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Osaka, J. Math.
32 (1995), 467–473

A RESULT ON BICANONICAL MAPS OF SURFACES OF GENERAL TYPE

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(Received September 13, 1993)

Pluricanonical maps of surfaces of general type have been studied for quite a long time. After Bombieri's remarkable work [2], recently, I. Reider [5] uses a new method, i.e., residues, zero cycles and rank two vector bundles, to deal with them successfully. Now such problems are left open only for bicanonical maps with small $K_S^2(\leq 4)$, and for canonical maps. For bicanonical maps, the non-trivial cases are these surfaces with $p_g(S)=0$ and $K_S^2=3,4$. As there are no examples and suitable methods, in these cases, we have the following

Conjecture. If S is a minimal surface of general type with $p_g(S)=0$ and $K_S^2=3$ or 4, then the bicanonical map $\Phi_{|2K_S|}$ has no fixed points..

So far, we have had little knowledge about this conjecture: among others, we even do not know whether there is no fixed components. In this paper, we start with such a study. We will control the fixed part, say, the number of irreducible components, etc.. As always, in these cases the most difficult part is about the (-2)-curves. Our first result then is the following

Theorem. *Let S be a minimal surface of general type with $p_g(S)=0$ and $K_S^2=3$ or 4. Suppose C is a (-2)-curve in S. Then C cannot be an irreducible component of the fixed part of the bicanonical map $\Phi_{|2K_S|}$.*

Proof. We will use the rank two vector bundle technique, developed by Bogomolov.

Otherwise, we have an exact sequence

$$0 \to \mathcal{O}_S(2K_S-C) \to \mathcal{O}_S(2K_S) \to \mathcal{O}_C \to 0.$$

By the fact that C is a fixed component of $|2K_S|$ and $h^0(\mathcal{O}_C)\neq 0$, we see that $h^1(2K_S-C)\neq 0$. By Serre's duality, we have $h^1(C-K_S)\neq 0$. Now from the natural isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_S,\mathcal{O}_S(C-K_S)) \simeq H^1(S,\mathcal{O}_S(C-K_S)),$$

we obtain a nontrivial extension

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{O}_S(K_S - C) \to 0. \tag{$*$}$$

As $c_1(\mathcal{E}) = (K_S) - C$, $c_2(\mathcal{E}) = 0$ and $(K_S - C)^2 = K_S^2 - 2 > 0$, we have the crucial relation that

$$c_1^2(\mathcal{E}) > 4c_2(\mathcal{E}).$$

In particular, we know that $\mathcal{E}$ is un-stable. In other words, by the Bogomolov lemma [4], we know that there is a sub-line bundle $L \subsetneq \mathcal{E}$ and a cluster ζ on S such that

(1) There exists a diagram with row and column being exact:

$$\begin{tikzcd}
& & 0 & & \\
& & \mathcal{I}_\zeta \otimes \mathcal{O}_S(K_S - C - L) \arrow[u] & & \\
0 \arrow[r] & \mathcal{O}_S \arrow[r] & \mathcal{E} \arrow[u] \arrow[r] & \mathcal{O}_S(K_S - C) \arrow[r] & 0, \\
& & \mathcal{O}_S(L) \arrow[u] \arrow[ur, bend right] & & \\
& & 0 \arrow[u] & &
\end{tikzcd}$$

where $\mathcal{I}_\zeta$ denotes the ideal sheaf of the 0-cycle, L is a line bundle on S;

(2) $(K_S - C - L)L + |\zeta| = 0$;

(3) $(L - (K_S - C - L))H > 0$ for any ample line bundle H over S.

In particular from (3), we know that

$$LK_S \geq \frac{1}{2}(K_S - C)K_S. \tag{4}$$

Indeed, K_S is big and nef, so for any $\varepsilon > 0$, we have $K_S + \varepsilon H$ is ample for any ample line bundle H. Thus if we choose ε small enough, then we have the wanted relation for the intersection with K_S. That is, we have

$$LK_S \geq \frac{1}{2}K_S^2 > 0.$$

As a corollary, we also have the oblique imbedding.

On the other hand, as $(*)$ is a nontrivial extension, there is a non-zero effective divisor $E > 0$ such that

$$L+E=K_S-C.$$

In fact, $E=0$ implies that the exact sequence (*) is split.

Next, we study the situation case by case.

(I) Case $K_S^2=3$. By (4),

$$3=K_S^2=K_S(K_S-C)=K_S(L+E)\geq K_SE+\frac{3}{2}.$$

From this, we have

$$K_SL=2,\quad K_SE=1;\quad \text{or}\quad K_SL=3,\quad K_SE=0.$$

(A) If $K_SL=2, K_SE=1$, we have $K_S(E+C)=1$. By the algebraic index theorem and the adjunction formula, we know that

$$E^2\leq -1,\quad (E+C)^2\leq -1.$$

Note that, by (2) and the definition of E,

$$2+|\zeta|=(C+L)L=1+E(E+C),$$

hence we have

$$E(E+C)=1+|\zeta|,$$

which implies $EC\geq 2$. Therefore

$$-1\geq (E+C)^2=E(E+C)+C(E+C)\geq E(E+C)+2-2=1+|\zeta|.$$

This is a contradiciton.

