Motivic Euler Product and Its Applications

Lin Weng

Abstract

We begin with a construction of non-abelian motivic zeta functions for curves over any base field, using moduli stacks of semi-stable bundles. As an application, we define motivic Euler products. Then, we introduce genuine zeta functions for Riemann surfaces and establish their convergences, based on the theory of Ray-Singer analytic torsions. To understand common features of these zetas, we next introduce natural motivic measures for the associated adelic spaces and hence obtain a motivic Siegel-Weil formula for the total mass of $\mathcal{G}$-torsors in terms of special values of motivic zetas, using the newly defined motivic Euler product. Moreover, we, using parabolic reduction and stability, obtain natural decompositions for moduli stacks of $\mathcal{G}$-torsors, and prove the parabolic reduction, stability and the mass conjecture for $\mathcal{G}$-torsors relating the total mass and the semi-stable masses. Finally, with Atiyah-Bott's analogue between Riemann surfaces and curves over finite fields and the conformal field theory in mind, we conjecture that our analytic zeta functions for Riemann surfaces are motivic, and hence unify our algebraic and analytic zetas.

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1 Motivic Euler Products

1.1 Motivic Zeta Functions

Let k be a field, and X an irreducible, reduced regular projective curve of genus g. Denote by $M_{X,n}(d)$ the moduli stack of k-rational semi-stable bundles on X. We want to study its class in the Grothendieck ring of k-stacks $K_0^\omega(Str_k)$.

For this purpose, introduce the *non-abelian rank n (complete) motivic zeta function for X* by

$$\begin{aligned}\widehat{Z}_{X,n}(u) &:= \sum_{m\geq 0}\int_{M_{X,n}(mn)} \frac{\mu(H^0(X,V)\backslash\{0\})}{\mu(\mathrm{Aut}(V))}\cdot u^{\chi(X,V)}d\mu(V)\\ &= \sum_{m\geq 0}\int_{M_{X,n}(mn)} \frac{\mathbb{L}^{h^0(X,V)}-1}{\mu(\mathrm{Aut}(V))}\cdot u^{\chi(X,V)}d\mu(V)\end{aligned}$$

viewed as an element in $K_0^\omega(Str_k)((u))$. Here, for a stack $\mathfrak{X}$ (over k), denote by $\mu(\mathfrak{X})$ its class in $K_0^\omega(Str_k)$, and write $\mathbb{L} := \mu(\mathbb{A}_k^1)$. Then it is well known that $K_0^\omega(Str_k) = \widehat{K_0^\omega(Var_k)}[\frac{1}{\mathbb{L}}]$ where $\widehat{\ }$ denotes the $\mathbb{L}$-adic completion and $K_0^\omega(Var_k)$ denotes the Grothendieck ring of k-varieties. Consequently, the motivic zetas are well-defined.[1] Quite often we also use the non-abelian rank n motivic zeta function for X defined by:

$$Z_{X,n}(u) := \sum_{m\geq 0}\int_{M_{X,n}(mn)} \frac{\mathbb{L}^{h^0(X,V)}-1}{\mu(\mathrm{Aut}(V))}\cdot u^{\deg(V)}d\mu(V)$$

Motivated by [Weng] and [HN], introduce the motivic α and β-invariants of X by

$$\alpha_{X,n}^\omega(d) := \int_{V\in\mathbb{M}_{X,n}(d)} \frac{\mathbb{L}^{h^0(X,V)}-1}{\mu(\mathrm{Aut}(V))}d\mu(V)$$

and

$$\beta_{X,n}^\omega(d) := \int_{V\in\mathbb{M}_{X,n}(d)} \frac{1}{\mu(\mathrm{Aut}(V))}d\mu(V).$$

Then, tautologically, by a direct computation using the vanishing theorem for semi-stable bundles, the duality and the Riemann-Roch theorem, we have

[1] For this reason, it would be better to call the above zetas $\mathbb{A}^1$-homological zetas. We notice then that our studies here and that of [ABK] can be unified.

Theorem 1. *We have*

$$\widehat{Z}_{X,n}(u) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn) \cdot \Big(\big(\frac{1}{u^n}\big)^{(g-1)-m} + \big((\mathbb{L}u)^n\big)^{(g-1)-m} \Big)$$
$$+ \alpha_{X,n}(n(g-1)) + \beta_{X,n}(0) \cdot \frac{(\mathbb{L}^n - 1)u^n}{(1-\mathbb{L}^n u^n)(1-u^n)}$$

From this, we see easily that $\widehat{Z}_{X,n}(u)$ is a rational function in u^n, satisfies the standard functional equation

$$\widehat{Z}_{X,n}(\frac{1}{\mathbb{L}u}) = \widehat{Z}_{X,n}(u)$$

and its residue to $u^n = 1$ is (essentially) given by $\beta^{\omega}_{X,n}(0)$. For this reason, we sometimes also write $\widehat{Z}_{X,n}(1) = \beta^{\omega}_{X,n}(0)$.

1.2 Motivic Euler Product for Curves

In the discussion above, we have used the fact that there is always a degree 1 line bundle over X/k. This can be proved by a standard discussion on the poles of the rank one zeta functions with the help of the motivic Euler product to be introduced in this section: Indeed, if $k = \bar{k}$, there is nothing to prove. Otherwise, there exists a certain prime l such that for a certain degree l extension k_l of k, we have the decomposition $Z_{X_l}(u) = \prod_{\xi\in\mu_l} Z_X(\xi u)$. Here, as usual, $X_l := X \times_k k_l$.

When $n = 1$, we have

$$Z_{X,1}(u) := \sum_{d\geq 0} \int_{\mathrm{Pic}_X(d)} \frac{\mu(H^0(X,L)\backslash\{0\})}{\mu(\mathrm{Aut}(L))} \cdot u^{\deg(L)} d\mu(L)$$

where as usual $\mathrm{Pic}_X(d)$ denotes the degree Picard variety of X. Then by the fact that non-zero global sections, up to a non-zero constant scalar factor, correspond in one-to-one to effective divisors on X, we see that

$$Z_{X,1}(u) = \sum_{d\geq 0} \mu(\mathrm{Eff}_X^{(0)}(d)) \cdot u^d,$$

where $\mathrm{Eff}_X^{(0)}(d)$ denotes the stack of degree d effective 0-cycles on X.

Accordingly, introduce the *motivic Euler product* by

$$\prod\nolimits^{\omega}_{x\in X} \frac{1}{1-u_x} = \prod\nolimits^{\omega}_{x\in X} \frac{1}{1-u^{\deg(x)}} := \sum_{d\geq 0} \mu(\mathrm{Eff}_X^{(0)}(d)) \cdot u^d.$$

Here as usual $d(x) := [k(x) : k]$ with $k(x)$ the residue field of the closed point $x \in X$ and $u_x := u^{\deg(x)}$.

