tau}. \tag{2.1.4}$$

Then it follows from a result of Bers [Be] that the function $\rho(z_i, t, \tau)$ on W_i' is locally uniformly continuous in all the variables.

Proposition 2.1.3. (*i*) *For each* $1 \le i \le n$, *there exist constants* $C_1, C_2 > 0$ *such that for all* $(t, \tau) \in U$, *one has*

$$\frac{C_1}{|z_i|^2(\log|z_i|)^2} \le \rho_{t,\tau}(z_i) \le \frac{C_2}{|z_i|^2(\log|z_i|)^2} \quad \textit{on } W_{i,t,\tau}. \tag{2.1.5}$$

(*ii*) (*Strengthened version of* (*i*)) *If, in addition,* z_i *is a standard local holomorphic coordinate function for* X_0 (*cf. Remark-Definition 2.1.2*), *then the inequality in* (2.1.5) *remains valid with the constants* C_1, C_2 *replaced by positive continuous functions* $C_{1,t,\tau}$, $C_{2,t,\tau}$ *(depending on* t, τ*) respectively and satisfying*

$$C_{1,t,\tau},\ C_{2,t,\tau} \to 1 \quad \textit{as } (t, \tau) \to (0, 0). \tag{2.1.6}$$

Proof. For simplicity, we will drop the subscript i, so that $W = W_i$, $W_{t,\tau} = W_{i,t,\tau}$, $W'_{t,\tau} = W'_{i,t,\tau}$, $z = z_i$, etc. First we remark that it is well-known (and follows also from the arguments in Lemma 2.1.1) that (2.1.5) holds for a fixed punctured Riemann surface; in other words, there exist constants $C_{1,0,0},\ C_{2,0,0} > 0$ such that

$$\frac{C_{1,0,0}}{|z|^2(\log|z|)^2} \le \rho_{0,0}(z) \le \frac{C_{2,0,0}}{|z|^2(\log|z|)^2} \quad \text{on } W'_{0,0} \simeq \Delta^*(R'). \tag{2.1.7}$$

For each $(t, \tau) \in U'$, since $\rho_{t,\tau}$ is of constant sectional curvature -1, it follows that one has

$$\Delta_0 \log \rho(z, t, \tau) = 2\rho(z, t, \tau) \quad \text{on } \Delta^*(R'), \tag{2.1.8}$$

where $\Delta_0 := \partial^2/\partial x^2 + \partial^2/\partial y^2$ (with $z = x + iy$) is the Euclidean Laplacian. Consider the continuous function

$$f(z, t, \tau) = \log \frac{\rho(z, t, \tau)}{\rho_{0,0}(z)} \quad \text{on } W' \simeq \Delta^*(R') \times U'. \tag{2.1.9}$$

We extend f to a function on $\Delta(R') \times U'$ by letting

$$f(0, t, \tau) = 0 \quad \text{for all } (t, \tau) \in U'. \tag{2.1.10}$$

Then it follows from Lemma 2.1.1 that for fixed $(t,\tau)\in U'$, $f(z,t,\tau)$ is continuous in the variable $z\in\Delta(R')$. By applying the Mean Value Theorem to the real exponential function, one easily sees that for $(t,\tau)\in U'$ and $z\in\Delta^*(R)$,

$$\begin{aligned}\Delta_0 f(z,t,\tau) =& 2\left(\rho(z,t,\tau)-\rho_{0,0}(z)\right) \quad \text{(by (2.1.8), (2.1.9))}\\ =& 2\left(e^{\log\rho(z,t,\tau)}-e^{\log\rho_{0,0}(z)}\right)\\ =& 2e^{\eta} f(z,t,\tau)\end{aligned} \tag{2.1.11}$$

for some real number $\eta=\eta(z,t,\tau)$ between $\log\rho(z,t,\tau)$ and $\log\rho_{0,0}(z)$. By the maximum principle, it follows from (2.1.10) and (2.1.11) that for each $(t,\tau)\in U'$, one has

$$\max_{z\in\bar\Delta(R)} f(z,t,\tau)\le \max\{0,\max_{z\in\partial\Delta(R)} f(z,t,\tau)\}, \tag{2.1.12}$$

where $\bar\Delta(R)=\{z\in\mathbb{C}: |z|\le R\}$ and $\partial\Delta(R)=\{z\in\mathbb{C}: |z|=R\}$. By applying the above arguments to the function $-f$, one also easily sees that for each $(t,\tau)\in U'$,

$$\min_{z\in\bar\Delta(R)} f(z,t,\tau)\ge \min\{0,\min_{z\in\partial\Delta(R)} f(z,t,\tau)\}. \tag{2.1.13}$$

Observe also that $f(z,0,0)=0$ for all $z\in\Delta(R')$. Together with the uniform continuity of $f(z,t,\tau)$ on the compact set $\partial\Delta(R)\times\overline{U}\subset W'$, where $\overline{U}\simeq\bar\Delta^m(r)\times\bar\Delta^{3g-3+n-m}\subset U'$, it follows readily that there exists positive continuous functions $C_{1,t,\tau}$, $C_{2,t,\tau}$ on $\overline{U}$ (which can be taken to be the exponential of the right-hand side of (2.1.13) and (2.1.12) respectively) such that $C_{1,0,0}=C_{2,0,0}=1$ and

$$C_{1,t,\tau}\rho_{0,0}(z)\le\rho(z,t,\tau)\le C_{2,t,\tau}\rho_{0,0}(z) \quad \text{for all } (t,\tau)\in\overline{U} \text{ and } z\in\Delta^*(R), \tag{2.1.14}$$

which, together with (2.1.7), lead to Proposition 2.1.3(i). Proposition 2.1.3(ii) is an immediate consequence of (2.1.14). □

2.2. Next we consider the behavior of the family of hyperbolic metrics near the nodes. Let $U=\Delta^m(r)\times V$ be as in Sect. 2.1, and fix a node q_j of X_0, where $1\le j\le m$. Then it follows readily from Sect. 1.3 (and with slight abuse of notation (cf. (1.3.1)) that there exists a local coordinate neighborhood $N_j=\Delta^{m+1}(r)\times V$ of q_j in $\mathcal{X}$ such that for fixed $(t,\tau)\in U$ with $t=(t_1,\dots,t_m)$, the set $N_{j,t,\tau}:=N_j\cap X_{t,\tau}$ is given by

$$\begin{aligned}N_{j,t,\tau} =& \{(t_1,\dots,t_{j-1},z_j,w_j,t_{j+1},\dots,t_m,\tau)\in N_j: z_jw_j=t_j,\ \frac{|t_j|}{r}<|z_j|<r\}\\ =& \{(t_1,\dots,t_{j-1},z_j,w_j,t_{j+1},\dots,t_m,\tau)\in N_j: z_jw_j=t_j,\ \frac{|t_j|}{r}<|w_j|<r\}.\end{aligned} \tag{2.2.1}$$

When $t_j\neq 0$, one can identify $N_{j,t,\tau}$ as an annulus via coordinate projections as

$$N_{j,t,\tau}\leftrightarrow\{z_j\in\mathbb{C}: \frac{|t_j|}{r}<|z_j|<r\}\leftrightarrow\{w_j\in\mathbb{C}: \frac{|t_j|}{r}<|w_j|<r\}. \tag{2.2.2}$$

Note that when $t_j=0$, $N_{j,t,\tau}$ consists of two open coordinate discs of radius r corresponding to the cases when $|z_j|<r$, $w_j=0$ and when $|w_j|<r$, $z_j=0$ respectively.

In terms of the coordinates t, τ and either z_j or w_j, we may also write $N_j = N_j^1 \cup N_j^2$, where

$$N_j^1 := \{(z_j, t, \tau) \in \Delta(r) \times U \,\big|\, |t_j|^{\frac{1}{2}} \le |z_j| < r\}, \quad \text{and}$$
$$N_j^2 := \{(w_j, t, \tau) \in \Delta(r) \times U \,\big|\, |t_j|^{\frac{1}{2}} \le |w_j| < r\}. \tag{2.2.3}$$

For each $(t, \tau) \in U$, we also denote

$$N_{j,t,\tau}^1 := N_j^1 \cap X_{t,\tau} \simeq \{z_j \in \mathbb{C}\colon |t_j|^{\frac{1}{2}} \le |z_j| < r\}, \quad \text{and}$$
$$N_{j,t,\tau}^2 := N_j^2 \cap X_{t,\tau} \simeq \{w_j \in \mathbb{C}\colon |t_j|^{\frac{1}{2}} \le |w_j| < r\}. \tag{2.2.4}$$

Recall also from Sect. 2.1 that, shrinking r if necessary, we will assume without loss of generality that each $N_j \subset\subset N_j'$ for some similarly defined local coordinate neighborhood $N_j' = \Delta^{m+1}(r') \times V$ of q_j in $\mathcal{X}$ with $r < r' < 1$, and thus we have corresponding similarly defined sets $N_j^{1,\prime}$, $N_j^{2,\prime}$, $N_{j,t,\tau}^{1,\prime}$, $N_{j,t,\tau}^{2,\prime}$, etc. For $(t, \tau) \in U'$ with $t_j \ne 0$, we define the function on $N_{j,t,\tau}$ given by

$$\rho_{j,t,\tau}^*(z_j) := \left(\frac{\pi}{|z_j| \log |t_j|} \csc \frac{\pi \log |z_j|}{\log |t_j|}\right)^2 \tag{2.2.5}$$

via the first identification of (2.2.2). Observe that the expression for $\rho_{j,t,\tau}^*$ actually does not depend on τ or t_k for $k \ne j$. It is also easy to see that $\rho_{j,t,\tau}^*$ is given by a similar expression in terms of the coordinate w_j. For $(t, \tau) \in U'$ with $t_j = 0$, we define $\rho_{j,t,\tau}^*$ by

$$\rho_{j,t,\tau}^*(z_j) = \frac{1}{|z_j|^2 (\log |z_j|)^2} \quad \text{on the } z_j\text{-coordinate disc,} \tag{2.2.6}$$

and by a similar expression on the w_j-coordinate disc. Then it is well-known and easy to see that the $\rho_{j,t,\tau}^*$'s glue together to form a continuous function on N_j' (with singularity along the complex analytic subset $z_j = w_j = 0$ of complex codimension two), which we denote by ρ_j^*. Moreover, for each $(t, \tau) \in U$ with $t_j \ne 0$,

$$\rho_{j,t,\tau}^* := \rho_{j,t,\tau}^*(z_j) dz_j \otimes d\bar{z}_j = \rho_{j,t,\tau}^*(w_j) dw_j \otimes d\bar{w}_j \tag{2.2.7}$$

is the restriction of the complete hyperbolic metric on the annulus $\{z_j \in \mathbb{C}\colon |t_j| < |z_j| < 1\} (\supset N_{j,t,\tau}')$; when $t_j = 0$, similar statements also hold for the two corresponding punctured coordinate discs. For fixed $(t, \tau) \in U'$ with $t_j \ne 0$, we write

$$\rho_{t,\tau} = \rho_{t,\tau}(z_j) dz_j \otimes d\bar{z}_j = \rho_{t,\tau}(w_j) dw_j \otimes d\bar{w}_j \quad \text{on } N_{j,t,\tau}'. \tag{2.2.8}$$

Proposition 2.2.1. *(i) For each $1 \le j \le m$, there exist constants $C_3, C_4 > 0$ such that for all $(t, \tau) \in U$ with $t_j \ne 0$, one has*

$$C_3 \rho_{j,t,\tau}^*(z_j) \le \rho_{t,\tau}(z_j) \le C_4 \rho_{j,t,\tau}^*(z_j) \quad \textit{on } N_{j,t,\tau}'. \tag{2.2.9}$$

A similar inequality also holds for the coordinate w_j. In particular, there exist constants $C_5, C_6 > 0$ such that for all $(t, \tau) \in U$ with $t_j \ne 0$, one has

$$\frac{C_5}{|z_j|^2 (\log |z_j|)^2} \le \rho_{t,\tau}(z_j) \le \frac{C_6}{|z_j|^2 (\log |z_j|)^2} \quad \textit{on} \quad N_{j,t,\tau}^1. \tag{2.2.10}$$

A similar inequality (with z_j replaced by w_j) also holds for the region $N_{j,t,\tau}^2$.

(ii) (Strengthened version of (i)) If, in addition, z_j and w_j are standard local holomorphic coordinate functions for X_0 (cf. Remark-Definition 2.1.2), then the inequalities in (2.2.9) and (2.2.10) remain valid with the constants C_3, C_4, C_5, C_6 replaced by positive continuous functions $C_{3,t,\tau}$, $C_{4,t,\tau}$, $C_{5,t,\tau}$, $C_{6,t,\tau}$ (depending on $(t,\tau) \in U$ (with $t_j \neq 0$)) respectively and satisfying

$$C_{3,t,\tau},\ C_{4,t,\tau},\ C_{5,t,\tau},\ C_{6,t,\tau} \to 1 \quad \text{as } (t,\tau) \to (0,0). \tag{2.2.11}$$

Proof. The proof of (i) for $T_{g,n}$ with $n > 0$ is the same as the case of $T_{g,0}$ given in [M, p. 632]. Next we recall Bers' result [Be] which implies that $\rho(z_j, t, \tau)$ is locally uniformly continuous in all variables at points where $z_j \neq 0$. Then the proof of (ii) follows from this result and a simple adaptation of that of (i) in a manner similar to Proposition 2.1.3, which will be left to the reader. □

3. Regular Quadratic Differentials and the Weil-Petersson Metric

3.1. To facilitate the ensuing discussion, we recall in this section Masur's construction in [M] of a certain local basis of regular quadratic differentials for a degenerating family of punctured Riemann surfaces. The concept of regular quadratic differentials dates back to earlier works of Bers (see e.g. [Be]). Since only the case of a degenerating family of compact Riemann surfaces was explicitly discussed in [M], we will indicate briefly the necessary modifications arising from the punctures of the Riemann surfaces for the convenience of the reader. Similar to [M, p. 627], we first have

Definition 3.1.1. (*a*) *Let X be a Riemann surface with possibly both punctures and nodes, and denote the smooth part of X by X^o. For $k = 1, 2$, a* **regular** k**-differential** ϕ *on X is a holomorphic section of $K^k_{X^o}$ such that (i) ϕ has at most a simple pole at each puncture of X; and (ii) ϕ has at most a pole of order k at each of the two punctures of X^o associated to a node of X; moreover, the residues of ϕ at each such pair of punctures are equal if $k = 2$ and opposite if $k = 1$. Here we recall that the residue of $\phi = \phi(z)dz^k$ at a point $z = 0$ is given locally by the residue of the abelian differential $\phi(z)z^{k-1}dz$.*
(*b*) *For $k = 1, 2$, a* **regular** k**-differential** *on a family of Riemann surfaces with punctures and nodes is a holomorphic function element on the total space which restricts to a regular k-differential on each fiber.*

We remark that when X has no punctures, the above definition is standard and well-known (see e.g. [M, §4]). When X has no nodes, the space of regular 2-differential on X coincides with the space of integrable holomorphic quadratic differentials on X.