(B) If $K_SL=3$, $K_SE=0$. Then E is a positive combination of (-2)-curves. Hence $E^2\leq -2$ and $(E+C)^2\leq -2$. (Otherwise, by the algebraic index theorem, E is numerically equivalent to zero. In particular, $HE=0$ for any ample line bundle, which contradicts to the fact that E is a non-zero effective divisor.) As before,

$$3+|\zeta|=(C+L)L=3+E(C+E),$$

hence we have $|\zeta|=E(C+E)$. Thus

$$CE=|\zeta|-E^2\geq 2.$$

Therefore

$$-2\geq (E+C)^2=E(E+C)+C(E+C)\geq E(E+C)+2-2=|\zeta|\geq 0.$$

We also get a contradiction.

So we know that if $K_S^2=3$, then C is not a component of the fixed part of the bicanonical map.

(II) Case $K_S^2=4$. In this case, we have, by (4),

$$4=K_S^2=K_S(K_S-C)=K_S(L+E)\geq K_SE+2.$$

Therefore,

$$K_SL=2,\quad K_SE=2;\quad \text{or}\quad K_SL=3,\quad K_SE=1;\quad \text{or}\quad K_SL=4,\quad K_SE=0.$$

In any case, by the algebraic index theorem, we know that $E^2\leq 0$ and $(E+C)^2\leq 0$.

(A) Subcase $K_SL=2$, $K_SE=2$. As before, in this subcase, we have, by (2) and the definition of E,

$$\begin{aligned}0\geq(E+C)^2&=E(E+C)+C(E+C)=(K_S-L-C)(K_S-L)+(CE+C^2)\\&=(2+|\zeta|)+(CE-2)=|\zeta|+CE.\end{aligned}$$

On the other hand,

$$EC\geq E(E+C)=(K_S-L-C)(K_S-L)=2+|\zeta|.$$

This contradicts the above result.

(B) Subcase $K_SL=3$, $K_SE=1$. The same calculation as in (II.A) leads to

$$1-|\zeta|\geq CE\geq 1+|\zeta|.$$

Hence $CE=1$, $|\zeta|=1$. But $E(C+E)=1+|\zeta|=1$, so $E^2=0$, which contradicts the fact that

$$E^2+K_SE\equiv 0 \pmod 2.$$

(C) Subcase $K_SL=4$, $K_SE=0$. In this case, by the algebraic index theorem, we know that $E^2\leq -2$ and $(E+C)^2\leq -2$. (Otherwise, we get a contradicion from that fact that an effective divisor cannot be numerically equivalent to zero.) Hence, by a similar discussion as in (II.A), we have

$$-|\zeta|\geq CE\geq|\zeta|.$$

So $CE=|\zeta|=0$. But $E(E+C)=E(K_S-L)=-EL=-(K_S-L-C)L=|\zeta|=0$, hence we get $E^2=0$. This is a contradiction too.

So we also know that if $K_S^2=4$, then C is not a component of the fixed part of the bicanonical map. This proves the theorem. Q.E.D.

Next, we will use the above theorem to control the fixed component of the bicanonical map. As we said before, the most difficult point is about the part consisting of (-2)-curves. The reason is that for (-2)-curves C, $K_SC=0$, so we cannot use the intersection information to control the appearence of them. Now as we know that there is no (-2)-curve in the fixed part of $|2K_S|$. So we could control it in a good way.

Let $|2K_S|=|M|+V$ be the decomposition of the bicanonical system into the moving part $|M|$ and the fixed part V. We assume that $V\neq 0$. Obviously, $K_SM\geq 1$. On the other hand, by our previous theorem, we know that $K_SV\geq 1$. Therefore, we know that $MV\geq 3$. In fact, it is immediate from proposition 6.2 of [1], p.219, which is a direct consequence of the algebraic index theorem. Now, by the work of Xiao [6], we know that only when $K_S^2=1,2$, the bicanonical map is composed with a pencil, provided $p_g=0$. Therefore we know that $\Phi_{|2K_S|}(S)$ is a non-degenerate surface in $P^{K_S^2}$, with $M^2\geq 4$. Here the number K_S^2 is just $h^0(2K_S)-1$, (since by the assumption that $p_g(S)=0$ and S is a surface of general type, we get $q(S)=0$,) while $M^2\geq 4$ comes from the fact that S is a surface of general type. Indeed, if the map is generically 1-1, then the degree of the image surface is at least 4, while if the morphism is generically n to 1 with $n\geq 2$, then, by the fact that the image surface is not degenerate, we know that the degree of the image surface is at least 2, hence $M^2\geq 2n\geq 4$. In particular, we get

$$2K_SM=M^2+MV\geq 4+3=7, \text{ i.e., } K_SM\geq 4.$$

So, if $K_S^2=3$ (resp. 4), using the relation $MK_S+VK_S=2K_S^2$, we have

$$(MK_S, VK_S)=(4,2), \text{ or } (5,1) \text{ (resp. } (4,4), (5,3), (6,2), \text{ or } (7,1)).$$