Example 1. *When $k = \mathbb{F}_q$, then $\prod^{\omega}_{x\in X}$ is the standard Euler product $\prod_{x\in X}$, since closed points on $X/\mathbb{F}_q$ are only countable. However, our motivic Euler product is much more general. In the case when $k = \overline{k}$ is algebraically closed, e.g., $k = \mathbb{C}$, the points involved in the motivic product are not countable. Moreover, in this case, $\mathrm{Eff}_X^{(0)}(d)$ is nothing but $\mathrm{Sym}^{(d)}X$, the d-th symmetric product of X. Consequently, our zeta function $Z_X(u)$ coincides with motivic zeta function $Z_X(u)$ in [K] used by geometers.*

1.3 Motivic Euler Product in General

More generally, for any algebraic variety X over a field k we define its motivic zeta function by

$$Z_X(u) := \sum_{n\geq 0} \mu(\mathrm{Eff}_X^{(0)}(n))u^n$$

where $\mathrm{Eff}_X^{(0)}(n)$ denotes the stack of effective 0-cycles of degree n on X. Accordingly, define the motivic Euler product $\prod_{x\in X}^{\omega}$ by

$$\prod\nolimits_{x\in X}^{\omega} \frac{1}{1-u_x} = \sum_{n\geq 0} \mu(\mathrm{Eff}_X^{(0)}(n))u^n.$$

For examples, if $k=\overline{k}$, then $\mathrm{Eff}_X^{(0)}(n)$ is nothing but $\mathrm{Sym}^{(n)}X$, the n-th symmetric product of X. So,

$$\prod\nolimits_{x\in X}^{\omega} \frac{1}{1-u_x} = \sum_{n\geq 0} \mu(\mathrm{Sym}^{(n)}X)u^n$$

the geometrical motivic zeta; And, when $k=\mathbb{F}_q$, since there are only countable algebraic points, our motivic Euler product coincides with the standard one, and the above product simply recovers Artin-Weil's zeta function.

Note that for a closed $Y\hookrightarrow X$ with $U=X\backslash Y$, hence

$$Z_X(u) = Z_Y(u)\cdot Z_U(u),$$

we have

$$\prod\nolimits_{x\in X}^{\omega} = \prod\nolimits_{x\in Y}^{\omega} \cdot \prod\nolimits_{x\in U}^{\omega}.$$

This latest property is basic to the theory. For example, we have the associative and commutative law and hence a well-defined motivic product:

$$\prod\nolimits_{i\in I}\prod\nolimits_{x\in X_i}^{\omega}\cdots \qquad \forall i\in I,\quad \#I<\infty. \tag{$*$}$$

2 Analytic Torsion and Genuine Zetas for Riemann surfaces

2.1 Regularized Integrations

2.1.1 Uniformizing Metrics

Let M be a compact Riemann surface of genus g. Denote its fundamental group by $\pi_1(M)$ and the associated uniformizing map by $\pi:\widetilde{M}\to M$. As a simply connected $\pi_1(M)$-space, $\widetilde{M}$ is well known to be isomorphic to either the complex projective line $\mathbb{P}^1$, or the complex plane $\mathbb{C}$, or the upper half complex plane $\mathbb{H}$, depending on whether $g=0$ or 1 or ≥ 2. Accordingly, put, on $\widetilde{M}$, the standard Fubini-Study metric, or the canonical flat metric, or the standard hyperbolic metric. Induced from the uniformization map π, we then obtain a natural metric μ on M which we call standard. Denote its normalized volume form by ω (so that $\int_M \omega = 1$). For a line bundle L of degree d, by definition, an

ω-admissible metric h on L is a hermitian metric on L such that $c_1(L,h)=d\cdot\omega$. One checks that ω-admissible metrics always exist and for a fixed line bundle they are parametrized by positive reals. For our purpose, fix a point $P_0\in M$ and use the normalized Green function ([L]) to define the ω-admissible metric h_A on the line bundle $A=A_M$ corresponding to the invertible sheaf $\mathcal{O}_M(P_0)$. (From now on, we will not make distinctions between bundles and locally free sheaves.) Moreover, for the metric μ on M, denote its induced ω-admissible metric on the canonical line bundle K_M by $\tau=\tau_\mu$.

Let $\mathcal{M}_{M,r}(d)$ be the moduli stack of semi-stable bundles of rank r and degree d on M. For all $m\in\mathbb{Z}$, there are natural isomorphisms

$$\mathcal{M}_{M,r}(0)\simeq\mathcal{M}_{M,r}(mr),\qquad V\mapsto V\otimes A^{\otimes m}.$$

By a result of Narasimhan-Seshadri [NS], stable points V of $\mathcal{M}_{M,r}(0)$ are in one-to-one correspondence with irreducible unitary representations ρ of $\pi_1(M)$. Consequently,

$$V\simeq V_\rho:=\widetilde{M}\times\mathbb{C}^r/\sim,\quad \text{with } (x,v)\sim(gx,\rho(g)v),\ \forall(x,v)\in\widetilde{M}\times\mathbb{C}^r,\ g\in\pi_1(M).$$

Hence, for such a V_ρ, from the uniformization, we get a natural induced hermitian metric h_ρ (from the standard one on $\mathbb{C}^r$).

To go further, recall that if the bundle extension on M

$$0\to V_1\to V_e\to V_3\to 0$$

is corresponding to $e\in\mathrm{Ext}^1(V_2,V_1)$, then for any fix hermitian metrics h_i on $V_i, i=1,2$, there exists a unique metric h_e on V_e defined using h_i's and the standard metric on the off diagonal entries. Since each degree zero semi-stable bundle is obtained canonically, up to isomorphism (in particular, not up to S-equivalence,) as successive extensions of stable bundles of degree zero, from above, we then obtain a unique hermitian metric h on a fixed degree zero semi-stable bundle V of rank r. Hence, by tensoring with the metric $h_A^{\otimes m}$ on $A^{\otimes m}$ ($m\in\mathbb{Z}$), we get a unique hermitian metric h on every degree mr semi-stable bundles V of rank r on M. For simplicity, denote such a metrized semi-stable bundle (resp. Riemann surface M) by $\overline{V}$ (resp. $\overline{M}$).

2.1.2 Analytic Torsions

To go further, we next recall some basic facts about analytic torsions. Let $\overline{V}/\overline{M}$ be a metrized vector bundle over a metrized Riemann surface. Denote the associated Laplacian by D_V on the associated space of L^2 sections $L^2(\overline{M},\overline{V})$ of V on M. From the Fredholm theory, the spectrum of D_V is a purely discrete sequence $0\leq\lambda_1\leq\lambda_2\leq\lambda_3\leq\dots,\quad \lambda_n\sim\frac{1}{r(g-1)}n$ with corresponding eigenfunctions $\{e_n(z,V)\}$ forming a complete orthonormal basis for $L^2(\overline{M},\overline{V})$. Accordingly, following Ray-Singer [RS], define for $\mathrm{Re}(\lambda)>0$ and $\mathrm{Re}(s)>1$, the spectrum zeta function $\zeta_\lambda(s,V):=\mathrm{Tr}(D_L+\lambda)^{-s}=\sum_{n=1}^\infty\frac{1}{(\lambda+\lambda_n)^s}$ and more generally for any $c\geq 0$, $\zeta_\lambda^c(s,V):=\sum_{\lambda>c}\frac{1}{(\lambda+\lambda_n)^s}$.