Let $X_0 \in \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ be a Riemann surface with punctures $p_1, \ldots, p_n$ and nodes $q_1, \ldots, q_m$, and let $\hat{U}$ be an open neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}$ with a local uniformizing chart $\psi\colon U \simeq \Delta^m(r) \times V \to \hat{U}$, where $V \simeq \Delta^{3g-3+n-m}$ is a domain in a suitable boundary Teichmüller space $\delta_{\gamma_1,\ldots,\gamma_m} T_{g,n}$ as in Sect. 1.3. Also, we let $\pi\colon \mathcal{X} = \{X_{t,\tau}\}_{(t,\tau)\in U} \to U$ be the corresponding degenerating family of Riemann surfaces associated to a choice of Beltrami differentials $\nu_1, \ldots, \nu_{3g-3+n-m}$ on X_0 as in Sect. 1.3. As in the case of $\mathcal{M}_{g,0}$ in [M, p. 625-626] and for each $1 \leq i \leq 3g-3+n-m$, the coordinate tangent vector $\partial/\partial\tau_i$ at $(t,\tau) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m}) \in U$ is identified with the Beltrami differential

$$\frac{\nu_i}{1-|\mu(\tau)|^2} \cdot \frac{w_z^{\mu(\tau)}}{\bar{w}_{\bar{z}}^{\mu(\tau)} \circ (w^{\mu(\tau)})^{-1}} \quad \text{on } X_{t,\tau}, \tag{3.1.1}$$

where $w^{\mu(\tau)}$ is as in Sect. 1.3, and like the ν_i's, it is easily seen to be of compact support away from the punctures and nodes of $X_{t,\tau}$. In addition, for each $1 \leq j \leq m$, the tangent vector $\partial/\partial t_j$ at $(t, \tau) \in U$ is identified with the Beltrami differential

$$\frac{\partial}{\partial t_j}(t, \tau) \leftrightarrow \frac{1}{2t_j \log|t_j|} \frac{z_j}{\bar{z}_j} \frac{d\bar{z}_j}{dz_j} = \frac{1}{2t_j \log|t_j|} \frac{w_j}{\bar{w}_j} \frac{d\bar{w}_j}{dw_j} \tag{3.1.2}$$

supported on $N_{j,t,\tau} \subset X_{t,\tau}$, where $N_{j,t,\tau}$ is as in (2.2.1) (cf. [M, p. 626]). Recall from (2.2) the open coordinate neighborhood $N_j = \Delta^{m+1}(r) \times V$ of each node q_j of X_0 in $\mathcal{X}$ with the corresponding decomposition $N_j = N_j^1 \cup N_j^2$ given in (2.2.3). Recall also from (2.1) the open coordinate neighborhood $W_i = \Delta^*(R) \times U$ of each puncture p_i of X_0 in $\mathcal{X}$. It is also clear from the constructions in (1.3) that $\mathcal{X} \setminus (\cup_{i=1}^n \{W_i\} \cup \cup_{j=1}^m \{N_j\})$ can be covered by a finite number of coordinate neighborhoods $\{A_\ell\}_{1 \leq \ell \leq \ell_o}$ of $\mathcal{X}$, where each A_ℓ is of the form

$$A_\ell = \Delta(r_\ell) \times U \quad \text{with } A_{\ell,t,\tau} := A_\ell \cap X_{t,\tau} = \Delta(r_\ell) \times \{(t, \tau)\} \tag{3.1.3}$$

for each $(t, \tau) \in U$. Here, $\ell_o \in \mathbb{Z}^+$, and $\Delta(r_\ell) = \{z_\ell \in \mathbb{C} \mid |z_\ell| < r_\ell\}$ with $r_\ell > 0$. For each non-empty subset $J \subset \{1, 2, \ldots, m\}$, let $B(J) = \{(t, \tau) \in U \mid t_j = 0 \text{ for all } j \in J\}$, and let $T(J) = \{\partial/\partial t_j \mid 1 \leq j \leq m,\ j \notin J\} \cup \{\partial/\partial \tau_\ell \mid 1 \leq \ell \leq 3g-3+n-m\}$. Let $U^* \simeq (\Delta^*(r))^m \times V \subset U$ be as in Theorem 1. Shrinking U if necessary, one has

Proposition 3.1.2. ([M]). *There exist regular* 2*-differentials* $\phi_k = \phi_k(z, t, \tau)dz^2$, $k = 1, 2, \ldots, 3g - 3 + n$, *on* $\mathcal{X} = \{X_{t,\tau}\}_{(t,\tau)\in U}$ *satisfying the following properties:*
(*i*) *At each* $(t, \tau) \in U^*$, $\{\phi_k\}_{1\leq k\leq 3g-3+n}$ *forms a basis of regular* 2*-differentials on* $X_{t,\tau}$ *dual to the ordered set of tangent vectors*

$$\{\partial/\partial t_j\}_{1\leq j\leq m} \cup \{\partial/\partial \tau_\ell\}_{1\leq \ell\leq 3g-3+n-m}$$

via the identifications (3.1.1), (3.1.2) *and with respect to the pairing in* (1.1.3).
(*ii*) *For each non-empty subset* $J \subset \{1, 2, \ldots, m\}$, $\phi_k \equiv 0$ *on* $B(J)$ *for each* $k \in J$, *and* $\{\phi_k\}_{k\in\{1,\ldots,3g-3+n\}\setminus J}$ *is dual to the ordered set* $T(J)$ *on* $B(J)$ *with respect to the pairing in* (1.1.3).
(*iii*) *For each* $1 \leq k, j \leq m$, *one has, on* N_j^1,

$$\phi_k(z_j, t, \tau) = -\frac{t_k}{\pi}\left[\frac{\delta_{kj}}{z_j^2} + a_{-1}(z_j, t, \tau) + \frac{1}{z_j^2}\sum_{\ell=1}^{\infty}\left(\frac{t_k}{z_j}\right)^{\ell} t_j^{\kappa(\ell)} a_\ell(t, \tau)\right], \tag{3.1.4}$$

where δ_{kj} *is the Kronecker symbol, each integer* $\kappa(\ell) \geq 0$, a_{-1} *has at most a simple pole at* $z_j = 0$, *and* a_ℓ $(\ell \geq 1)$ *is holomorphic. In particular, there exist constants* $C_1, C_2, C_3 > 0$ *such that on* N_j^1, *one has*

$$\begin{cases} C_1\dfrac{|t_j|}{|z_j|^2} \leq |\phi_j(z_j, t, \tau)| \leq C_2\dfrac{|t_j|}{|z_j|^2} & \text{if } 1 \leq j \leq m, \\ \qquad\qquad |\phi_k(z_j, t, \tau)| \leq C_3\dfrac{|t_k|}{|z_j|} & \text{if } 1 \leq k \neq j \leq m. \end{cases} \tag{3.1.5}$$

Similar expressions hold on N_j^2 *with respect to the* (w_j, t, τ)*-coordinates.*

(*iv*) *For each* $m+1 \le k \le 3g-3+n$ *and* $1 \le j \le m$*, one has, on* N_j^1*,*

$$\phi_k(z_j,t,\tau) = \phi_k(z_j,0,0) + \frac{1}{z_j^2}\sum_{\ell=1}^{\infty}\left(\frac{t_j}{z_j}\right)^{\ell} t_j^{\tilde{\kappa}(\ell)} b_\ell(t,\tau) + \sum_{\ell=-1}^{\infty} z_j^{\ell} c_\ell(t,\tau), \qquad (3.1.6)$$

where each integer $\tilde{\kappa}(\ell) \ge 0$*,* $\phi_k(z_j,0,0)$ *has at most a simple pole at* $z_j = 0$*, and* b_ℓ*,* c_ℓ *are holomorphic with* $c_\ell(0,0) = 0$*. In particular, there exists a constant* $C_4 > 0$ *such that on* N_j^1*, one has*

$$|\phi_k(z_j,t,\tau)| \le \frac{C_4}{|z_j|} \quad \text{if } m+1 \le k \le 3g-3+n \text{ and } 1 \le j \le m. \qquad (3.1.7)$$

Similar expressions hold on N_j^2 *with respect to the* (w_j,t,τ)*-coordinates.*
(*v*) *For each* $1 \le i \le n$*, one has, on* $W_i = \Delta^*(R) \times U$*,*

$$\phi_k(z_i,t,\tau) = \begin{cases} -\dfrac{t_k}{\pi}\dfrac{d_k(z_i,t,\tau)}{z_i} & \text{if } 1 \le k \le m, \\ \dfrac{d_k(z_i,t,\tau)}{z_i} & \text{if } m+1 \le k \le 3g-3+n, \end{cases} \qquad (3.1.8)$$

where each $d_k(z_i,t,\tau)$ *is holomorphic on* W_i*. In particular, there exist constants* C_5*,* $C_6 > 0$ *such that on* W_i*, one has*

$$|\phi_k(z_i,t,\tau)| \le \begin{cases} C_5\dfrac{|t_k|}{|z_i|} & \text{if } 1 \le k \le m, \\ \dfrac{C_6}{|z_i|} & \text{if } m+1 \le k \le 3g-3+n. \end{cases} \qquad (3.1.9)$$

(*vi*) *For each* $1 \le \ell \le \ell_o$ *and* $1 \le k \le 3g-3+n$*,* $\phi_k(z_\ell,t,\tau)$ *is holomorphic on* $A_\ell = \Delta(r_\ell) \times U$*. Moreover, for* $1 \le k \le m$*, one has, on* A_ℓ*,*

$$\phi_k(z_\ell,t,\tau) = -\frac{t_k}{\pi} e_k(z_\ell,t,\tau) \qquad (3.1.10)$$

for some holomorphic function $e_k(z_\ell,t,\tau)$*. In particular, upon shrinking* r_ℓ *if necessary, there exist constants* C_7*,* $C_8 > 0$ *such that on* A_ℓ*, one has*

$$|\phi_k(z_\ell,t,\tau)| \le \begin{cases} C_7|t_k| & \text{if } 1 \le k \le m, \\ C_8 & \text{if } m+1 \le k \le 3g-3+n. \end{cases} \qquad (3.1.11)$$

Proof. The proof in the general case when $n > 0$ follows mutatis mutandis from the discussions of the case when $n = 0$ in [M, §4, §5 and §7], to which we refer the reader for details. Here we only indicate the necessary modifications arising from the punctures. By adjoining n points to each fiber $X_{t,\tau}$ corresponding to the punctures, one has an associated family of compact Riemann surfaces $\bar{\pi}: \bar{\mathcal{X}} \to U$, where the punctures of the $X_{t,\tau}$'s correspond to n non-intersecting holomorphic sections of $\bar{\pi}$, which we denote by $\sigma_1^{(p)}, \dots, \sigma_n^{(p)}$. Applying the arguments of [M, Lemma 4.3], one can produce a regular 1-differential $\psi = \psi(z,t,\tau)dz$ and $2g-2$ disjoint holomorphic sections $\sigma_1 \dots, \sigma_{2g-2}$ of the family $\bar{\pi}: \bar{\mathcal{X}} \to U$ such that each $\sigma_i(t,\tau)$ is a zero of $\psi(z,t,\tau)dz$ and each $\sigma_i(t,\tau)$ misses the nodes and the punctures of $X_{t,\tau}$. Then using ψ and the

$2g-2+n$ disjoint sections $\sigma_1, \ldots, \sigma_{2g-2}$, $\sigma_1^{(p)}, \ldots, \sigma_n^{(p)}$, one can produce the desired regular 2-differentials by following the arguments in [M, §5 and §7]. Finally we remark that (3.1.5) (resp. (3.1.7)) follows readily from (3.1.4) (resp. (3.1.6)) and the inequality $|t_j|^{\frac{1}{2}} \leq |z_j| < r$ which holds on N_j^1 (cf. (2.2.3)). □

3.2. Next we recall the well-known result of Masur [M] on the asymptotic behavior of the Weil-Petersson metric g^{WP} on $T_{g,n}$. It should be remarked that this result has been improved recently by Wolpert [Wolp5] and Obitsu-Wolpert [OW], where information on higher order terms are obtained. Masur's original result will be sufficient for our purpose. As in Sect. 3.1, since only the case when $n = 0$ was explicitly discussed in [M], we will indicate briefly the modifications needed for the case when $n > 0$ for the convenience of the reader.