Then we may study each case. As an illustration, we here only give the full details for the following two typical cases:

(I) Case $MK_S=4$, $VK_S=2$. We have

$$8=2K_SM=M^2+MV, \quad 4=2K_SV=MV+V^2.$$

Thus, by using

$$M^2+K_SM\equiv MV+V^2\equiv 0 \pmod 2,$$

we get

$$(M^2, MV, V^2)=(4,4,0).$$

(II) Case $MK_S=7$, $VK_S=1$. In this case, there is only one irreducible component in V, since V does not contain any (-2)-curves. Thus

$$K_SV+V^2=2p_a(V)-2\geq -2, \text{ i.e., } V^2\geq -3.$$

Furthermore, by the algebraic index theorem, we get $1=(K_SV)^2\geq K_S^2V^2=4V^2$. Hence, $V^2=-3$ or -1. Therefore

$$(M^2, MV, V^2) = (9, 5, -3), \quad \text{or} \quad (11, 3, -1).$$

The discussion for other cases are just the same, we leave the details to the reader. So, we get the following

Corollary. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $|2K_S| = |M| + V$ the decomposition of the bicanonical system into the moving part $|M|$ and the fixed part V. We assume that $V \neq 0$.*

(1) *Case $K_S^2 = 3$. We only have the following possibilities:*
   (*a*) $K_SM = 4$, $K_SV = 2$, $M^2 = 4$, $MV = 4$, $V^2 = 0$;
   (*b*) $K_SM = 5$, $K_SV = 1$, $M^2 = 5$, $MV = 5$, $V^2 = -3$;
   (*c*) $K_SM = 5$, $K_SV = 1$, $M^2 = 7$, $MV = 3$, $V^2 = -1$.

(2) *Case $K_S^2 = 4$. We only have the following possibilities:*
   (*a*) $K_SM = 4$, $K_SV = 4$, $M^2 = 4$, $MV = 4$, $V^2 = 4$;
   (*b*) $K_SM = 5$, $K_SV = 3$, $M^2 = 5$, $MV = 5$, $V^2 = 1$;
   (*c*) $K_SM = 6$, $K_SV = 2$, $M^2 = 4$, $MV = 8$, $V^2 = -4$;
   (*d*) $K_SM = 6$, $K_SV = 2$, $M^2 = 6$, $MV = 6$, $V^2 = -2$;
   (*e*) $K_SM = 6$, $K_SV = 2$, $M^2 = 8$, $MV = 4$, $V^2 = 0$;
   (*f*) $K_SM = 7$, $K_SV = 1$, $M^2 = 9$, $MV = 5$, $V^2 = -3$;
   (*g*) $K_SM = 7$, $K_SV = 1$, $M^2 = 11$, $MV = 3$, $V^2 = -1$.

In particular, note the fact that $K_SD > 0$ if D is an irreducible curve, which is not (-2)-curve, we know that the number of the irreducible components of the fixed part of $|2K_S|$ is at most 2, (resp. 4) when $K_S^2 = 3$ (resp. 4).

On the other hand, with the same technique as in the proof of the theorem, we may know that (2.g) in the above corollary does not really occur, since we know that

$$1 = (K_S - V)^2 > 0,$$

we have another un-stable rank-two vector bundle $\mathscr{E}_1$, and the same method applies.

As a final remark, we would like to point out the following fact: One may go further with the above results. For example, in the cases (1.c), (2.a) and (2.e) in the corollary, we may show that there is a non-trivial extension of vector bundles

$$0 \to \mathcal{O}_S \to \mathscr{E}_2 \to \mathcal{O}_S(K_S - V) \to 0$$

with $\mathscr{E}_2$ coming from a non-trivial PU(2)-representation of $\pi_1(S)$. For more details, please see my MPI preprint 90–40. We do not state such a partial result here, because we believe that the conjecture mentoined at the beginning of this paper holds, and understand that we really need a method to completely solve this problem, with the help of the result here. Another partial result in this direction may be found in [3].

ACKNOWLEDGEMENT. Thank Dr. K. Zuo for his helpful discussion when I was at MPI fur Mathematik, Bonn. We also want to thank the referee for his (or her) suggestions about the presentation.

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