We have the following

Theorem 2. *([RS]) (i) For fixed $c\geq0$, $\zeta_\lambda^c(s,V)$ has an analytic continuation to the half plane $\mathrm{Re}(\lambda)\geq c$ and a meromorphic continuation to the whole s-plane with only a simple pole at $s=1$ with residue $r(g-1)$.*

(ii) For $\mathrm{Re}(\lambda) > \lambda_*$ *the smallest non-zero eigenvalue of* D_V*,*

$$\zeta_\lambda^0(s,V) = \zeta_\lambda(s,V) - h^0(M,V)\lambda^{-s}$$

has an analytic continuation through the s*-plane with*

$$\zeta_\lambda^0(0,V) + h^0(M,V) = -\Big(\lambda + \frac{1}{3}\Big)n(g-1) + \frac{1}{2}\deg(V).$$

(iii) (Duality) *Equipped the dual bundle* $V^\vee$ *with the dual metric, then*

$$\zeta_\lambda^c(s,V) = \zeta_\lambda^c(s, K_M \otimes V^\vee)$$

Based on this, define the *(Ray-Singer) analytic torsion* for $\overline{V}(/\overline{M})$ by

$$T(V) := T(\overline{M};\overline{V}) := e^{-\tau(V)}$$

with

$$\tau(V) := \tau(M,V) := \tau(\overline{M};\overline{V}) := \frac{d}{ds}\zeta_\lambda^0(s,V)|_{s=\lambda=0}.$$

It is well known then that $T(V)$ may be viewed as a regularized determinant of the Laplacian D_V. That is to say, formally, we have

$$T(V) = \det{}'(D_V) = \prod\nolimits'_{n:\,\lambda_n>0}\lambda_n.$$

The biggest advantage of using analytic torsions is that together with (determinant of) L^2-metrics, we get a smooth metric, the Quillen metric, on the so-called determinant bundles λ on $\mathcal{M}_{M,r}(rm)$, whose fiber at $V \in \mathcal{M}_{M,r}(rm)$ is given by $\det H^0(M,V)\otimes\det H^1(M,V)^{\otimes -1}$ and that its associated Chern form can be calculated via the so-called local family index theorem. For our own use, denote this metrized line bundle on $\mathcal{M}_{M,r}(rm)$ by $\overline{\lambda}$.

2.1.3 Stractifications of Moduli Stacks: Determinant Varieties Structures

Let $\mathcal{V}_m$ be the universal Poincaré bundle on the moduli stack $\mathcal{M}_{M,r}(rm)$. Then with respect to the projection $q : M \times \mathcal{M}_{M,r}(rm) \to \mathcal{M}_{M,r}(rm)$, for sufficiently larger d, we may *assume* that the direct images with respect to q associated to the short exact sequence of coherent sheaves on $M \times \mathcal{M}_{M,r}(rm)$:

$$0 \to \mathcal{V}_m \to \mathcal{V}_m \otimes A^{\otimes d} \to \mathcal{V}_m \otimes A^{\otimes d}/\mathcal{V}_m \to 0$$

yields an exact sequence of *vector bundles* on $\mathcal{M}_{M,r}(rm)$:

$$0 \to q_*\mathcal{V}_m \to q_*\Big(\mathcal{V}_m \otimes A^{\otimes d}\Big) \xrightarrow{\gamma} q_*\Big(\mathcal{V}_m \otimes A^{\otimes d}/\mathcal{V}_m\Big) \to R^1q_*\mathcal{V}_m \to 0.$$

Denote by $W_{M,r}^{\geq i}(rm)$ the determinantal variety associated to γ. Then one checks that $W_{M,r}^{\geq i}(rm)$ is well-defined, namely, independent of the choices of d, and the support of $W_{M,r}^{\geq i}(rm)$ coincides with the so-called *Brill-Noether locus* consisting of these whose h^0 are at least i. That is,

$$\mathrm{SuppW_{M,r}^{\geq i}(rm) = \{V \in W_{M,r}(rm) : h^0(M,V) \geq i\}}.$$

(See e.g., [ACGH, p.176] when $r = 1$.) Moreover, $W^{\geq i}_{\bar{M},r}(rm)$ are normal subvarieties of $\mathcal{M}_{M,r}(rm)$ and analytic torsion $T(V, M)$ defines a smooth function on

$$W^{i}_{M,r}(rm) = W^{\geq i}_{\bar{M},r}(rm) \setminus W^{\geq (i+1)}_{\bar{M},r}(rm)$$

when $m \in \mathbb{Z}$. By an abuse of notation, denote by $d\mu$ the volume forms on $W^{\geq i}_{\bar{M},r}(mn)$ induced by the polarization λ.

2.1.4 Regularized Integration

With all this, we are ready to introduce the regularized integration by

$$\begin{aligned}
&\int^{\#}_{\mathcal{M}_{M,r}(rm)} \Big(e^{T(M,V)} - 1\Big)\big(e^{-s}\big)^{\chi(M,V)} d\mu \\
&\quad := \sum_{i=0}^{\infty} \int_{W^{i}_{M,r}(rm)} \Big(e^{T(M,V)}\Big)\big(e^{-s}\big)^{\chi(M,V)} d\mu - \int_{\mathcal{M}_{M,r}(rm)} \big(e^{-s}\big)^{\chi(M,V)} d\mu.
\end{aligned}$$

2.2 Zeta Facts I: Formal Aspect

Let X be a compact Riemann surface of genus g. Using the uniformization, for any stable bundle V of rank n and degree mn, $m \in \mathbb{Z}$, we can canonically associate the analytic torsion $\tau(\bar{M}, \bar{V})$ to the metrized $\bar{M}$ and metrized $\bar{V}$. As such define the associated rank n zeta function of M by

$$\widehat{\zeta}_{X,n}(s) := \sum_{m=-\infty}^{\infty} \int^{\#}_{\mathcal{M}_{X,n}(mn)} \Big(e^{e^{-\tau(\bar{X},\bar{V})} + h^0(X,V)} - 1\Big)\Big(e^{-s}\Big)^{\chi(X,V)} d\mu(V).$$

Before justifying the convergence of our regularized integrations appeared in the definition of new zeta functions, let us formally establish the functional equation and find out the singularities of these zetas and hence calculate the associated residues when applicable.

Then

$$\widehat{\zeta}_{X,n}(s) = I(s) + II(s) + III(s) - IV(s)$$

with

$$\begin{aligned}
I(s) &= \sum_{m=0}^{2g-2} \int^{\#}_{\mathcal{M}_{X,n}(mn)} e^{e^{-\tau(\bar{X},\bar{V})} + h^0(X,V)} \cdot \Big(e^{-s}\Big)^{\chi(X,V)} d\mu(V), \\
II(s) &= \sum_{m<0} \int^{\#}_{\mathcal{M}_{X,n}(mn)} \Big(e^{e^{-\tau(\bar{X},\bar{V})} + h^0(X,V)} - 1\Big)\Big(e^{-s}\Big)^{\chi(X,V)} d\mu(V), \\
III(s) &= \sum_{m>2g-2} \int^{\#}_{\mathcal{M}_{X,n}(mn)} e^{e^{-\tau(\bar{X},\bar{V})} + h^0(X,V)} \cdot \Big(e^{-s}\Big)^{\chi(X,V)} d\mu(V), \\
IV(s) &= \sum_{m\geq 0} \int^{\#}_{\mathcal{M}_{X,n}(mn)} \Big(e^{-s}\Big)^{\chi(X,V)} d\mu(V).
\end{aligned}$$

We next study each of these functions.