Proposition 3.2.1. ([M]). *For $g \geq 0$ and $n > 0$, let $X_0 \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ with local uniformizing chart $\psi : U \simeq \Delta^m(r) \times V \to \hat{U}$, where $V \simeq \Delta^{3g-3+n-m} \subset \delta_{\gamma_1,\ldots,\gamma_m} T_{g,n}$, $U^* \simeq (\Delta(r)^*)^m \times V \subset U$, and corresponding local coordinates*

$$(s_1, \ldots, s_{3g-3+n}) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m}) = (t, \tau)$$

be as in Theorem 1. Denote the components of the Weil-Petersson metric g^{WP} by

$$g_{i\bar{j}}^{WP} = g^{WP}\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right), \quad 1 \leq i, j \leq 3g-3+n,$$

on U^*. Then the following statements hold:
(i) For each $1 \leq j \leq m$, one has

$$\begin{aligned} 0 < & \liminf_{(t,\tau)\in U^* \to (0,0)} |t_j|^2(-\log|t_j|)^3 g_{j\bar{j}}^{WP}(t,\tau) \\ \leq & \limsup_{(t,\tau)\in U^* \to (0,0)} |t_j|^2(-\log|t_j|)^3 g_{j\bar{j}}^{WP}(t,\tau) < \infty. \end{aligned} \tag{3.2.1}$$

(ii) For each $1 \leq j, k \leq m$ with $j \neq k$, one has

$$\left|g_{j\bar{k}}^{WP}(t,\tau)\right| = O\left(\frac{1}{|t_j|\,|t_k|\,(\log|t_j|)^3(\log|t_k|)^3}\right) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{3.2.2}$$

(iii) For each $j, k \geq m+1$, one has

$$\lim_{(t,\tau)\in U^* \to (0,0)} g_{j\bar{k}}^{WP}(t,\tau) = g_{j\bar{k}}^{WP}(0,0), \tag{3.2.3}$$

where $g_{j\bar{k}}^{WP}(0,0)$ denotes the $(j,k)^{th}$ component of the Weil-Petersson metric on the boundary Teichmüller space $\delta_{\gamma_1,\ldots,\gamma_m} T_{g,n}$ at X_0.
(iv) For each $1 \leq j \leq m$ and $k \geq m+1$, one has

$$\left|g_{j\bar{k}}^{WP}(t,\tau)\right| = O\left(\frac{1}{|t_j|(-\log|t_j|)^3}\right) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{3.2.4}$$

Proof. The proof in the general case when $n > 0$ follows mutatis mutandis from the arguments for the case when $n = 0$ in [M, §7, proof of Theorem 1] with [M, Prop. 7.1] replaced by Proposition 3.1.2. For $1 \le i \le n$ and $(t, \tau) \in U$, let $W_{i,t,\tau}$ be as in (2.1). We remark that the only extra integral estimates needed are those on the $W_{i,t,\tau}$'s as follows:

$$\int_{W_{i,t,\tau}} \frac{\phi_k \overline{\phi_\ell}}{\rho_{t,\tau}} = \begin{cases} O(|t_k||t_\ell|) & \text{if } 1 \le k, \ell \le m, \\ O(|t_k|) & \text{if } 1 \le k \le m \text{ and } m+1 \le \ell \le 3g-3+n, \end{cases} \tag{3.2.5}$$

as $(t, \tau) \to (0, 0)$, and for $m + 1 \le k, \ell \le 3g - 3 + n$,

$$\lim_{(t,\tau)\to(0,0)} \int_{W_{i,t,\tau}} \frac{\phi_k \overline{\phi_\ell}}{\rho_{t,\tau}} = \int_{W_{i,0,0}} \frac{\phi_k \overline{\phi_\ell}}{\rho_{0,0}}. \tag{3.2.6}$$

The estimates in (3.2.5) and the limit in (3.2.6) follow readily from a straightforward calculation using Proposition 2.1.3(i), Proposition 3.1.2 and the dominated convergence theorem. □

4. Estimates on the Eisenstein Series

In this section, we are going to obtain some estimates on the Eisenstein series $E(z, s)$ in the setting of holomorphic families of degenerating punctured Riemann surfaces, which will be needed for ensuing discussion in §5. We should clarify that by an Einsenstein series, we will mean here the real-analytic (non-holomorphic) series as in (1.1.1) or [Ku] rather than more customary holomorphic ones. Our approach is geometrical in nature, and it consists largely of constructing suitable germs of comparison functions for the Eisenstein series near the nodes and punctures. Starting from Sect. 4.2, we will restrict our discussions to $E(z, 2)$, although most of our discussions will also be valid for $E(z, s)$ with $\mathrm{Re}\, s > 1$.

4.1. First we extend the definition of Eisenstein series to the case of punctured Riemann surfaces with nodes.

Let X be a stable connected Riemann surface with n punctures $p_1, \dots, p_n$ and m nodes $q_1, \dots, q_m$. Then $X^o := X \setminus \{q_1, \dots, q_m\}$ is a smooth punctured Riemann surface with $n + 2m$ punctures, and we denote the connected components of X° by S_α, $\alpha = 1, \dots, d$ (cf. Sect. 1.2). We denote the new punctures by $p_{n+1}, \dots, p_{n+2m}$. Each old or new puncture p_i, $1 \le i \le n + 2m$, of X° is a puncture of a unique $S_{\alpha(i)}$ for some $1 \le \alpha(i) \le d$.

Definition 4.1.1. *For $1 \le i \le n + 2m$ and $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$, the* **Eisenstein series** *$E_i(\cdot, s)$* **on** *X attached to p_i is defined by*

$$E_i(z, s) = \begin{cases} E_{i,S_{\alpha(i)}}(z, s) & \text{if } z \in S_{\alpha(i)}, \\ 0 & \text{if } z \in X \setminus S_{\alpha(i)}, \end{cases} \tag{4.1.1}$$

where $E_{i,S_{\alpha(i)}}(\cdot, s)$ is the corresponding Eisenstein series on $S_{\alpha(i)}$ attached to p_i given as in (1.1.1).

In the case when X has no nodes, it is well known that for $1 \leq i, j \leq n$ and in terms of the Euclidean coordinate $Z = X + iY$ on $\mathbb{H}$ with p_j corresponding to ∞ (as in the proof of Lemma 2.1.1), there exists some constant $c > 0$ such that

$$E_i(Z, s) = \delta_{ij} Y^s + \phi_{ij}(s) Y^{1-s} + o(e^{-cY}) \quad \text{as } Y \to \infty, \tag{4.1.2}$$

where δ_{ij} is the Kronecker symbol, $(\phi_{ij}(s))$ is a symmetric $n \times n$ matrix (cf. e.g. [Ku] and [Wolp4, p.260]). In this section, we are going to give a variant version of (4.1.2) for a Riemann surface X with nodes. For a point $z \in X^\circ$, we denote by injrad(z) the injectivity radius of X° at z with respect to the complete hyperbolic metric on X°.

Proposition 4.1.2. *Notation as above. Fix an integer* $1 \leq i \leq n + 2m$*, and let* $s \in \mathbb{C}$ *be a fixed number with* $\mathrm{Re}\, s > 1$.
(*i*) *Let* z_i *be a standard local holomorphic coordinate function around* p *(cf. Remark-Definition 2.1.2 (i) and (ii)). Then for any* $\epsilon > 0$*, there exists a constant* $C_{s,\epsilon} > 0$ *(depending only on* s*,* ϵ *and indpendent of* X*) such that*

$$\Big| E_i(z_i, s) - \left(-\frac{\log |z_i|}{2\pi}\right)^s \Big| \leq C_{s,\epsilon} \quad \text{on } \Delta^*(e^{-2\pi e^\epsilon}) := \{z_i \in \mathbb{C} \,\big|\, 0 < |z_i| < e^{-2\pi e^\epsilon}\}. \tag{4.1.3}$$

(*ii*) *For any* $\kappa > 0$*, there exists a constant* $C'_{s,\kappa} > 0$ (*depending only on* s *and* κ) *such that*

$$|E_i(z, s)| \leq C'_{s,\kappa} \quad \text{for any } z \in X^\circ \text{ with injrad}(z) \geq \kappa. \tag{4.1.4}$$

(*iii*) *For* $1 \leq i \neq j \leq n + 2m$*, one has*

$$E_i(z, s) \to 0 \quad \text{as } z \to p_j. \tag{4.1.5}$$

Proof. We remark that to prove (i), (ii) and (iii), it follows readily from (4.1.1) that we may assume without loss of generality that X° is connected with p_i (and possibly p_j) as one of its punctures. To prove (i), we recall from (1.1) that we may write $X^\circ = \mathbb{H}/\Gamma$, where Γ is a Fuchsian group which uniformizes X with ∞ corresponding to p_i, and the infinite cyclic subgroup $\Gamma_\infty \subset \Gamma$ generated by $Z \to Z + 1$, $Z \in \mathbb{H}$, corresponds to the parabolic transformations of Γ fixing p_i. Let $\mathcal{C}_\infty := \{Z \in \mathbb{C} \mid \mathrm{Im}\, Z \geq 1\}$ be a horoball around ∞ in $\mathbb{H}$. As mentioned in Remark-Definition 2.1.2(ii), it follows from the collar lemma for non-compact surfaces ([Bu, Theorem 4.4.6, p.112]) that $\mathcal{C}_\infty$ descends under the projection map

$$z_i = e^{2\pi i Z} \tag{4.1.6}$$

to the punctured coordinate neighborhood $\Delta^*(e^{-2\pi}) := \{z_i \in \mathbb{C} \,\big|\, 0 < |z_i| < e^{-2\pi}\}$ around p_i. Let $\epsilon > 0$ be a given constant. We recall from [Ku, Sect. 1.3] (cf. also [O p.146]) the following integral representation for the Eisenstein series: For $Z \in \mathbb{H}$, one has

$$\Lambda_\epsilon(s) E_i(Z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \iint_{B(\gamma Z, \epsilon)} Y'^{s-2}\, dX' dY', \quad \text{where} \tag{4.1.7}$$

$$\Lambda_\epsilon(s) := \iint_{B(i,\epsilon)} Y'^{s-2}\, dX' dY'. \tag{4.1.8}$$

Here $Z' = X' + iY'$ denotes the Euclidean coordinate function on $\mathbb{H}$, and $B(Z', \epsilon)$ denotes the hyperbolic geodesic ball in $\mathbb{H}$ of radius ϵ and with center at Z'. From (4.1.7), we have

$$\begin{aligned}\Lambda_\epsilon(s)E_i(Z,s) &= \iint_{B(Z,\epsilon)} Y'^{s-2} dX'dY' + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \gamma\neq id}} \iint_{B(\gamma Z,\epsilon)} Y'^{s-2} dX'dY' \\ &= \Lambda_\epsilon(s)(\mathrm{Im}\ Z)^s + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \gamma\neq id}} \iint_{B(\gamma Z,\epsilon)} Y'^{s-2} dX'dY', \end{aligned} \tag{4.1.9}$$

where the second line is obtained by making the change of variable $Z'' = (\mathrm{Im}\, Z)^{-1} \cdot (Z' - \mathrm{Re}\, Z)$ in the first integral of the first line and then invoking the definition of $\Lambda_\epsilon(s)$ in (4.1.8). (Note that the above change of variable corresponds to a hyperbolic isometry on $\mathbb{H}$.) Next we find an absolute bound for the last term of (4.1.9) by adapting the proof of Theorem 2.1.2 in [Ku, p.12]. Let $\mathcal{C}_\infty^\epsilon := \{Z \in \mathbb{C} \mid \mathrm{Im}\ Z \geq e^\epsilon\}$, which is easily seen to descend under the map in (4.1.6) to the punctured coordinate neighborhood $\Delta^*(e^{-2\pi e^\epsilon})$ $(\subset \Delta^*(e^{-2\pi}))$ in (4.1.3). It is easy to see that for any $Z \in \mathcal{C}_\infty^\epsilon$ and $\gamma \in \Gamma$, one has

$$B(Z,\epsilon) \subset \mathcal{C}_\infty, \quad \text{and} \quad B(\gamma Z,\epsilon) = \gamma(B(Z,\epsilon)) \subset \gamma(\mathcal{C}_\infty) = \mathcal{C}_{\gamma(\infty)}, \tag{4.1.10}$$

where $\mathcal{C}_{\gamma(\infty)}$ denotes the corresponding horoball around the cusp $\gamma(\infty)$, which is isometric to $\mathcal{C}_{(\infty)}$ via γ. By the collar lemma mentioned above, all the horoballs $\mathcal{C}_{\gamma(\infty)}$, $\gamma \in \Gamma_\infty \setminus \Gamma$, are mutually disjoint. It follows that all the hyperbolic geodesic balls $B(\gamma Z, \epsilon)$, $id \neq \gamma \in \Gamma_\infty\backslash\Gamma$, are mutually disjoint, and thus they may be considered to be disjoint subsets of

$$\{Z \in \mathbb{H} \mid -1 \leq \mathrm{Re}\ Z \leq 2,\ 0 < \mathrm{Im}\, Z \leq e^\epsilon\}, \tag{4.1.11}$$

after we choose suitable representatives in the coset decomposition of $\Gamma_\infty\backslash\Gamma$. Together with (4.1.9), it follows that for any $Z \in \mathcal{C}_\infty^\epsilon$, one has

$$\begin{aligned}\left|\Lambda_\epsilon(s)\right| \left|E_i(Z,s) - (\mathrm{Im}\ Z)^s\right| &\leq \iint_{\substack{-1\leq X'\leq 2 \\ 0\leq Y'\leq e^\epsilon}} \left|Y'^{s-2}\right| dX'dY' \\ &= \frac{3}{\mathrm{Re}\, s - 1} e^{\epsilon(\mathrm{Re}\, s-1)}. \end{aligned} \tag{4.1.12}$$

Observe from (4.1.8) that $\left|\Lambda_\epsilon(s)\right| = \iint_{B(i,\epsilon)} Y'^{\mathrm{Re}\, s-2} dX'dY' > 0$ and it depends only on s and ϵ. By descending the inequality in (4.1.12) on $\mathcal{C}_\infty^\epsilon$ to the corresponding inequality on $\Delta^*(e^{-2\pi e^\epsilon})$ via the map in (4.1.6), one easily sees that (4.1.3) holds with the constant given by

$$C_{s,\epsilon} = \frac{3}{\left|\Lambda_\epsilon(s)\right|(\mathrm{Re}\, s - 1)} e^{\epsilon(\mathrm{Re}\, s-1)}, \tag{4.1.13}$$

and this finishes the proof of (i). Next we proceed to give the proof of (ii), which is similar to that of (i). Let $p\colon \mathbb{H} \to X^\circ$ denote the covering space projection. Let $z \in X^\circ$ be a point with $\mathrm{injrad}(z) \geq \kappa$, and fix a point $Z \in \mathbb{H}$ such that $p(Z) = z$. With $\Gamma_\infty \subset \Gamma$ and other notations as in (i) above, it is easy to see that $\mathrm{injrad}(p(Z')) \geq \frac{\kappa}{2}$ for any $Z' \in B(\gamma Z, \frac{\kappa}{2})$ and any $\gamma \in \Gamma_\infty\backslash\Gamma$. For any $Z' \in \mathbb{H}$, it is easy to calculate that the hyperbolic length of the horizontal line segment from Z' to $Z'+1$ is $\frac{1}{\operatorname{Im} Z'}$, which implies readily that $\operatorname{injrad}(p(Z')) \le \frac{1}{\operatorname{Im} Z'}$ (since $p(Z') = p(Z'+1)$). Hence we have

$$\operatorname{Im} Z' \le \frac{2}{\kappa} \quad \text{for all } Z' \in B(\gamma Z, \frac{\kappa}{2}),\ \gamma \in \Gamma_\infty \backslash \Gamma. \tag{4.1.14}$$