Set $\mu^{\rm ss}_{X,n}(0) = \mathrm{Vol}\mathcal{M}_{X,n}(0)$. Then

$$\begin{aligned} IV(s) =&\mu^{\rm ss}_{X,n}(0)\sum_{m=0}^{\infty}(e^{-s})^{n[m-(g-1)]} \\ =&\mu^{\rm ss}_{X,n}(0)\cdot\frac{e^{ns(g-1)}}{1-e^{-sn}}, \qquad \mathrm{Re}(s)>0, \end{aligned} \tag{1}$$

a meromorphic function with simple poles at $s = 2\pi i\cdot\frac{1}{n}\mathbb{Z}$, whose residues are $\frac{1}{n}\mu^{\rm ss}_{X,n}(0)$.

To understand $I(s)$, note that

$$\tau(\bar X,\bar V)=\tau(\bar X,\bar K_X\otimes\bar V^\vee),\qquad \chi(X,K_X\otimes V^\vee)=-\chi(X,V).$$

We get

$$\begin{aligned} I(s)=&\sum_{m=2g-2}^{0}\int^{\#}_{\mathcal{M}_{X,n}(mn)} e^{e^{-\tau(\bar X,\bar V)}+h^1(X,V)}\cdot\Big(e^{-s}\Big)^{-\chi(X,V)}d\mu(V) \\ =&\sum_{m=0}^{2g-2}\int^{\#}_{\mathcal{M}_{X,n}(mn)} e^{e^{-\tau(\bar X,\bar V)}+h^0(X,V)}\cdot\Big(e^{-(1-s)}\Big)^{\chi(X,V)}d\mu(V) \\ =&I(1-s) \end{aligned} \tag{2}$$

a holomorphic function in s. (In fact, $I(s)$ is a rational function in $T=t^n$ with $t=e^{-s}$.)

To see $II(s)$, using the fact that $\tau(\bar X,\bar V)=\tau(\bar X,\bar K_X\otimes\bar V^\vee)$ again, we get

$$II(s)=\sum_{m>2g-2}\int^{\#}_{\mathcal{M}_{X,n}(mn)}\Big(e^{e^{-\tau(\bar X,\bar V)}+h^0(X,V)}-1\Big)\Big(e^{s}\Big)^{\chi(X,V)}d\mu(V).$$

By Thm 9 of [W], we know that $\tau(\bar X,\bar V)=O(m^n\log m)$. Thus the convergence of $II(s)$ comes form that for the series

$$\sum_{m>2g-2}\Big(e^{1/(m^{m^n})}-1\Big)\cdot\big(e^{ns}\big)^m\qquad\forall s.$$

In particular, $II(s)$ is holomorphic in s.

Similarly, if we set

$$V(s):=\sum_{m>2g-2}\int^{\#}_{\mathcal{M}_{X,n}(mn)}\Big(e^{1-s}\Big)^{\chi(X,V)}d\mu(V),$$

then

$$\begin{aligned} III(s)=&\sum_{m>2g-2}\int^{\#}_{\mathcal{M}_{X,n}(mn)} e^{e^{-\tau(\bar X,\bar V)}}\Big(e^{1-s}\Big)^{\chi(X,V)}d\mu(V) \\ =&\sum_{m>2g-2}\int^{\#}_{\mathcal{M}_{X,n}(mn)}\Big(e^{e^{-\tau(\bar X,\bar V)}}-1\Big)\Big(e^{1-s}\Big)^{\chi(X,V)}d\mu(V)+V(s) \\ =&II(1-s)+V(s). \end{aligned}$$

Thus $III(s) - V(s) = II(1-s)$ is holomorphic in s.

Finally,

$$\begin{aligned} V(s) =& \mu_{X,n}^{\rm ss}(0) \sum_{m>2g-2} \left(e^{1-s}\right)^{n[m-(g-1)]} \\ =& \mu_{X,n}^{\rm ss}(0) \cdot e^{(1-s)ng} \sum_{m\geq 0} \left(e^{1-s}\right)^{nm} \\ =& \mu_{X,n}^{\rm ss}(0) \cdot \frac{e^{(1-s)ng}}{1-e^{(1-s)n}} \\ =& \mu_{X,n}^{\rm ss}(0) \cdot \frac{e^{(1-s)n(g-1)}}{e^{(s-1)n}-1} = -IV(1-s) \qquad \mathrm{Re}(s)>1, \end{aligned}$$

a meromorphic function in s with simple poles at $s = 1+2\pi i \cdot \frac{1}{n}\mathbb{Z}$, whose residues are $\frac{1}{n}\mu_{X,n}^{\rm ss}(0)$. Thus all in all, we have just established the following

Theorem 3. *Let $\widehat{\zeta}_{X,n}(s)$ be the rank n zeta function for compact Riemann surface X. Then*
(i) it is a well-defined meromorphic function for $\mathrm{Re}(s) > 1$ *and admits a meromorphic continuation in the whole complex s-plane;*
(ii) $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$;
(iii) the singularities are concentrated on $2\pi i \cdot \frac{1}{n}\mathbb{Z}$ and $1 + 2\pi i \cdot \frac{1}{n}\mathbb{Z}$, all simple poles with residues $\frac{1}{n}\mu_{X,n}^{\rm ss}(0)$ (at $s = 1$).

2.3 Zeta Facts II: Analytic Aspect

To establish the zeta facts for our zeta functions, two types of convergences should be justified properly. Namely, the one for regularized integrations over the moduli spaces $\mathcal{M}_{M,r}(rm)$ for a fixed m, and the other for the infinite sum on $m, m > 2g-2$ appeared in $III(s)$. As we will see below, these two are very different in nature: Technically, for the first type, we need to see how the analytic torsions $T(V)$ degenerate when h^0 jump; while for the second, we need to understand how the analytic torsions $T(V \otimes A^{\otimes m})$ behave when $m \to \infty$.

2.3.1 Degenerations of Analytic Torsions

In this subsection, we will establish the convergence of the regularized integration

$$\int_{\mathcal{M}_{M,r}(rm)}^{\#} \left(e^{T(M,V)+h^0(M,V)} - 1\right) \left(e^{-s}\right)^{\chi(M,V)} d\mu$$

for each fixed m. There are a few issues here: First, $\mathcal{M}_{M,r}(rm)$ are not compact; second, $W_{M,r}^{\geq i}(rm)$ are not smooth; and finally, $T(V)$ are not smooth on $W_{M,r}^{\geq i}(rm)$.

A. Non-Compactness This is not really that serious, thanks to the classical works done. In fact, using Mumford's GIT, we now have a natural compactification $\overline{\mathcal{M}_{M,r}(rm)}$ in terms of Seshadri's equivalences of semi-stable bundles. Denote by $\partial\Big(\mathcal{M}_{M,r}(rm)\Big) := \overline{\mathcal{M}_{M,r}(rm)} \backslash \mathcal{M}_{M,r}(rm)$ the associated boundary.

It is well-known that this boundary is of much higher co-dimension, and hence does not cause any serious trouble for the integration.

B. Geometric Singularities

Note that for any open subset $U \subset \mathcal{M}_{M,r}(rm)$, if h^0 is a constant on U, then $T(V)$ is smooth on U. Thus we only need to consider the case when $0 \leq m \leq g-1$ with the duality for analytic torisons in mind.

From the structure of determinantal varieties, it is well-known that the singularities of $W_{M,r}^{\geq i}(rm)$ is contained in $W_{M,r}^{\geq (i+1)}(rm)$, which has codimension at least 2 except in the case when $m = g-1$. Thus, in pour discussion, we will use the most complicated level, i.e., $m = g-1$, to show how the convergence can be established. For other levels, which are much simpler, a consideration using the following insertion formula for analytic torsions ([AGBMNV]) is sufficient to complete the argument. (For unknown notations, please consult [F].)

Theorem 4. *(see e.g. [F, Thm 4.13]) Let L be a line bundle of degree $d \geq g$ with an admissible metric h. Then for all stable bundles $V = V_\rho$ with uniformizing metric, $h^1(M, V_\rho \otimes L) = 0$ and for any points $x_1, \ldots, x_{N:=d+1-g} \in M$:*

$$T(V_\rho \otimes L) = \varepsilon_d(M) \frac{T(V_\rho \otimes L(-\sum_{i=1}^N P_i))}{\det(B(x_i, \overline{x}_j; V_\rho \otimes L))} \cdot \frac{\prod_{i<j}^N P(x_i, x_j)^{2r}}{\prod_{i=1}^m h(x_i)^r}$$

where $\varepsilon_d(\rho)$ is a constant depending only on d and M.

C. Analytic Singularities

So from now on, we concentrate on the level $g-1$. For this, we recall some facts on both abelian and non-abelian theta functions. The explosion here follows closely that of Fay [F] (please check the meaning of the unknown notations below in [F] as well).

Theorem 5. *(**Theta Functions** [F, Thm 1.6]) Let L be a fixed line bundle with $h^0(\chi(s) \otimes L)$ constant for s in some neighborhood V containing $0 \in \mathbb{C}^N$; choose $\{\omega_i\}$, $\{\omega_i^*\}$ bases for $H^0(\chi(s) \otimes L)$, $H^0(\chi^*(s) \otimes KL^{-1})$ with $\{M^{-1}(z,s)\omega_i\}$, $\{{}^tM(z,s)\omega_i^*\}$ holomorphic in $s \in V$. Then*
(i) for $s \in V$,

$$T(\chi \otimes L) = U(\chi(s))|f(s)|^2 \det(\langle \omega_i, \omega_j \rangle_{\chi(s)\otimes L}) \det(\langle \omega_i^*, \omega_j^* \rangle_{\chi^*(s)\otimes KL^{-1}})$$

where $f(s)$ is a holomorphic function on V depending on L, the fixed potential U, the bases $\{\omega_i\}$, $\{\omega_j^*\}$, and the metrics $h, I \otimes h$ and ρ. In particular,
(ii) in a neighborhood of any point $\chi(0)$ where $h^0(\chi(s) \otimes \Delta) = 0$ with Δ a Riemann divisor class satisfying $h^0(M, \Delta) = 0$,

$$T(\chi(s) \otimes L) = c_0^l(h, \rho)\, U(\chi(s))\, |\theta(\chi(s))|^2$$

with $\theta(\chi(s)) = \theta(s)$ holomorphic in s and independent of the metrics h, ρ.

Theorem 6. *(**Vanishing of Non-Abelian Theta**, [F, Prop 4.7, Thm 4.8]) There exists a holomorphic section θ_r of the determinant line bundle λ on $\mathcal{M}_{M,r}(0)$ such that*

(i) $\theta_r(V_\rho)=0$ *if and only if* $h^0(M,\mathrm{End}V_\rho\otimes\Delta)>0$*;*
(ii) $\frac{\theta_r(E_\rho)\theta(\det E_\rho)^2}{\theta(\rho)^{2r}}$ *is a meromorphic function on* $\mathcal{M}_{M,r}(0)$*;*
(iii) As a bundle on $\mathcal{M}_{M,r}(0)$*,* $\lambda\simeq K^\vee_{\mathcal{M}_{M,r}(0)}$*, the dual of the canonical line bundle of* $\mathcal{M}_{M,r}(0)$*.*
(iv) The section θ_r *vanishes to order* n *at any representation* $V_\rho\in\mathcal{M}_{M,r}(0)$ *with* $h^0(V_\rho\otimes\Delta)=n$*. The tangent cone to* (θ_r) *at* V_ρ *is the sub variety of* $\overline{w}=\sum_{i=1}^{\dim\mathcal{M}_{M,r}(0)}s_i\overline{w}_i(z,\mathrm{End}V_\rho)\in\overline{H^0(M,K_M\otimes\mathrm{End}V_\rho)}$ *given by*

$$\det_{1\leq i,j\leq n}\Big(\int_M {}^te_i(z;V_\rho\otimes\Delta)\overline{w(z)}e_j(z;V_\rho^\vee\otimes\Delta)\widehat{dz}\Big)\equiv 0$$

for any fixed bases $\{e_i(z;V_\rho^{(*)}\otimes\Delta)\}$ *of* $H^0(M,V^{(*)}\otimes\Delta)$*.*

This generalizes the standard theory of abelian theta functions and the Brill-Noether loci to non-abelian setting. For example, when $r=1$, on the i-th Brill-Noether locus, the analogue of (iv) says that the analytic torsion may be calculated via the norm of the i-th partial derivatives of the standard theta functions. For details, see [F, Thm 4.9] and [ACGR].

Put all this together, we have then justified the convergence in the abelian case, namely, $r=1$. As for general non-abelian cases, such a strong result has yet been obtained. Fortunately, what needed is a much weak result which we recall below:

Theorem 7. *(***Degenerations of Analytic Torisons** *[F, Thm 4.12]) Let* L *be a line bundle of degree* $d\geq g-1$ *such that* $h^1(V_\rho\otimes L)=n>0$ *for a fixed* $V_\rho\in\mathcal{M}_{M,r}(0)$*. Then within a neighborhood* U *of* V_ρ *in* $\mathcal{M}_{M,r}(0)$*, the analytic torsions* $T(V_{\rho(s)}\otimes L)$*, which is positive whenever* $h^1(V_{\rho(s)}\otimes L)=0$*, vanishes to order* $2n$ *at* $V_\rho=V_{\rho(0)}$*. In particular, near* $s=0$*,*

$$T(V_{\rho(s)}\otimes L)=4^nT(V_\rho\otimes L)\det[\overline{{}^tC(s)}C(s)]+O(\|s\|^{2n+1})$$

where for any orthonormal basis $\{e_i\},\{e_j^*\}$ *for* $H^0(M,V_\rho\otimes L)$*,* $H^0(M,K_M\otimes(V_\rho\otimes L)^\vee)$ *respectively:*

$$C_{ij}(s):=\int_M {}^te_i(z;V_\rho\otimes L)\overline{w}e_j^*(z;K_M\otimes(V_\rho\otimes L)^\vee)\widehat{dz}.$$