The condition $\operatorname{injrad}(z) \ge \kappa$ also implies readily that the geodesic balls $B(\gamma Z, \frac{\kappa}{2})$, $\gamma \in \Gamma_\infty \backslash \Gamma$, are mutually disjoint, and thus similar to (4.1.11), they may be regarded as disjoint subsets of

$$\{Z \in \mathbb{H} \mid -1 \le \operatorname{Re} Z \le 2,\ 0 < \operatorname{Im} Z \le \frac{2}{\kappa}\}. \tag{4.1.15}$$

Together with (4.1.7) and (4.1.8) (and with $\epsilon = \frac{\kappa}{2}$), it follows as in (4.1.12) that one has

$$\begin{aligned} \left|\Lambda_{\frac{\kappa}{2}}(s)\right| \left|E_i(Z,s)\right| &\le \iint_{\substack{-1 \le X' \le 2 \\ 0 \le Y' \le \frac{2}{\kappa}}} \left|Y'^{s-2}\right| dX' dY' \\ &= \frac{3}{(\operatorname{Re} s - 1)} \cdot \left(\frac{2}{\kappa}\right)^{\operatorname{Re} s - 1}. \end{aligned} \tag{4.1.16}$$

By descending the above inequality to X^o, one easily sees that (4.1.4) holds with the constant given by

$$C'_{s,\kappa} = \frac{3}{\left|\Lambda_{\frac{\kappa}{2}}(s)\right| (\operatorname{Re} s - 1)} \cdot \left(\frac{2}{\kappa}\right)^{\operatorname{Re} s - 1}, \tag{4.1.17}$$

and this finishes the proof of (ii). Finally one easily sees that (4.1.5) is a direct consequence of (4.1.2), and this finishes the proof of Proposition 4.1.1. □

4.2. Upper bound of $E_{i,t,\tau}$ near a node. Notation as in §1. Let $X_0 \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ be a Riemann surface with n punctures at $p_1, \dots, p_n$ and m nodes at $q_1, \dots, q_m$, and let $\hat{U}$ be an open neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}$ together with a local uniformizing chart $\chi: U \to \hat{U}$, where $U \simeq \Delta^m(r) \times V = \{(t,\tau) = (t_1, \dots, t_m, \tau_1, \dots, \tau_{3g-3+n-m}) : t \in \Delta^m(r), \tau \in V\}$, and $V \simeq \Delta^{3g-3+n-m}$ is an open coordinate neighborhood of X_0 in $\delta_{\gamma_1,\dots,\gamma_m} T_{g,n}$ as in (1.3). Let $\mathcal{X} := \{X_{t,\tau}\}_{(t,\tau)\in U}$ be the corresponding family of Riemann surfaces parametrized by U with $X_0 = X_{(0,0)}$. Let $U^* \simeq (\Delta^*(r))^m \times V \subset U$ be as in Theorem 1. For each $1 \le i \le n$ and $(t,\tau) \in U$, we denote the Eisenstein series with $S = 2$ on $X_{t,\tau}$ associated to the puncture corresponding to p_i by $E_{i,t,\tau}$ (à la Definition 4.1.1 when some $t_j = 0$). It is well-known that $\{E_{i,t,\tau}\}_{(t,\tau)\in U^*}$ form a continuous family of functions on $\{X_{(t,\tau)}\}_{(t,\tau)\in U^*}$. The following proposition follows from previous work of Obitsu [O2]:

Proposition 4.2.1. ([O2]). *For each $i = 1, \dots, n$, $E_{i,t,\tau}$ converges uniformly on compact subsets of $X_0 \setminus \{p_1, \dots, p_n, q_1, \dots, q_m\}$ to $E_{i,0,0}$ as $(t,\tau) \in U^* \to (0,0)$.*

Here, it is easy to see that a compact set $K \subset X_0 \setminus \{p_1, \dots, p_n, q_1, \dots, q_m\}$ can be extended to a neighborhood of the form $K \times U$ in the total space of $\{X_{t,\tau}\}_{(t,\tau)\in U}$, shrinking U if necessary. Therefore, $E_{i,t,\tau}$ may be regarded as a function on K for (t,τ) sufficiently close to $(0,0)$.

For a fixed integer i with $1 \leq i \leq n$, we are going to give a pointwise upper bound of $E_{i,t,\tau}$ near a node q_j with $1 \leq j \leq m$. Let $N_j = \Delta^{m+1}(r) \times V \subset\subset \Delta^{m+1}(r') \times V' = N'_j$ (with $0 < r < r' < 1$), $N_{j,t,\tau}$, $N'_{j,t,\tau}$, $N^1_{j,t,\tau}$, $N^2_{j,t,\tau}$, z_j, w_j be as in Sect. 2.2. Motivated by (2.2.5) and (2.2.6), we consider a family of comparison functions for the $E_{i,t,\tau}$'s as follows: For each $(t,\tau) \in U$ with $t_j \neq 0$, we let

$$E^*_{t,\tau}(z_j) := -\frac{\pi}{\log|t_j| \cdot \sin\left(\frac{\pi \log|z_j|}{\log|t_j|}\right)} \quad \text{on } N'_{j,t,\tau}. \tag{4.2.1}$$

For each $(t,\tau) \in U$ with $t_j = 0$, recall that $N'_{j,t,\tau}$ consists of two discs $\{z_j \in \mathbb{C} \mid |z_j| < r'\}$ and $\{w_j \in \mathbb{C} \mid |w_j| < r'\}$, and we let

$$E^*_{t,\tau}(\cdot) := \begin{cases} -\dfrac{1}{\log|z_j|} & \text{on the } z_j\text{-disc}, \\ -\dfrac{1}{\log|w_j|} & \text{on the } w_j\text{-disc}. \end{cases} \tag{4.2.2}$$

As in Sect. 2.2, it is easy to see that the $E^*_{t,\tau}$'s glue together to form a positive continuous function on $N'_j \setminus \{\text{nodes}\}$. We write $\|(t,\tau)\| = \sqrt{\sum_{j=1}^m |t_j|^2 + \sum_{k=1}^{3g-3+n-m} |\tau_k|^2}$ for $(t,\tau) \in U$.

Proposition 4.2.2. *For fixed $1 \leq i \leq n$, $1 \leq j \leq m$ and $0 < \alpha < 1$, there exist constants C_1, C_2, $\delta > 0$ such that for all $(t,\tau) \in U$ with $t_j \neq 0$ and satisfying $\|(t,\tau)\| < \delta$, one has*

$$E_{i,t,\tau} \leq C_1 (E^*_{t,\tau})^\alpha \quad \text{on } N_{j,t,\tau}, \text{ so that} \tag{4.2.3}$$

$$E_{i,t,\tau}(z_j) \leq \frac{C_2}{(-\log|z_j|)^\alpha} \quad \text{on } N^1_{j,t,\tau}, \tag{4.2.4}$$

and a similar inequality (with z_j replaced by w_j) holds on $N^2_{j,t,\tau}$.

Proof. First we consider the special case when at $(t,\tau) = (0,0)$, z_j, w_j are standard local holomorphic coordinates for X_0 (cf. Remark-Definition 2.1.2). Consider the operator

$$\Delta_j := 4\frac{\partial^2}{\partial z_j \partial \bar{z}_j} \quad \text{on } N_{j,t,\tau}. \tag{4.2.5}$$

(Note that in terms of real coordinates, one has $\Delta_j = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}$, where $z_j = x_j + iy_j$.) By direct calculation, one can check that for $(t,\tau) \in U$ with $t_j \neq 0$,

$$\begin{aligned} \Delta_j E^*_{t,\tau}(z_j) &= \left(1 + \cos^2\left(\frac{\pi \log|z_j|}{\log|t_j|}\right)\right) E^*_{t,\tau}(z_j) \rho^*_{j,t,\tau}(z_j) \\ &\leq \frac{2}{C_{3,t,\tau}} E^*_{t,\tau}(z_j) \rho_{t,\tau}(z_j) \quad \text{on } N_{j,t,\tau} \quad \text{(by Proposition 2.2.1)}, \end{aligned} \tag{4.2.6}$$

where $\rho^*_{j,t,\tau}$, $C_{3,t,\tau}$ and $\rho_{t,\tau}(z_j)$ are as in (2.2.5), (2.2.10) and (2.2.7) respectively. For $(t,\tau)\in U$ with $t_j\neq 0$, it follows from the chain rule that

$$\begin{aligned}\Delta_j(E^*_{t,\tau})^\alpha &= 4\alpha(\alpha-1)(E^*_{t,\tau})^{\alpha-2}|\partial_{z_j}E^*_{t,\tau}|^2+\alpha(E^*_{t,\tau})^{\alpha-1}\Delta_j E^*_{t,\tau}\\ &\le \frac{2\alpha}{C_{3,t,\tau}}(E^*_{t,\tau}(z_j))^\alpha\rho_{t,\tau}(z_j)\quad \text{on } N_{j,t,\tau}\\ &\qquad\text{(by (4.2.6) and since } 0<\alpha<1). \end{aligned}\tag{4.2.7}$$

On the other hand, for $(t,\tau)\in U$ with $t_j\neq 0$, it follows from (1.1.2) and Definition 4.1.1 that

$$\Delta_j E_{i,t,\tau}(z_j)=2E_{i,t,\tau}(z_j)\rho_{t,\tau}(z_j)\quad \text{on } N_{j,t,\tau},\tag{4.2.8}$$

and a similar expression holds for the w_j-coordinate. For each $(t,\tau)\in U$, the boundary $\partial N_{j,t,\tau}$ of $N_{j,t,\tau}$ consists of two circles $|z_j|=r$ and $|w_j|=r$, and it is easy to see that $\cup_{(t,\tau)\in U}\partial N_{j,t,\tau}$ forms a compact subset of N'_j. It follows readily from Proposition 2.2.1 that for any $(t,\tau)\subset U$ and any point z on $\partial N_{j,t,\tau}$, the injectivity radius of $(X_{t,\tau},\ \rho_{t,\tau})$ at z is uniformly bounded below by some constant $\kappa>0$ independent of (t,τ). Thus, by Proposition 4.1.2(ii), there exists a constant $C>0$ such that

$$E_{i,t,\tau}(z)\le C\quad \text{for all } (t,\tau)\in U \text{ and } z\in\partial N_{j,t,\tau}.\tag{4.2.9}$$

It is also easy to see from (4.2.1) and (4.2.2) that there exists a constant $C^*>0$ such that

$$E^*_{t,\tau}(z)\ge C^*\quad \text{for all } (t,\tau)\in U \text{ and } z\in\partial N_{j,t,\tau}.\tag{4.2.10}$$

Let $C_1=\frac{C}{(C^*)^\alpha}>0$. Then it follows from (4.2.9) and (4.2.10) that for all $(t,\tau)\in U$, one has

$$E_{i,t,\tau}(z_j)-C_1(E^*_{t,\tau}(z_j))^\alpha\le 0\quad \text{on } \partial N_{j,t,\tau}.\tag{4.2.11}$$

Since $\alpha<1$, it follows from Proposition 2.2.1 that there exists a constant $\delta>0$ such that

$$C_{3,t,\tau}\ge\alpha\quad \text{for all } (t,\tau)\in U \text{ satisfying } t_j\neq 0 \text{ and } \|(t,\tau)\|<\delta.\tag{4.2.12}$$

Combining (4.2.7), (4.2.8) and (4.2.12), one easily sees that for all $(t,\tau)\in U$ satisfying $t_j\neq 0$ and $\|(t,\tau)\|<\delta$, one has

$$\Delta_j(E_{i,t,\tau}(z_j)-C_1(E^*_{t,\tau}(z_j))^\alpha)\ge 2(E_{i,t,\tau}(z_j)-C_1(E^*_{t,\tau}(z_j))^\alpha)\rho_{t,\tau}(z_j)\quad \text{on } N_{j,t,\tau}.\tag{4.2.13}$$

By using the maximum principle, one easily obtains (4.2.3) as a consequence of (4.2.11) and (4.2.13). Then (4.2.4) follows readily from (4.2.3), (2.2.4) and the boundedness of the function $x/\sin x$ for $0<x\le\frac{\pi}{2}$, and this finishes the proof of the proposition in the special case when at $(t,\tau)=(0,0)$, z_j, w_j are standard local holomorphic coordinates for X_0. Finally we remark that the general case of the proposition follows readily from the above special case by performing a change of variable and adjusting the values of C_1 and C_2 in (4.2.3) and (4.2.4) if necessary. □

4.3. Integral lower bound of $E_{i,t,\tau}$ near an adjacent node. Settings, notations and definitions are as in Sect. 4.2. We are going to derive a desired integral lower bound for $E_{i,t,\tau}$ on the region $N_{j,t,\tau}$ associated to any node q_j adjacent to p_i (cf. Remark 4.3.2).