And for any $x_1,\dots,x_{d+1-g}$*,*

$$\begin{aligned}&\det[\overline{{}^tC(s)}C(s)]\det[B(x_i,\overline{x_j};V_{\rho(s)}\otimes L)]\\ =&\left|\det\begin{pmatrix}{}^te_1(x_1:V_\rho\otimes L)&\cdots&{}^te_1(x_1:V_\rho\otimes L)&C_{11}(s)&\cdots&C_{1n}(s)\\ &\cdots&&&&\\ {}^te_p(x_1:V_\rho\otimes L)&\cdots&{}^te_p(x_1:V_\rho\otimes L)&C_{p1}(s)&\cdots&C_{pn}(s)\end{pmatrix}\right|^2\\ &+O(\|s\|^{2n+1})\end{aligned}$$

as $\rho(s)\to\rho$ *along any smooth curve transverse at* ρ *to the subvariety* V_1 *of all* $V_\rho\in\mathcal{M}_{M,r}(0)$ *with* $h^1(M,V_\rho\otimes L)>0$*. Here* $p=n+r(d+1-g)$*.*

Thus by the fact that the Bergman kernel admits logarithmic degeneration, we complete the proof of the convergence of the regularized integrations appeared in the proof of our zeta functions.

2.3.2 Asymptotics of Analytic Torsions

To establish the convergence of the infinite sum appeared in the definition of our zeta functions on m, we only need to understand

$$III(s) = \sum_{m>2g-2} \int_{\mathcal{M}_{M,r}(rm)} \Big(e^{T(M,V)+h^0(M,V)} - 1\Big)(e^{-s})^{\chi(M,V)} d\mu,$$

by the discussion in §1.3. Note that for stable bundle of rank r and degree mr with $m > 2g-2$, $h^0(M,V) = \chi(M,V) = r[m-(g-1)]$ is a constant. So $T(V)$ is a constant function on $\mathcal{M}_{M,r}(mr)$. Thus using the natural isomorphism

$$\mathcal{M}_{M,r}(0) \simeq \mathcal{M}_{M,r}(mr), \qquad V \mapsto V \otimes A^{\otimes m},$$

we have

$$III(s) = \sum_{m>g-1} \int_{V\in\mathcal{M}_{M,r}(0)} \Big(e^{T(M,V\otimes A^{\otimes(m+g-1)})+h^0(M,V\otimes A^{\otimes(m+g-1)})}-1\Big)(e^{-s})^{rm} d\mu.$$

As such, then the convergence is guaranteed by the following results of Faltings, Miyaoka, Bismut-Vasserot:

Theorem 8. *(See e.g., [BV, Thm 8]) As $m \to \infty$,*

$$\frac{d}{ds}\zeta_\lambda^0(s, V\otimes A^{\otimes(m+1-1)})\Big|_{\lambda=s=0} = O(m^r \log m).$$

Thus, in essence, we are dealing with the infinite summation

$$\sum_{m>g-1} \int_{M_{m,r}(0)} \Big(e^{e^{-m^r \log m}} - 1\Big)(e^{-s})^{rm} d\mu$$

which is clearly convergent. All this then justify that our analytic zeta functions for Riemann surfaces are well-defined. In particular, we have proved the following

Theorem 9. *Let M be a compact Riemann surfaces. Then the rank n analytic zeta function of M constructed by*

$$\widehat{\zeta}_{M,n}(s) := \sum_{m=-\infty}^{\infty} \int_{\mathcal{M}_{M,n}(mn)}^{\#} \Big(e^{e^{-\tau(\bar{X},\bar{V})}+h^0(M,V)} - 1\Big)\Big(e^{-s}\Big)^{\chi(M,V)} d\mu(V)$$

is well-defined. Furthermore, it satisfies the following standard zeta facts.
(i) it defines meromorphic function for $\mathrm{Re}(s) > 1$ *and admits an infinite order meromorphic continuation in the whole complex s-plane;*
(ii) $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$;
(iii) the singularities are concentrated on $2\pi i \cdot \frac{1}{n}\mathbb{Z}$ and $1 + 2\pi i \cdot \frac{1}{n}\mathbb{Z}$, all simple poles with residues $\frac{1}{n}\mu_{X,n}^{\mathrm{ss}}(0)$ (at $s=1$).

3 Adelic Motivic Measures

3.1 Motivic Measure: Adelic Spaces

Let $F = k(X)$ be the function field of an irreduciblele, reduced regular projective curve X of genus g over the base field k. Denote by $\mathbb{A} = \mathbb{A}_F = \prod_{x\in X}^{\omega,'}(F_x, \mathcal{O}_x)$ its associated adelic ring. Namely, the restricted product of the local fields K_x of X at x with respect to the rings of integers $\mathcal{O}_x$. Here, as usual, for $x \in X$, K_x denotes (the completion of) the local field of X at x, and $\mathcal{O}_x$ its ring of integers.

For $x \in X$, let $|\omega_x|$ be the motivic measure on F_x such that

$$\mu(\mathcal{O}_x) = \int_{\mathcal{O}_x} |\omega_x| = 1.$$

With this, we claim that the theory of motivic integrations on $\mathbb{A}$ can be developed, with the help of two mutually compatible topologies, i.e., ind-pro topology and locally linearly compact topology on $\mathbb{A}$. To justify this, recall that from the ind-pro topology, we have the following natural diagram

$$\begin{array}{ccccccc}
 & & & \mathcal{O}_x = \lim_{\leftarrow n} \mathcal{O}_x/\mathbf{m}_x^n & \longrightarrow & \mathcal{O}_x/\mathbf{m}_x = k(x) \\
 & & & \downarrow & \searrow & \\
\{0\} \subset \mathbf{m}_x^{n-1}/\mathbf{m}_x^n \subset \cdots \subset & & \mathbf{m}_x^1/\mathbf{m}_x^n \subset & \mathcal{O}_x/\mathbf{m}_x^n & \ldots & \mathcal{O}_x/\mathbf{m}_x^2 \\
 & & & & & \\
k(x) \qquad k(x)\ldots k(x) & & k(x) & & &
\end{array}$$

Consequently,

$$\mu(\mathbf{m}_x^n) = \mathbb{L}_x^{-n}$$

with $\mathbb{L}_x = \mathbb{L}^{\deg(x)}$ since from above $\mu(\mathcal{O}_x) = \mu(\mathbf{m}_x^n)\cdot\mu(\mathcal{O}_x/\mathbf{m}_x^n) = \mu(\mathbf{m}_x^n)\cdot\mathbb{L}_x^n$.

More generally, for a divisor $D = \sum_x n_x x$ on X, set

$$\mathbb{A}(D) := \{(a_x) \in \mathbb{A} : \mathrm{ord}_x(a_x) + n_x \geq 0 \forall x\}.$$

Then

$$\mathbb{A} = \lim_{\rightarrow D_1} \lim_{\leftarrow D_2 : D_2 \leq D_1} \mathbb{A}(D_1)/\mathbb{A}(D_2).$$

This then gives the ind-prod topology on $\mathbb{A}$ which is compatible the locally linearly compact topology since (i) $\mathbb{A}(D)$ is linearly compact an (ii) $\mathbb{A}(D_1)/\mathbb{A}(D_2)$ is a finite dimensional vector space, being discrete and linearly compact.