Proposition 4.3.1. *Let $1 \leq i \leq n$, $1 \leq j \leq m$ be such that the node q_j of X_0 is adjacent to the puncture p_i, and let $\phi_j = \phi_j(z, t, \tau)dz^2$ be as in Proposition 3.1.2. Then for any fixed $\beta > 1$, there exist constants $C = C(\beta)$, $\delta = \delta(\beta) > 0$ such that for all $(t, \tau) \in U^*$ satisfying $\|(t, \tau)\| < \delta$, one has*

$$\int_{N_{j,t,\tau}} E_{i,t,\tau} \frac{\phi_j \overline{\phi_j}}{\rho_{t,\tau}} \geq C|t_j|^2(-\log|t_j|)^{3-\beta}. \tag{4.3.1}$$

Proof. As in Proposition 4.2.2, we will assume without loss of generality that at $(t, \tau) = (0, 0)$, z_j, w_j are standard local holomorphic coordinates for X_0. Consider the biholomorphism from N_j onto itself given by

$$\sigma(t_1, \ldots, t_{j-1}, z_j, w_j, t_{j+1}, \ldots, t_m, \tau) = (t_1, \ldots, t_{j-1}, w_j, z_j, t_{j+1}, \ldots, t_m, \tau)$$

in terms of the coordinates in (2.2.1). For each fixed $(t, \tau) \in U$ with $t_j \neq 0$ (and upon suppressing the coordinates $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m, \tau$), it is easy to see that σ restricts to a biholomorphism $\sigma_{j,t,\tau}\colon N^1_{j,t,\tau} \to N^2_{j,t,\tau}$ given by

$$\sigma_{j,t,\tau}(z_j, w_j) = (w_j, z_j) \quad \text{with } z_j w_j = t_j \tag{4.3.2}$$

(cf. (2.2.3) and (2.2.4)). With $\rho^*_{j,t,\tau}$ as given in (2.2.7) (see also (2.2.5)), it is easy to see that $\sigma_{j,t,\tau}$ induces the following isometry between $N^1_{j,t,\tau}$ and $N^2_{j,t,\tau}$:

$$\sigma^*_{j,t,\tau}\rho^*_{j,t,\tau} = \rho^*_{j,t,\tau}. \tag{4.3.3}$$

Let $C_1 > 0$ be as in (3.1.5). Then it follows from Proposition 3.1.2 and (4.3.2) that for all $(t, \tau) \in U$, one has

$$|\phi_j(z_j, t, \tau)|,\ |\phi_j(\sigma_{j,t,\tau}(z_j), t, \tau)| \geq C_1 \frac{|t_j|}{|z_j|^2} \quad \text{on } N^1_{j,t,\tau}. \tag{4.3.4}$$

Using the limit $\lim\limits_{x\to 0} x \csc x = 1$, it is easy to see from (2.2.5) that for all $(t, \tau) \in U$ with $t_j \neq 0$, one has

$$\rho^*_{j,t,\tau}(z_j) \leq \frac{C_{t_j}}{|z_j|^2(\log|z_j|)^2} \quad \text{on } N^1_{j,t,\tau}, \tag{4.3.5}$$

where C_{t_j} is a positive continuous function in the variable t_j such that

$$C_{t_j} \to 1 \quad \text{as } t_j \to 0. \tag{4.3.6}$$

Let $C_{4,t,\tau}$ be as in (2.2.11). By Proposition 2.2.1, we have

$$\begin{aligned}
&\int_{N_{j,t,\tau}} E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho_{t,\tau}}\\
&\geq \frac{1}{C_{4,t,\tau}}\left(\int_{N^1_{j,t,\tau}} E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho^*_{j,t,\tau}}+\int_{N^2_{j,t,\tau}} E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho^*_{j,t,\tau}}\right)\\
&= \frac{1}{C_{4,t,\tau}}\int_{N^1_{j,t,\tau}}\left(E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho^*_{j,t,\tau}}+\sigma^*_{j,t,\tau}E_{i,t,\tau}\frac{\sigma^*_{j,t,\tau}\phi_j\overline{\sigma^*_{j,t,\tau}\phi_j}}{\sigma^*_{j,t,\tau}\rho^*_{j,t,\tau}}\right)\\
&= \frac{1}{C_{4,t,\tau}}\int_{N^1_{j,t,\tau}}\left(\frac{E_{i,t,\tau}(z_j)|\phi_j(z_j,t,\tau)|^2}{\rho^*_{j,t,\tau}(z_j)}\right.\\
&\quad\left.+\frac{E_{i,t,\tau}(\sigma_{j,t,\tau}(z_j))|\phi_j(\sigma_{j,t,\tau}(z_j),t,\tau)|^2}{\rho^*_{j,t,\tau}(z_j)}\right)dz_jd\overline{z_j}\quad\text{(by (4.3.3))}\\
&\geq \frac{C_1^2}{C_{4,t,\tau}C_{t_j}}\int_{N^1_{j,t,\tau}}\big(E_{i,t,\tau}(z_j)+E_{i,t,\tau}(\sigma_{j,t,\tau}(z_j))\big)\cdot\frac{|t_j|^2}{|z_j|^4}\cdot|z_j|^2(\log|z_j|)^2\,dz_jd\overline{z_j}\\
&\quad\text{(by (4.3.2), (4.3.4) and (4.3.5)).}
\end{aligned}\tag{4.3.7}$$

In polar coordinates, we write $z_j = r_je^{i\theta_j}$, $t_j = |t_j|e^{i\psi_j}$, and write $E_{i,t,\tau}(z_j) = E_{i,t,\tau}(r_j,\theta_j)$, so that $E_{i,t,\tau}(\sigma_{j,t,\tau}(z_j)) = E_{i,t,\tau}(\frac{|t_j|}{r_j},\psi_j-\theta_j)$. Then (4.3.7) can be re-written in the following form:

$$\begin{aligned}
\int_{N_{j,t,\tau}} E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho_{t,\tau}} &\geq \frac{C_1^2|t_j|^2}{C_{4,t,\tau}C_{t_j}}\int_{|t_j|^{\frac12}}^{r} f_{t,\tau}(r_j)\cdot\frac{(\log r_j)^2}{r_j}\,dr_j,\quad\text{where}\\
f_{t,\tau}(r_j) &:= \int_0^{2\pi}\left(E_{i,t,\tau}(r_j,\theta_j)+E_{i,t,\tau}(\frac{|t_j|}{r_j},\psi_j-\theta_j)\right)d\theta_j\\
&= \int_0^{2\pi}\left(E_{i,t,\tau}(r_j,\theta_j)+(\sigma^*_{j,t,\tau}E_{i,t,\tau})(r_j,\theta_j)\right)d\theta_j.
\end{aligned}\tag{4.3.8}$$

It is easy to see that the $f_{t,\tau}$'s (with $(t,\tau)\in U^*$) form a continuous family of functions and each $f_{t,\tau}$ is a smooth function in the variable r_j. Moreover, one also has $f_{t,\tau}(r_j) = f_{t,\tau}(\frac{|t_j|}{r_j})$ for all $\frac{|t_j|}{r}\leq r_j\leq r$, which implies readily that

$$f'_{t,\tau}(|t_j|^{\frac12}) = 0. \tag{4.3.9}$$

Consider the differential operator

$$\widetilde{\Delta}_j := \frac{1}{r_j}\frac{\partial}{\partial r_j}\left(r_j\frac{\partial}{\partial r_j}\right),\quad\text{so that } \Delta_j = \widetilde{\Delta}_j + \frac{1}{r_j^2}\frac{\partial^2}{\partial\theta_j^2}, \tag{4.3.10}$$

where Δ_j is as in (4.2.5). Also, we denote the hyperbolic Laplacian on $N_{j,t,\tau}$ with respect to $\rho^*_{j,t,\tau}$ by $\Delta^*_{j,t,\tau}$, so that in terms of the z_j-coordinate, one has

$$\Delta^*_{j,t,\tau} = (\rho^*_{j,t,\tau}(z_j))^{-1}\Delta_j, \tag{4.3.11}$$

and a similar expression holds for the w_j-coordinate. The isometric property of $\sigma_{j,t,\tau}$ in (4.3.3) implies readily that

$$\Delta^*_{j,t,\tau}(\sigma^*_{j,t,\tau}E_{i,t,\tau}) = \sigma^*_{j,t,\tau}(\Delta^*_{j,t,\tau}E_{i,t,\tau}). \tag{4.3.12}$$

From the analogues of (4.2.8) and (4.3.11) for the w_j-coordinate, one has

$$\Delta^*_{j,t,\tau}E_{i,t,\tau}(w_j) = 2E_{i,t,\tau}(w_j)\cdot\frac{\rho_{t,\tau}(w_j)}{\rho^*_{j,t,\tau}(w_j)} \quad \text{on } N^2_{j,t,\tau}. \tag{4.3.13}$$

Upon pulling back by $\sigma_{j,t,\tau}$ and using Proposition 2.2.1, one obtains from (4.3.12) and (4.3.13) that on $N^1_{j,t,\tau}$,

$$\begin{aligned} &\Delta^*_{j,t,\tau}(\sigma^*_{j,t,\tau}E_{i,t,\tau})(z_j) \le 2\sigma^*_{j,t,\tau}E_{i,t,\tau}(z_j)\cdot C_{4,t,\tau}\\ \Longrightarrow\ &\Delta_j(\sigma^*_{j,t,\tau}E_{i,t,\tau})(z_j) \le 2\sigma^*_{j,t,\tau}E_{i,t,\tau}(z_j)\cdot C_{4,t,\tau}\cdot\rho^*_{j,t,\tau}(z_j) \quad \text{(by (4.3.11))}\\ &\qquad\qquad\le 2\sigma^*_{j,t,\tau}E_{i,t,\tau}(z_j)\cdot C_{4,t,\tau}\cdot\frac{C_{t_j}}{|z_j|^2(\log|z_j|)^2} \quad \text{(by (4.3.5))} \end{aligned} \tag{4.3.14}$$

Similarly, it follows from Proposition 2.2.1, (4.2.8) and (4.3.5) that one has

$$\Delta_j E_{i,t,\tau}(z_j) \le 2E_{i,t,\tau}(z_j)\cdot C_{4,t,\tau}\cdot\frac{C_{t_j}}{|z_j|^2(\log|z_j|)^2} \quad \text{on } N^1_{j,t,\tau}. \tag{4.3.15}$$

It follows readily from (4.3.8) and (4.3.10) that

$$\begin{aligned} \widetilde{\Delta}_j f_{t,\tau}(r_j) &= \int_0^{2\pi}\widetilde{\Delta}_j\left(E_{i,t,\tau}+\sigma^*_{j,t,\tau}E_{i,t,\tau}\right)(r_j,\theta_j)\,d\theta_j\\ &= \int_0^{2\pi}\Delta_j\left(E_{i,t,\tau}+\sigma^*_{j,t,\tau}E_{i,t,\tau}\right)(r_j,\theta_j)\,d\theta_j\\ &\quad -\frac{1}{r_j^2}\int_0^{2\pi}\frac{\partial^2}{\partial\theta_j^2}\left(E_{i,t,\tau}(r_j,\theta_j)+E_{i,t,\tau}\left(\frac{|t_j|}{r_j},\psi_j-\theta_j\right)\right)d\theta_j. \end{aligned} \tag{4.3.16}$$

Observe that

$$\int_0^{2\pi}\frac{\partial^2}{\partial\theta_j^2}\left(E_{i,t,\tau}(r_j,\theta_j)+E_{i,t,\tau}\left(\frac{|t_j|}{r_j},\psi_j-\theta_j\right)\right)d\theta_j = 0, \tag{4.3.17}$$

since the expression $E_{i,t,\tau}(r_j,\theta_j)+E_{i,t,\tau}(\frac{|t_j|}{r_j},\psi_j-\theta_j)$ is periodic in θ_j with period 2π. By (4.3.14) and (4.3.15), we also have

$$\begin{aligned} &\int_0^{2\pi}\Delta_j\big(\left(E_{i,t,\tau}+\sigma^*_{j,t,\tau}E_{i,t,\tau}\right)(r_j,\theta_j)\,d\theta_j\\ &\quad\le\int_0^{2\pi}(E_{i,t,\tau}+\sigma^*_{j,t,\tau}E_{i,t,\tau})(r_j,\theta_j)\cdot\frac{2C_{4,t,\tau}C_{t_j}}{r_j^2(\log r_j)^2}\,d\theta_j\\ &\quad=\frac{2C_{4,t,\tau}C_{t_j}}{r_j^2(\log r_j)^2}\cdot f(r_j). \end{aligned} \tag{4.3.18}$$

Combining (4.3.16), (4.3.17) and (4.3.18), it follows that we have

$$\widetilde{\Delta}_j f_{t,\tau}(r_j) \le \frac{2C_{4,t,\tau}C_{t_j}}{r_j^2(\log r_j)^2} \cdot f(r_j). \tag{4.3.19}$$

For any fixed number $\beta > 1$, a direct calculation gives

$$\frac{d}{dr_j}\left(\frac{1}{(-\log r_j)^\beta}\right) = \frac{\beta}{r_j(-\log r_j)^{\beta+1}} > 0 \quad \text{for } 0 < r_j < 1, \text{ and} \tag{4.3.20}$$

$$\widetilde{\Delta}_j\left(\frac{1}{(-\log r_j)^\beta}\right) = \frac{\beta(\beta+1)}{r_j^2(-\log r_j)^{\beta+2}}. \tag{4.3.21}$$

Since the node q_j of X_0 is adjacent to the puncture p_i, it follows that $E_{i,0,0}$ is positive (and thus bounded below by some constant $C_2 > 0$) on at least one of the boundary circles of $N_{i,0,0}$, namely $|z_j| = r$ or $|w_j| = r$. (We remark that $E_{i,0,0}$ may be identically zero on the other boundary circle of $N_{i,0,0}$.) Together with Proposition 4.2.1, it follows that there exists a constant $\delta_1 > 0$ such that $E_{i,t,\tau} \ge \frac{C_2}{2}$ on the corresponding boundary circle of $N_{i,t,\tau}$ for all $(t,\tau) \in U^*$ satisfying $\|(t,\tau)\| < \delta_1$. Together with (4.3.8), it follows that

$$f_{t,\tau}(r) \ge 2\pi \cdot \frac{C_2}{2} = \pi C_2 \quad \text{for all } (t,\tau) \in U^* \text{ satisfying } \|(t,\tau)\| < \delta_1. \tag{4.3.22}$$