Set now $|\omega_{\mathbb{A}}| := \prod_{x\in X}^{\omega} |\omega_x|$. Then, by definition,

$$\mu(\prod_{x\in X}^{\omega} \mathcal{O}_x) = \prod_{x\in X}^{\omega} \mu(\mathcal{O}_x) = \prod_{x\in X}^{\omega} 1 = 1$$

where in the last step, the canonical property of motivic Euler product has been used. To apply this to $\mathbb{A}$, we recall the following key

Lemma 10. *We have*
(i) $\mathbb{A}(0) = \prod_{x\in X}^{\omega} \mathcal{O}_x$.
(ii) For an effective divisor D, $\mathbb{A}(D)/\mathbb{A}(0) \simeq \prod_{x\in X}^{\omega} \mathbb{L}_x^{n_x} = \mathbb{L}^{\deg(D)}$.
(iii) $H^0(X, D) = F \cap \mathbb{A}(D)$, $\qquad H^1(X, D) = \mathbb{A}/(\mathbb{A}(D) + F)$.

Theorem 11. *For the linear compact quotient space $\mathbb{A}/F$, its motivic class is given by*

$$\mu(\mathbb{A}/F) = \mathbb{L}^{g-1}.$$

Proof. First, we have the following 9 diagram with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F \cap \mathbb{A}(D) & \to & \mathbb{A}(D) & \to & \mathbb{A}(D)/F \cap \mathbb{A}(D) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & \mathbb{A} & \to & \mathbb{A}/F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F/F \cap \mathbb{A}(D) & \to & \mathbb{A}/\mathbb{A}(D) & \to & \mathbb{A}/(\mathbb{A}(D)+F) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

Concentrating on the part

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & H^0(X,D) & \to & \mathbb{A}(D) & \to & \mathbb{A}(D)/F \cap \mathbb{A}(D) & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathbb{A}/F & & \\ & & & & & & \downarrow & & \\ & & & & & & H^1(X,D) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0, & & \end{array}$$

we have

$$\mu(\mathbb{A}/F) = \mu(\mathbb{A}(D)) \cdot \mathbb{L}^{-\chi(X,D)} = \mathbb{L}^{\deg(D)} \cdot \mathbb{L}^{-\deg(D)+(g-1)} = \mathbb{L}^{g-1}$$

since

$$\mu(\mathbb{A}(D)) = \mu(\mathbb{A}(D)/\mathbb{A}(0)).$$

Here we have used the Riemann-Roch theorem, which itself can be proved using the language of locally linearly compact topology.

3.2 Motivic Measure: Reductive Algebraic Groups

Denote by $K = k(X)$ the field of rational functions of X, and $\mathbb{A}$ its adelic ring. Let G be a semi-simple algebraic group over K, and $\mathcal{G}$ a smooth connected reductive group scheme over X with G its generic fiber.

Before introducing our motivic measure on $G(\mathbb{A})$, let us make some preparations, following [Weil] and [O, §2].

Let ω be an algebraic differential form on G over K of degree $N = \dim_K G$. Let $x^0 \in G$ and $x_1, \ldots, x_N$ be a local coordinate system at $x^0 \in G$. Then $\omega = f(x)\, dx_1 \ldots dx_N$ with $f(x)$ a rational function defined at x^0 which can be written as a formal power series

$$f(x) = \sum a_{i_1,\ldots,i_N} (x - x_1^0)^{i_1} \ldots (x_N - x_N^0)^{i_N}, \tag{1}$$

convergent in a neighborhood of x^0 in G. Accordingly, for any fixed closed point $v \in X$, if we assume that x^0 is a regular point of the analytic space $\mathcal{G}_v(K_v)$ and that $x_i^0 \in K_v$, then f is a power series with coefficients in K_v which converges in some neighborhood of the origin in K_v^N. Moreover, if x_i^0 are in in $\mathbb{A}$, then the $a_{i_1,\dots,i_N}$ are integers for almost all v. Hence, by a suitable linear change of coordinates, we can assume that all of them are v-adic integers for all v. Consequently, for each v, (1) converges for $x_i \equiv x_i^0(\mathbf{m}_v)$. Moreover, for each v, there exist a neighborhood U of x^0 in $\mathcal{G}_v(K_v)$ such that we have a natural measure $|f(x)|_v(dx_1)_v \dots (dx_N)_v$ which in turn through the pull-back, we get a well-defined local motivic measure ω_v, or better, $d_{\omega,v}\mu$ or even $d_\omega\mu$ on U and hence on the open subset of $\mathcal{G}_v(K_v)$ consisting of the regular points where ω is regular and not zero. For details, please refer [Weil, 2.2.1].

To go further, assume that for $p \in X$, $\mathcal{G}_p(K_p)$ is regular. Then there exists an algebraic differential form on $\mathcal{G}_p(K_p)$ which is regular and no where vanishing, the so-called gauge form. Fix a translation-invariant gauge form ω, which always exists. Then, by a result of Rosenlicht, any gauge form on $\mathcal{G}_p(K_p)$ can be written as $\lambda\,\chi(x)\,\omega$ with $\lambda \in K_p^*$ a constant and $\chi : G \to \mathbb{G}_m$ a character.

Let $\mathcal{G}_p(\mathcal{O}_p)$ be the group scheme associated to $\mathcal{G}$ with generic fiber $\mathcal{G}_p(K_p)$ and special fiber G_p. Then the reduction modulo $\mathbf{m}_p$ gives a one-to-one correspondence between equivalence classes modulo $\mathbf{m}_p$ of $\mathcal{G}_p(\mathcal{O}_p)$ and the $k(p)$ rational points of $G_{k(p)}$. Indeed, locally, at each good reduction point $p \in X$ for $\mathcal{G}$, we have natural morphisms $\mathcal{O}_p/\mathbf{m}_p^{n+1} \to \mathcal{O}_p/\mathbf{m}_p^n$. They form a projective system $\mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^{n+1}) \to \mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^n)$ and $\lim_{\longleftarrow n} \mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^n)$ is nothing but the space $\mathcal{G}_p(\mathcal{O}_p)$ of $\mathcal{O}_p$-valued points of $\mathcal{G}_p$. Moreover, one checks that $\pi_n : \mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^{n+1}) \to \mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^n)$ is a $k(p)^{\dim G}$-fiber space. Indeed, the morphism π_n is a $\mathrm{Hom}_{k(p)}(\Omega^1_{\mathcal{G}_p/\mathcal{O}_p} \otimes_{\mathcal{O}_p} k(p), \mathbf{m}_p^n/\mathbf{m}_p^{n+1})$-fiber space. Thus the smoothness of $\mathcal{G}$ at p implies that $\Omega^1_{\mathcal{G}_p/\mathcal{O}_p} \otimes_{\mathcal{O}_p} k(p) \simeq k(p)^{\dim G}$ and hence the assertion above since $\mathbf{m}_p^n/\mathbf{m}_p^{n+1} \simeq k(p)$.