Let $C_3 := \pi C_2 \cdot (-\log r)^\beta > 0$. For each $(t,\tau) \in U^*$, consider the function

$$F_{t,\tau}(r_j) := \frac{C_3}{(-\log r_j)^\beta} - f_{t,\tau}(r_j), \quad |t_j|^{\frac{1}{2}} \le r_j \le r. \tag{4.3.23}$$

Then it follows from (4.3.22) that

$$F_{t,\tau}(r) \le 0 \quad \text{for all } (t,\tau) \in U^* \text{ satisfying } \|(t,\tau)\| < \delta_1. \tag{4.3.24}$$

Moreover, it follows from (4.3.9) and (4.3.20) that for all $(t,\tau) \in U^*$, one has

$$F'_{t,\tau}(|t_j|^{\frac{1}{2}}) \ge 0. \tag{4.3.25}$$

Since $\beta > 1$, we have $\beta(\beta+1) > 2$. Together with (2.2.11) and (4.3.6), it follows that there exists a constant $\delta_2 > 0$ such that

$$2C_{4,t,\tau}C_{t_j} < \beta(\beta+1) \quad \text{for all } (t,\tau) \in U^* \text{ satisfying } \|(t,\tau)\| < \delta_2. \tag{4.3.26}$$

Together with (4.3.19), (4.3.21) and (4.3.23), it follows that

$$\widetilde{\Delta}_j F_{t,\tau}(r_j) \ge \frac{\beta(\beta+1)}{r_j^2(-\log r_j)^2} F_{t,\tau}(r_j) \quad \text{for all } (t,\tau) \in U^* \text{ satisfying } \|(t,\tau)\| < \delta_2. \tag{4.3.27}$$

Regarding $F_{t,\tau}$ also as a function in the new variable $s_j = \log r_j$, one easily sees from (4.3.10) that the inequality in (4.3.27) can be re-written as

$$\frac{d^2}{ds_j^2} F_{t,\tau} \ge \frac{\beta(\beta+1)}{s_j^2} F_{t,\tau}. \tag{4.3.28}$$

By using the maximum principle, one easily sees from (4.3.24), (4.3.25) and (4.3.28) that for all $(t,\tau)\in U^*$ satisfying $\|(t,\tau)\|<\min\{\delta_1,\ \delta_2\}$, one has

$$F_{t,\tau}(r_j)\le 0,\quad \text{or equivalently, } f_{t,\tau}(r_j)\ge \frac{C_3}{(-\log r_j)^{\beta}}\quad \text{for all } |t_j|^{\frac{1}{2}}\le r_j\le r. \tag{4.3.29}$$

We remark that to prove (4.3.1), it is clear that we may assume without loss of generality that $\beta<3$. From (4.3.8) and (4.3.29), one has

$$\begin{aligned}\int_{N_{j,t,\tau}} E_{i,t,\tau}\frac{\phi_j\overline{\phi_j}}{\rho_{t,\tau}} &\ge \frac{C_1^2|t_j|^2}{C_{4,t,\tau}C_{t_j}}\int_{|t_j|^{\frac{1}{2}}}^{r}\frac{C_3}{(-\log r_j)^{\beta}}\cdot\frac{(\log r_j)^2}{r_j}\,dr_j\\ &=\frac{C_3C_1^2|t_j|^2}{C_{4,t,\tau}C_{t_j}}\cdot\frac{\left((-\log|t_j|^{\frac{1}{2}})^{3-\beta}-(-\log r)^{3-\beta}\right)}{3-\beta}.\end{aligned} \tag{4.3.30}$$

It follows from (2.2.11) and (4.3.6) that there exists $\delta_3>0$ such that $C_{4,t,\tau}\le 2$ and $C_{t_j}\le 2$ for all $(t,\tau)\in U^*$ satisfying $\|(t,\tau)\|<\delta_3$. Clearly there also exists $\delta_4>0$ such that $(-\log r)^{3-\beta}<\frac{1}{2}(-\log|t_j|^{\frac{1}{2}})^{3-\beta}$ if $0<|t_j|<\delta_4$. Now let $\delta=\min\{\delta_1,\ \delta_2,\ \delta_3,\ \delta_4\}>0$. Then it follows readily from (4.3.30) that (4.3.1) holds for all $(t,\tau)\in U^*$ satisfying $\|(t,\tau)\|<\delta$ (and with the constant $C=\frac{C_3C_1^2}{8\cdot 2^{3-\beta}\cdot(3-\beta)}>0$). □

Remark 4.3.2. The proof of Proposition 4.3.1 does not work for the case when the node q_j is not adjacent to the puncture p_i, since $E_{i,0,0}$ is identically zero near such q_j (cf. (4.1.1)). One expects that (4.3.1) will not hold for such q_j. For a similar reason, one expects that a pointwise lower bound in the spirit of Proposition 4.2.2 will not hold on the entire $N_{j,t,\tau}$ even in the case when q_j is adjacent to p_i (unless both branches of $N_{j,0,0}$ are "adjacent to" p_i).

4.4. Upper and lower bounds of $E_{i,t,\tau}$ near p_i. For a fixed integer i with $1\le i\le n$, we are going to give a pointwise upper bound of $E_{i,t,\tau}$ near the puncture p_i. Let $W_i=\Delta^*(R)\times U$, $W_i'=\Delta^*(R')\times U'$ (with $0<R<R'<1$), $W_{i,t,\tau}$, $W'_{i,t,\tau}$, z_i be as in (2.1).

Proposition 4.4.1. *There exist constants $\delta,\ c_1,\ c_2>0$ such that for all $(t,\tau)\in U$ satisfying $\|(t,\tau)\|<\delta$, one has*

$$\left(\frac{\log|z_i|}{2\pi}\right)^2+c_1\log|z_i|\le E_{i,t,\tau}(z_i)\le\left(\frac{\log|z_i|}{2\pi}\right)^2-c_2\log|z_i|\quad \textit{on } W_{i,t,\tau}, \tag{4.4.1}$$

shrinking R if necessary. In particular, there exist constants $c_3,\ c_4>0$ such that for all $(t,\tau)\in U$ satisfying $\|(t,\tau)\|<\delta$, one has

$$c_3\,(\log|z_i|)^2\le E_{i,t,\tau}(z_i)\le c_4\,(\log|z_i|)^2\quad \textit{on } W_{i,t,\tau}. \tag{4.4.2}$$

Proof. For any $(t,\tau)\in U$, we let $z_{i,t,\tau}$ be a standard local holomorphic coordinate function around the puncture p_i on $X_{t,\tau}$ (cf. Remark-Definition 2.1.2). As mentioned in Remark-Definition 2.1.2(ii), $\Delta^*_{t,\tau}(e^{-\pi}):=\{z_{i,t,\tau}\in\mathbb{C}\,\big|\,0<|z_{i,t,\tau}|<e^{-\pi}\}$ is a bona-fide local coordinate neighborhood around p_i in $X_{t,\tau}$. It follows readily from the collar lemma for non-compact surfaces ([Bu, Theorem 4.4.6, p.112]) and Proposition 2.1.3(ii) that there exists $\delta>0$ such that for all $(t,\tau)\in U$ satisfying $\|(t,\tau)\|<\delta$, one has

$W'_{i,t,\tau} \subset \Delta^*_{t,\tau}(e^{-\pi})$, upon shrinking R' (and possibly also R) if necessary; in particular, $z_{i,t,\tau}$ (in addition to z_i) provides a holomorphic coordinate function on $W'_{i,t,\tau}$ vanishing only at p_i. In terms of the Euclidean coordinate $Z = X + iY$ on the upper half plane $\mathbb{H}$, it is easy to calculate that the hyperbolic distance from a point Z to $Z+1$ is given by $2\coth^{-1}(\sqrt{4Y^2+1})$ (with the hyperbolic geodesic joining Z to $Z+1$ given by the Euclidean circular arc joining the two points and with center at $X+\frac{1}{2}$). Shrinking R again if necessary, it follows that for all $(t,\tau) \in U$ satisfying $\|(t,\tau)\| < \delta$, the injectivity radius of $W'_{i,t,\tau}$ at any point $a \in W_{i,t,\tau}$ with respect to the restriction of the hyperbolic metric $\rho_{t,\tau}$ on $X_{t,\tau}$ is given by $f(|z_{i,t,\tau}(a)|)$, where $f(t) := 2\coth^{-1}\left(\sqrt{1+(\frac{1}{\pi}\log t)^2}\right)$. Next we recall from Proposition 2.1.3(i) the comparison of $\rho_{t,\tau}$ with the two model hyperbolic metrics on $W'_{i,t,\tau}$ with respect to the z_i-coordinate (the comparison was stated for $W_{i,t,\tau}$ there, but clearly it holds for $W'_{i,t,\tau}$ here). Upon shrinking R further if necessary, one easily sees that it leads readily to a corresponding comparison of the injectivity radii of $W'_{i,t,\tau}$ at the point a with respect to these metrics given by

$$\sqrt{C_1}\,f(R) \leq f(|z_{i,t,\tau}(a)|) \leq \sqrt{C_2}\,f(R), \tag{4.4.3}$$

where $C_1,\ C_2 > 0$ are as in Proposition 2.1.3(i). It is easy to see that $f\colon (0,1) \to (0,\infty)$ is a continuous strictly increasing and bijective function. Since z_i and $z_{i,t,\tau}$ are both coordinate functions on $W'_{i,t,\tau}$ vanishing at p_i, it follows that the function $h_{t,\tau} = z_{i,t,\tau}/z_i$ extends across p_i as a non-vanishing holomorphic function. By applying the maximum and minimum modulus principles to the (extended) function $h_{t,\tau}$ on the disc $|z_i| \leq R$ and varying (t,τ), it follows from (4.4.3) that for all $(t,\tau) \in U$ satisfying $\|(t,\tau)\| < \delta$, one has

$$C_3\,|z_i| \leq |z_{i,t,\tau}| \leq C_4\,|z_i| \qquad \text{on } W_{i,t,\tau}, \text{ where} \tag{4.4.4}$$

$$C_3 = \frac{f^{-1}(\sqrt{C_1}\,f(R))}{R} > 0 \quad \text{and} \quad C_4 = \frac{f^{-1}(\sqrt{C_2}\,f(R))}{R} > 0. \tag{4.4.5}$$

Fix a number $\epsilon > 0$. Then shrinking R and δ if necessary, we may assume that $W_{i,t,\tau} \subset \Delta^*_{t,\tau}(e^{-2\pi e^{-\epsilon}})$ for all $(t,\tau) \in U$ satisfying $\|(t,\tau)\| < \delta$. Thus by Proposition 4.1.2(i) (with $s = 2$), there exists a constant $C_{2,\epsilon} > 0$ such that for all $(t,\tau) \in U$ satisfying $\|(t,\tau)\| < \delta$, one has

$$-C_{2,\epsilon} \leq E_{i,t,\tau}(z_{i,t,\tau}) - \left(\frac{\log|z_{i,t,\tau}|}{2\pi}\right)^2 \leq C_{2,\epsilon} \quad \text{on } W_{i,t,\tau}. \tag{4.4.6}$$

By replacing C_3 by $\min\{C_3, 1\}$, etc., we may assume that (4.4.4) holds with $C_3 \leq 1$ and $C_4 \geq 1$. Note also that $|z_i|,\ |z_{i,t,\tau}| < 1$ on $W_{i,t,\tau}$. Thus (4.4.4) leads to the inequality

$$(\log|z_i| + \log C_4)^2 \leq (\log|z_{i,t,\tau}|)^2 \leq (\log|z_i| + \log C_3)^2 \quad \text{on } W_{i,t,\tau}, \tag{4.4.7}$$

which, together with (4.4.6), lead readily to (4.4.1). We just remark that the constant terms in (4.4.6) and (4.4.7) can be absorbed by the terms linear in $\log|z_i|$ in (4.4.1) by adjusting c_1 and c_2 suitably, if necessary. Finally one easily sees that (4.4.2) is a direct consequence of (4.4.1). $\square$

4.5. Upper bound of $E_{i,t,\tau}$ near a puncture p_j with $j \neq i$. For fixed integers i, j with $1 \leq i \neq j \leq n$, we are going to give a pointwise upper bound of $E_{i,t,\tau}$ near the puncture p_j. Let $W_j = \Delta^*(R) \times U$ (with $0 < R < 1$), $W_{j,t,\tau}$, z_j be as in (2.1).

Proposition 4.5.1. *For fixed integers $1 \leq i \neq j \leq n$ and real number α satisfying $0 < \alpha < 1$, there exist constants C, $\delta > 0$ such that for all $(t, \tau) \in U$ satisfying $\|(t, \tau)\| < \delta$, one has*

$$E_{i,t,\tau}(z_j) \leq \frac{C}{(-\log|z_j|)^\alpha} \quad \text{on } W_{j,t,\tau}. \tag{4.5.1}$$

Proof. The proof is similar to that of Proposition 4.2.2, and as in there, we will assume without loss of generality that at $(t, \tau) = (0, 0)$, z_j is a standard local holomorphic coordinate for X_0. For each $(t, \tau) \in U$, the boundary $\partial W_{j,t,\tau}$ of $W_{j,t,\tau}$ consists of the circle $|z_j| = R$. As in (4.2.9), it follows from Proposition 2.1.3 and Proposition 4.1.2(ii) that there exists a constant $C_1 > 0$ such that for all $(t, \tau) \in U$, one has

$$E_{i,t,\tau}(z_j) \leq C_1 \quad \text{on } \partial W_{j,t,\tau}. \tag{4.5.2}$$

Thus for all $(t, \tau) \in U$, one has

$$E_{i,t,\tau}(z_j) - \frac{C}{(-\log|z_j|)^\alpha} \leq 0 \quad \text{on } \partial W_{j,t,\tau}, \quad \text{where } C := C_1 \cdot (-\log R)^\alpha > 0. \tag{4.5.3}$$

Since $0 < \alpha < 1$, it follows from Proposition 4.1.2(iii) that for all $(t, \tau) \in U$, one has

$$E_{i,t,\tau}(z_j) - \frac{C}{(-\log|z_j|)^\alpha} \to 0 \quad \text{as } z_j \to 0. \tag{4.5.4}$$

Let $\Delta_j := 4\frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ be as in (4.2.5). Then a direct calculation gives

$$\Delta_j \left(\frac{1}{(-\log|z_j|)^\alpha} \right) = \frac{\alpha(\alpha+1)}{|z_j|^2(-\log|z_j|)^{\alpha+2}}. \tag{4.5.5}$$

Let $\rho_{t,\tau}(z_j)$ be as in (2.1.4). For all $(t, \tau) \in U$, it follows from (1.1.2) that one has

$$\begin{aligned} \Delta_j E_{i,t,\tau}(z_j) &= 2E_{i,t,\tau}(z_j)\rho_{t,\tau}(z_j) \\ &\geq \frac{2\,C_{1,t,\tau} E_{i,t,\tau}(z_j)}{|z_j|^2(\log|z_j|)^2} \quad \text{on } W_{j,t,\tau} \quad \text{(by Proposition 2.1.3)}, \end{aligned} \tag{4.5.6}$$

where $C_{1,t,\tau}$ is as in (2.1.13). Since $\alpha(\alpha+1) < 2$, it follows from Proposition 2.1.3(ii) that there exists a constant $\delta > 0$ such that

$$C_{1,t,\tau} > \frac{\alpha(\alpha+1)}{2} \quad \text{for all } (t, \tau) \in U \text{ satisfying } \|(t, \tau)\| < \delta. \tag{4.5.7}$$

Together with (4.5.5) and (4.5.6), it follows that for all $(t, \tau) \in U$ satisfying $\|(t, \tau)\| < \delta$, one has

$$\begin{aligned} &\Delta_j \left(E_{i,t,\tau}(z_j) - \frac{C}{(-\log|z_j|)^\alpha} \right) \\ &\quad \geq \frac{\alpha(\alpha+1)}{|z_j|^2(\log|z_j|)^2} \cdot \left(E_{i,t,\tau}(z_j) - \frac{C}{(-\log|z_j|)^\alpha} \right) \quad \text{on } W_{j,t,\tau}. \end{aligned} \tag{4.5.8}$$

By using the maximum principle, one easily obtains (4.5.1) as a consequence of (4.5.3), (4.5.4) and (4.5.8). □

5. Asymptotic Behavior of the Takhtajan-Zograf Metric

5.1. Let $X_0 \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ be a stable Riemann surface with n punctures $p_1, \ldots, p_n$ and m nodes $q_1, \ldots, q_m$, and let $\psi\colon U \simeq \Delta^m \times V \to \hat{U}$ with $V \simeq \Delta^{3g-3+n-m}$ and coordinates $(s_1, \ldots, s_{3g-3+n}) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_{3g-3+n-m}) = (t, \tau)$ be as in Theorem 1. For $(t, \tau) \in U^* = (\Delta^*)^m \times V$, the components of the Takhtajan-Zograf metric given as in (1.4.1) form a matrix $G^{\mathrm{TZ}} := \left(g^{\mathrm{TZ}}_{j\bar{k}}\right)_{1\le j,k\le 3g-3+n}$. For the Weil-Petersson metric, we similarly denote the matrix $G^{\mathrm{WP}} := \left(g^{\mathrm{WP}}_{j\bar{k}}\right)_{1\le j,k\le 3g-3+n}$, where the $g^{\mathrm{WP}}_{j\bar{k}}$'s are as in Proposition 3.2.1. Let ϕ_k, $k = 1, \ldots, 3g-3+n$, be the regular 2-differentials given by Proposition 3.1.2. For $1 \le j, k \le 3g-3+n$, we define

$$h^{\mathrm{TZ}}_{j\bar{k}} = \sum_{i=1}^{n} \int_{X_{t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_k}}{\rho_{t,\tau}}, \tag{5.1.1}$$

and denote the corresponding matrix by $\Phi^{\mathrm{TZ}} := \left(h^{\mathrm{TZ}}_{j\bar{k}}\right)$. We remark that it is easy to see from Proposition 3.1.2 and Definition 4.1.1 that Φ^{TZ} is actually well-defined on the entirety of U; moreover, for each non-empty subset $J \subset \{1, \ldots, m\}$ and with $B(J) = \{(t, \tau) \in U \,|\, t_j = 0 \text{ for all } j \in J\}$ as defined in (3.1), one has, at any point $(t, \tau) \in B(J)$, $h^{\mathrm{TZ}}_{j\bar{k}} = 0$ whenever either $j \in J$ or $k \in J$ (cf. Proposition 3.1.2(ii)). In particular, at $(t, \tau) = (0, 0)$, we have

$$h^{\mathrm{TZ}}_{j\bar{k}}(0,0) = 0 \quad \text{if } j \le m+1 \text{ or } k \le m+1. \tag{5.1.2}$$

Proposition 5.1.1. *On U^*, we have*

$$G^{TZ} = G^{WP}\overline{\Phi^{TZ}}G^{WP}, \quad \textit{or equivalently,}$$
$$g^{TZ}_{j\bar{k}} = \sum_{\ell,r=1}^{3g-3+n} g^{WP}_{j\bar{\ell}}\overline{h^{TZ}_{\ell\bar{r}}}g^{WP}_{r\bar{k}}, \quad 1 \le j,k \le 3g-3+n. \tag{5.1.3}$$

Proof. For $(t, \tau) \in U^*$, let $\mu_k = \mu_k(z, t, \tau)d\bar{z}/dz$, $k = 1, \ldots, 3g-3+n$, be a basis of harmonic Beltrami differentials on $X_{t,\tau}$ dual to $\{\phi_k\}_{1\le k\le 3g-3+n}$ with respect to the pairing in (1.1.3). From the definition of harmonic Beltrami differentials in (1.1) and Proposition 3.1.2(i), one easily sees that for each $1 \le j \le 3g-3+n$, $\mu_j = \sum_{k=1}^{3g-3+n} c_{jk}\overline{\phi_k}/\rho_{t,\tau}$ for some constants c_{jk}. Now for each j, k, we have

$$g^{\mathrm{WP}}_{j\bar{k}} = \int_{X_{t,\tau}} \mu_j\overline{\mu_k}\rho_{t,\tau} = \sum_{\ell=1}^{3g-3+n} c_{j\ell}\int_{X_{t,\tau}} \overline{\phi_\ell}\overline{\mu_k} = \sum_{\ell=1}^{3g-3+n} c_{j\ell}\delta_{\ell k} = c_{jk}.$$

It follows that

$$\mu_j = \sum_{\ell=1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}}\frac{\overline{\phi_\ell}}{\rho_{t,\tau}} \tag{5.1.4}$$

for each j. Now, for each $1 \le j, k \le 3g-3+n$, we have

$$
\begin{aligned}
g^{\mathrm{TZ}}_{j\bar{k}} &= \sum_{i=1}^{n} \int_{X_{t,\tau}} E_{i,t,\tau}\mu_j \overline{\mu_k}\rho_{t,\tau} \\
&= \sum_{\ell,r=1}^{3g-3+n} \sum_{i=1}^{n} \int_{X_{t,\tau}} \frac{E_{i,t,\tau} g^{\mathrm{WP}}_{j\bar{\ell}} \overline{\phi_\ell} \overline{g^{\mathrm{WP}}_{k\bar{r}}} \phi_r}{\rho_{t,\tau}} \quad \text{(by (5.1.4))} \\
&= \sum_{\ell,r=1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}} \left(\sum_{i=1}^{n} \int_{X_{t,\tau}} \frac{E_{i,t,\tau}\phi_r \overline{\phi_\ell}}{\rho_{t,\tau}} \right) g^{\mathrm{WP}}_{r\bar{k}} \\
&= \sum_{\ell,r=1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}} \overline{h^{\mathrm{TZ}}_{\ell\bar{r}}} g^{\mathrm{WP}}_{r\bar{k}}.
\end{aligned}
$$

□

5.2. We obtain the asymptotic behavior of the matrix Φ^{TZ} as follows:

Proposition 5.2.1. *Notation as in Theorem* 1 *and* (5.1). *Then the following statements hold*:
(*i*) *For each* $1 \le j \le m$ *and any* $\varepsilon > 0$, *there exist constants* $C_1 > 0$ (*depending on* ε) *such that*

$$h^{TZ}_{j\bar{j}}(t,\tau) \le C_1 |t_j|^2 (-\log |t_j|)^{2+\varepsilon}) \tag{5.2.1}$$

for all $(t,\tau) \in U^*$.
(*ii*) *For each* $1 \le j \le m'$ *and any* $\varepsilon > 0$, *there exists a constant* $C_2 > 0$ (*depending on* ε) *such that*

$$h^{TZ}_{j\bar{j}}(t,\tau) \ge C_2 |t_j|^2 (-\log |t_j|)^{2-\varepsilon} \tag{5.2.2}$$

for all $(t,\tau) \in U^*$.
(*iii*) *For each* $1 \le j,\ k \le m$ *with* $j \ne k$,

$$\left| h^{TZ}_{j\bar{k}}(t,\tau) \right| = O\left(|t_j|\,|t_k| \right) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{5.2.3}$$

(*iv*) *For each* $j\,, k \ge m+1$,

$$\lim_{\substack{(t,\tau)\to(0,0) \\ (t,\tau)\in U^*}} h^{TZ}_{j\bar{k}}(t,\tau) = h^{TZ}_{j\bar{k}}(0,0). \tag{5.2.4}$$

(*v*) *For each* $j \le m$ *and* $k \ge m+1$,

$$\left| h^{TZ}_{j\bar{k}}(t,\tau) \right| = O\left(|t_j| \right) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{5.2.5}$$

Proof. First we prove (i). It is easy to see that we only need to verify (5.2.1) for $(t,\tau)\in U^*$ with small $\|(t,\tau)\|$. Recall from (1.3) and (3.1.3) the covering of $\mathcal{X}$ by coordinate neighborhoods $\{N_j\}_{1\le j\le m}$, $\{W_i\}_{1\le i\le n}$ and $\{A_\ell\}_{1\le\ell\le\ell_o}$, and the corresponding fibers $N_{j,t,\tau}$, $W_{i,t,\tau}$, $A_{\ell,t,\tau}$. For each $1\le j\le m$ and each $1\le i\le n$, we have

$$\begin{aligned}
&\int_{X_{t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}}\\
&\le \left(\int_{N_{j,t,\tau}} + \sum_{\substack{1\le j'\le n\\ j'\ne j}}\int_{N_{j',t,\tau}} + \int_{W_{i,t,\tau}} + \sum_{\substack{1\le i'\le n\\ i'\ne i}}\int_{W_{i',t,\tau}} + \sum_{1\le\ell\le\ell_o}\int_{A_{\ell,t,\tau}}\right)\frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}}\\
&=: I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{5.2.6}$$

Fix an ε with $0<\varepsilon<1$, and recall the decomposition $N_{j,t,\tau}=N^1_{j,t,\tau}\cup N^2_{j,t,\tau}$ in (2.2.4). By Proposition 4.2.2 (with $\alpha=1-\epsilon$), the first line of (3.1.5) in Proposition 3.1.2 and Proposition 2.2.1, it follows that there exist constants $C_1,\ \delta_1>0$ such that for all $(t,\tau)\in U$ with $t_j\ne 0$ and satisfying $\|(t,\tau)\|<\delta_1$, one has

$$\begin{aligned}
&\int_{N^1_{j,t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}}\\
&\le C_1\int_{|t_j|^{\frac12}<|z_j|<r}\frac{1}{(-\log|z_j|)^{1-\varepsilon}}\cdot\frac{|t_j|}{|z_j|^2}\cdot\frac{|t_j|}{|z_j|^2}\cdot|z_j|^2(\log|z_j|)^2dz_jd\bar z_j\\
&= C_1|t_j|^2\int_0^{2\pi}\int_{|t_j|^{\frac12}}^r\frac{(-\log r_j)^{1+\epsilon}}{r_j}dr_jd\theta_j \quad (\text{with } z_j=r_je^{i\theta_j})\\
&= \frac{2\pi C_1}{2+\varepsilon}\cdot|t_j|^2\cdot\left((-\log|t_j|^{\frac12})^{2+\varepsilon}-(-\log r)^{2+\varepsilon}\right).
\end{aligned}\tag{5.2.7}$$

A similar estimate holds on $N^2_{j,t,\tau}$, and thus we get an estimate of the form

$$I_1=\int_{N_{j,t,\tau}}\frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}}=O\left(|t_j|^2(-\log|t_j|)^{2+\varepsilon}\right)\quad\text{as } (t,\tau)\in U^*\to(0,0).\tag{5.2.8}$$

For each $1\le j'\le n$ with $j'\ne j$, one easily performs a computation similar to (5.2.7) with the first line of (3.1.5) replaced by the second line of (3.1.5) to see that there exist constants $C_2,\delta_2>0$ such that for all $(t,\tau)\in U$ with $t_j\ne 0$ and satisfying $\|(t,\tau)\|<\delta_2$, one has

$$\begin{aligned}
\int_{N^1_{j',t,\tau}}\frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}} &\le C_2|t_j|^2\int_{|t_{j'}|^{\frac12}<|z_{j'}|<r}|z_{j'}|^2(\log|z_{j'}|)^{1+\varepsilon}dz_{j'}d\bar z_{j'}\\
&\le C_3|t_j|^2
\end{aligned}\tag{5.2.9}$$

for some constant $C_3>0$, and a similar estimate holds on $N^2_{j',t,\tau}$. By summing (5.2.9) over the j''s, we get an estimate of the form

$$I_2=\sum_{\substack{1\le j'\le n\\ j'\ne j}}\int_{N_{j',t,\tau}}\frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}}=O(|t_j|^2)\quad\text{as } (t,\tau)\in U^*\to(0,0).\tag{5.2.10}$$

Using Proposition 4.4.1, the first line of (3.1.9) in Proposition 3.1.2 and Proposition 2.1.3, one easily checks that for each $1 \le i \le n$ and each $1 \le j \le m$, there exist constants $C_4, \delta_3 > 0$ such that for all $(t, \tau) \in U$ satisfying $\|(t, \tau)\| < \delta_3$, one has

$$\begin{aligned} I_3 &= \int_{W_{i,t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}} \\ &\le C_4 \int_{0<|z_i|<R} (\log|z_i|)^2 \cdot \frac{|t_j|^2}{|z_i|^2} \cdot |z_i|^2 (\log|z_i|)^2 dz_i d\bar{z}_i \\ &\le C_5 |t_j|^2 \end{aligned} \tag{5.2.11}$$

for some constant $C_5 > 0$. A calculation similar to (5.2.11) (using Proposition 4.5.1 in place of Proposition 4.4.1) easily shows that

$$I_4 = \sum_{\substack{1\le i'\le n \\ i'\neq i}} \int_{W_{i',t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}} = O(|t_j|^2) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{5.2.12}$$

For each $1 \le \ell \le \ell_o$, it follows readily from the result of Bers [Be] mentioned in (2.1) that there exist constants C_5, $C_6 > 0$ such that for all $(t, \tau) \in U$, one has

$$C_5 dz_\ell \otimes d\bar{z}_\ell \le \rho_{t,\tau} \le C_6 dz_\ell \otimes d\bar{z}_\ell \quad \text{on } A_{\ell,t,\tau}. \tag{5.2.13}$$

Together with the first line of (3.1.11) in Proposition 3.1.2 and Proposition 4.2.1, it follows easily that for each $1 \le i \le n$, there exist constants C_7, $\delta_4 > 0$ such that for all $(t, \tau) \in U^*$ satisfying $\|(t, \tau)\| < \delta_4$, one has

$$I_5 = \sum_{1\le\ell\le\ell_o} \int_{A_{\ell,t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_j}}{\rho_{t,\tau}} \le C_7\, |t_j|^2. \tag{5.2.14}$$

By using (5.1.1), (5.2.8), (5.2.10), (5.2.11), (5.2.12) and (5.2.14), one easily sees that (5.2.1) can be obtained readily by summing (5.2.6) with the index i running from 1 to n, and this finishes the proof of (i). We remark that I_1 is the dominant term on the right-hand side of (5.2.6). Next one easily sees that (5.2.2) is a direct consequence of Proposition 4.3.1 (by setting $\beta = 1 + \epsilon$ in (4.3.1)), which gives (ii). The proof of (iii) is similar to that of (i), and thus it will be skipped. To prove (iv), we first observe from (2.2.4) that for each $(t, \tau) \in U$, $N^1_{j,t,\tau}$ can be identified with the subset $|t_j|^{\frac{1}{2}} \le |z_j| < r$ in $N^1_{j,0,0}$ via the projection map in the z_j-coordinate, and similar description holds for $N^2_{j,t,\tau}$. Similarly, each $W_{i,t,\tau}$ and $A_{\ell,t,\tau}$ can be identifed with $W_{i,0,0}$ and $A_{\ell,0,0}$ respectively. Next we recall the pointwise upper bounds for the $E_{i,t,\tau}$'s in Proposition 4.2.2, Proposition 4.4.1 and Proposition 4.5.1, the pointwise upper bounds for the ϕ_j's (with $j \ge m+1$) in (3.1.7), the second line of (3.1.9) and that of (3.1.11) in Proposition 3.1.2, and the pointwise lower bounds for the $\rho_{t,\tau}$'s in Proposition 2.1.3, Proposition 2.2.1 and (5.2.13). Recall also the pointwise convergence of the $E_{i,t,\tau}$'s given by Proposition 4.2.1, that of the ϕ_k's given by Proposition 3.1.2 and that of the $\rho_{t,\tau}$'s given by Bers' result [Be] as $(t, \tau) \in U^* \to (0, 0)$. Together with a partition of unity of $\mathcal{X}$ with respect to the coverings $\{N_j\}$, $\{W_i\}$ and $\{A_\ell\}$, one can easily apply the dominated convergence theorem to show that for each $1 \le i \le n$ and $j, k \ge m+1$, one has

$$\int_{X_{t,\tau}} \frac{E_{i,t,\tau}\phi_j\overline{\phi_k}}{\rho_{t,\tau}} \to \int_{X_{0,0}} \frac{E_{i,0,0}\phi_j\overline{\phi_k}}{\rho_{0,0}} \quad \text{as } (t,\tau) \in U^* \to (0,0), \tag{5.2.15}$$

which together with (5.1.1), leads to (5.2.4) readily, and this finishes the proof of (iv). Finally the proof of (v) is similar to those of (i) and (iii) (and involves the use of the pointwise upper bounds for the ϕ_j's with $j \leq m$ needed in (i) and those for the ϕ_k's with $k \geq m+1$ needed in (iv) above), and thus it will be skipped. □

5.3. Finally we are ready to give the proof of Theorem 1 as follows:

Proof of Theorem 1. We are going to deduce Theorem 1 from Proposition 3.2.1, Proposition 5.1.1 and Proposition 5.2.1, and it amounts to estimating terms of the form

$$g^{\mathrm{WP}}_{j\bar{\ell}}\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}g^{\mathrm{WP}}_{r\bar{k}}, \quad 1 \leq j, \ell, r, k \leq 3g-3+n$$

(cf. Proposition 5.2.1). To prove Theorem 1(i) or equivalently (1.4.7), we fix an ε with $0<\varepsilon<1$, and fix a j with $1 \leq j \leq m$. Then it follows from (3.2.1) and (5.2.1) that

$$\begin{aligned} g^{\mathrm{WP}}_{j\bar{j}}\overline{h^{\mathrm{TZ}}_{j\bar{j}}}g^{\mathrm{WP}}_{j\bar{j}} &= O\left(\frac{1}{|t_j|^2(-\log|t_j|)^3}\cdot|t_j|^2(-\log|t_j|)^{2+\varepsilon}\cdot\frac{1}{|t_j|^2(-\log|t_j|)^3}\right) \\ &= O\left(\frac{1}{|t_j|^2(-\log|t_j|)^{4-\varepsilon}}\right) \quad \text{as } (t,\tau)\in U^* \to (0,0). \end{aligned} \tag{5.3.1}$$

Similarly, it follows from (3.2.1), (3.2.2), (5.2.1) and (5.2.3) that

$$\begin{aligned} &g^{\mathrm{WP}}_{j\bar{\ell}}\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}g^{\mathrm{WP}}_{r\bar{j}} \\ &= \begin{cases} O\Bigg(\dfrac{1}{|t_j|^2(-\log|t_j|)^3}\cdot|t_j|\,|t_r| \\ \quad\cdot\dfrac{1}{|t_r|\,|t_j|\,(\log|t_r|)^3(\log|t_j|)^3}\Bigg) & \text{if } \ell=j \;\&\; 1\leq r\neq j\leq m, \\ O\Bigg(\dfrac{1}{|t_j|\,|t_\ell|\,(\log|t_j|)^3(\log|t_\ell|)^3} \\ \quad\cdot|t_\ell|^2(-\log|t_\ell|)^{2+\epsilon} \\ \quad\cdot\dfrac{1}{|t_\ell|\,|t_j|\,(\log|t_\ell|)^3(\log|t_j|)^3}\Bigg) & \text{if } 1\leq \ell=r\neq j\leq m, \\ O\Bigg(\dfrac{1}{|t_j|\,|t_\ell|\,(\log|t_j|)^3(\log|t_\ell|)^3}\cdot|t_\ell|\,|t_j| \\ \quad\cdot\dfrac{1}{|t_j|^2(-\log|t_j|)^3}\Bigg) & \text{if } 1\leq\ell\neq j\leq m \;\&\; r=j, \end{cases} \\ &= O\left(\frac{1}{|t_j|^2(\log|t_j|)^6}\right) \quad \text{as } (t,\tau)\in U^*\to(0,0) \quad \text{if } 1\leq \ell,\ r\leq m \;\&\; (\ell,r)\neq(j,j). \end{aligned} \tag{5.3.2}$$

Similarly one easily checks from (3.2.1), (3.2.2), (3.2.3), (3.2.4), (5.2.4), (5.2.5) that

$$\begin{aligned} &g^{\mathrm{WP}}_{j\bar{\ell}}\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}g^{\mathrm{WP}}_{r\bar{j}} \\ &= O\left(\frac{1}{|t_j|^2(\log|t_j|)^6}\right) \quad \text{as } (t,\tau)\in U^* \to (0,0) \quad \text{if } \ell\geq m+1 \text{ or } r\geq m+1. \end{aligned} \tag{5.3.3}$$

Combining Proposition 5.1.1, (5.2.1), (5.2.2) and (5.2.3), we have

$$\begin{aligned} g^{\mathrm{TZ}}_{j\bar{j}} &= g^{\mathrm{WP}}_{j\bar{j}} \overline{h^{\mathrm{TZ}}_{j\bar{j}}} g^{\mathrm{WP}}_{j\bar{j}} + \sum_{\substack{1\le \ell, r\le 3g-3+n \\ (\ell,r)\neq(j,j)}} g^{\mathrm{WP}}_{j\bar{\ell}} \overline{h^{\mathrm{TZ}}_{\ell\bar{r}}} g^{\mathrm{WP}}_{r\bar{j}} \\ &= O\left(\frac{1}{|t_j|^2(-\log|t_j|)^{4-\varepsilon}}\right) + O\left(\frac{1}{|t_j|^2(\log|t_j|)^6}\right) \\ &= O\left(\frac{1}{|t_j|^2(-\log|t_j|)^{4-\varepsilon}}\right) \quad \text{as } (t,\tau)\in U^*\to(0,0), \end{aligned} \tag{5.3.4}$$

and this gives Theorem 1(i). A calculation similar to (5.3.1) using (5.2.2) in place of (5.2.1) implies that for $1\le j\le m$ and $0<\varepsilon<1$, there exists a constant $C>0$ such that

$$g^{\mathrm{WP}}_{j\bar{j}} \overline{h^{\mathrm{TZ}}_{j\bar{j}}} g^{\mathrm{WP}}_{j\bar{j}} \ge \frac{C}{|t_j|^2(-\log|t_j|)^{4+\varepsilon}} \tag{5.3.5}$$

for all $(t,\tau)\in U^*$. Then a calculation similar to (5.3.4) using (5.2.3) in place of (5.2.1) leads readily to (1.4.8), which, in turn, leads to Theorem 1(ii). The proof of Theorem 1(iii) is similar to that of Theorem 1(i), and thus it will be skipped. To prove Theorem 1(iv), we first observe that from (1.2), (5.1.1), Proposition 3.1.2(ii) and using a calculation similar to Proposition 5.1.1, one has, for each $j,k\ge m+1$,

$$\hat{g}^{\mathrm{TZ},(\gamma_1,\ldots,\gamma_m)}_{j\bar{k}}(0,0) = \sum_{\ell,r=m+1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}}(0,0)\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}(0,0) g^{\mathrm{WP}}_{r\bar{k}}(0,0). \tag{5.3.6}$$

For each $j,k\ge m+1$ and each $1\le \ell, r\le m$, it follows from (3.2.4), (5.2.1), (5.2.3) that

$$\begin{aligned} & g^{\mathrm{WP}}_{j\bar{\ell}}(t,\tau)\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}(t,\tau) g^{\mathrm{WP}}_{r\bar{k}}(t,\tau) \\ &= \begin{cases} O\left(\dfrac{1}{|t_\ell|(-\log|t_\ell|)^3}\cdot|t_\ell||t_r|\cdot\dfrac{1}{|t_r|(-\log|t_r|)^3}\right) & \text{if } \ell\neq r, \\ O\left(\dfrac{1}{|t_\ell|(-\log|t_\ell|)^3}\cdot|t_\ell|^2(-\log|t_\ell|)^{2+\epsilon}\cdot\dfrac{1}{|t_\ell|(-\log|t_\ell|)^3}\right) & \text{if } \ell = r, \end{cases} \\ &\to 0 \quad \text{as } (t,\tau)\in U^*\to(0,0). \end{aligned} \tag{5.3.7}$$

Similarly, for each $j,k\ge m+1$, one also easily sees from (3.2.3), (3.2.4), (5.2.4) and (5.2.5) that

$$g^{\mathrm{WP}}_{j\bar{\ell}}(t,\tau)\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}(t,\tau) g^{\mathrm{WP}}_{r\bar{k}}(t,\tau)\to 0 \quad \text{as } (t,\tau)\in U^*\to(0,0), \text{ if } \ell\ge m+1 \text{ or } r\ge m+1. \tag{5.3.8}$$

Thus, one has, for $j,k\ge m+1$,

$$\begin{aligned} \lim_{(t,\tau)\in U^*\to(0,0)} g^{\mathrm{TZ}}_{j\bar{k}}(t,\tau) &= \lim_{(t,\tau)\in U^*\to(0,0)} \sum_{\ell,r=m+1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}}(t,\tau)\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}(t,\tau) g^{\mathrm{WP}}_{r\bar{k}}(t,\tau) \\ &\qquad \text{(by Proposition 5.1.1, (5.3.7), (5.3.8))} \\ &= \sum_{\ell,r=m+1}^{3g-3+n} g^{\mathrm{WP}}_{j\bar{\ell}}(0,0)\overline{h^{\mathrm{TZ}}_{\ell\bar{r}}}(0,0) g^{\mathrm{WP}}_{r\bar{k}}(0,0) \quad \text{(by (3.2.3), (5.2.4))} \\ &= \hat{g}^{\mathrm{TZ},(\gamma_1,\ldots,\gamma_m)}_{j\bar{k}}(0,0) \quad \text{(by (5.3.6))}, \end{aligned} \tag{5.3.9}$$

and this finishes the proof of Theorem 1(iv). Finally the proof of Theorem 1(v) is similar to that of Theorem 1(i) and (iv), and thus it will be skipped. □

Acknowledgements. The authors would like to thank Professor Ngaiming Mok and Professor Scott Wolpert for their interest, comments and suggestions related to this work.

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Communicated by L. Takhtajan