Consequently, we can introduce a compatible system of normalized motivic volume on $\mathcal{G}_p(\mathcal{O}_p)$ by the condition that each fiber of the projection $\mathcal{G}_p(\mathcal{O}_p) \to \mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^n)$ has the motivic volume $\mathbb{L}_p^{-n\dim G}$ with $\mathbb{L}_p = \mathbb{L}^{\deg(p)}$. In particular, by taking $n = 1$, we conclude that within this system, the total motivic volume $\mu(\mathcal{G}_p(\mathcal{O}_p)) := \int_{\mathcal{G}_p(\mathcal{O}_p)} \omega_p$ of $\mathcal{G}_p(\mathcal{O}_p)$ is $\mathbb{L}_p^{-\dim G}\mu(G_p(k(p)))$.

To summaries, recall that we start with a (non-zero) maximal degree algebraic differential form ω on G. Via functoriality, we then obtain an analytic differential form on the analytic variety $\mathcal{G}(K_x)$ and hence on its open sub variety $\mathcal{G}(\mathcal{O}_x)$. In particular, for the identity section $e \in \mathcal{G}(\mathcal{O}_x)$, the relative tangent $\mathcal{O}_x$-module $T_e(\mathcal{G})$ of the scheme $\mathcal{G}_x/O_x$ at the section e may be viewed as an $\mathcal{O}_x$-lattice of the tangent K_x-vector space $T_e(\mathcal{G}_x(\mathcal{O}_x))$ of the analytic variety $\mathcal{G}_x(\mathcal{O}_x)$ at e. Denote by ω_x the linear form on $\Lambda^{\dim G}T_e(\mathcal{G}_x(\mathcal{O}_x))$ induced from ω. The image of $\Lambda^{\dim G}T_e(\mathcal{G})$ of ω_x then gives a fractional ideal of K_x. Being principal, we may choose a generator θ_x for this fractional ideal. Define the *motivic module of* ω at x by $\|\omega\|(x) := \mathbb{L}_x^{-\mathrm{ord}_x(\theta_x)}$. Clearly, this is well-defined, i.e., independent of the choice of θ_x.

Even in general, $\mathbb{A}$ is by no means locally compact, it is locally linearly compact. So we may introduce motivic measure for the associated spaces as above using the ind-pro topological structure: Since $\pi_p^n\mathcal{O}_p$, $n \in \mathbb{Z}$, gives a local ind-pro topological basis for K_p, the motivic measure $\mu_p = \mu_{\omega_p}$ on $\mathcal{G}_p(K_p)$ is in fact determined by this condition if we set $|\pi_p|_{\mu_p} = \mu(k(p))^{-1}$. This gives a generalization for the classical Haar measure when the space is locally compact. Such a measure is unique once we assume that $\int_{\mathbb{A}/K} d\mu = \mathbb{L}^{g-1}$. Denote this canonically determined motivic measure by $d_{\mathrm{can}}\mu$. Moreover, as to be expected, for $a \in A$, we define its motivic module by $\|a\| = \prod_x \mathbb{L}_x^{-\mathrm{ord}_x(a_x)}$.

Lemma 12. *For good reduction closed point of $\mathcal{G}$, denote by* $\mod(\omega)$ *the measure on $\mathcal{G}_p(\mathcal{O}_p)$ the module of ω with respect to the motivic measure $d_\omega\mu$. Then the measure* $\mod(\omega)$ *on $\mathcal{G}_p(\mathcal{O}_p)$ is of density $\mu(\mathcal{O}_p)^{\dim G}\|\omega\|$ with respect to the canonical measure $d_\omega\mu$.*

Proof. This is a motivic version of Thm 2.4 of [Oe]. We check the details in that proof. Being a local problem, it suffices to prove the theorem when there exists an etale morphism u of $\mathcal{G}$ on an $\mathcal{O}_p$-scheme $\mathcal{G}'$ of the form $\mathrm{Spec}(\mathcal{O}_p[T_1, \dots, T_{\dim G}]$. Let $\omega' = dT_1 \wedge \cdots \wedge dT_{\dim G}$ and $d_{\omega'}\mu'$ be the canonical measure on $\mathcal{G}'(\mathcal{O}_p) = \mathcal{O}_p^{\dim G}$. Clearly, we have $\mod(\omega') = \mu(\mathcal{O}_p)^{\dim G} d'_\omega\mu'$. The analytic morphism u of $\mathcal{G}(\mathcal{O}_p)$ in $\mathcal{G}'(\mathcal{O}_p) = \mathcal{O}_p^{\dim G}$ associated to u is etale in analytic sense, i.e., for all $g \in \mathcal{G}'_p(\mathcal{O}_p) = \mathcal{O}_p^{\dim G}$, the linear morphism between tangent spaces $T_g(u) : T_g(\mathcal{G}_p(\mathcal{O}_p)) \to T_g(\mathcal{G}'_p(\mathcal{O}_p))$ is an isomorphism and since u is etale, it transforms $\mathcal{O}_p$-lattices of $T_g(\mathcal{G})$ to $T_g(\mathcal{G}')$. Now let f be an analytic function on $\mathcal{G}(\mathcal{O}_p)$ such that $\omega = fu^*(\omega')$, then

$$\|\omega\|(g) = |f(g)|\|u^*(\omega')\|(g) = |f(g)|.$$

Moreover, since u is etale, there exists a pair (m, n) of integers, $m \geq n \geq 1$, such that $\mathcal{G}_p(\mathcal{O}_p/\mathbf{m}_p^m)$ coincides with the product of $\mathcal{G}'_p(\mathcal{O}_p/\mathbf{m}_p^n)$ with its fiber. Hence

$$\begin{aligned}\mod(\omega) =& \mathrm{mod}(\mathrm{fu}^*(\omega')) = |\mathrm{f}|\mathrm{u}^{-1}(\mathrm{mod}(\omega'))\\ =& |f|u^{-1}(\mu(\mathcal{O}_p)^{\dim G} d'_\omega\mu) = |f|\mu(\mathcal{O}_p)^{\dim G} d_\omega\mu = \|\omega\|\mu(\mathcal{O}_p)^{\dim G} d_\omega\mu.\end{aligned}$$

Corollary 13. *If $\omega_e(\Lambda^{\dim G}(\mathrm{Lie}\,\mathcal{G}_p)) = \mathbf{m}_p^n$ and $d_\omega\mu$ is the canonical motivic measure on $\mathcal{G}(O_x)$ and μ the motivic measure on K_x. Then*

$$\mod(\omega) = \mu(\mathcal{O}_p)^{\dim G}\mathbb{L}_p^{-n} d_\omega\mu$$

and its total mass is given by $\mu(\mathcal{O}_p)^{\dim G}\mathbb{L}_p^{n-\dim G}\mu(\mathcal{G}(k(p)))$.

That is to say, if we fix a non-zero section ω of the line bundle $\wedge_{\mathcal{O}_X}^{\dim G}\mathrm{Lie}\,\mathcal{G}$, we get a motivic measure ω_x on $\mathcal{G}_x(K_x)$. In particular, when $x = p$ is with good reduction, this is realized via the identification $\mathrm{Lie}\,\mathcal{G}_p(K_p) = \mathrm{Lie}\,G \otimes_K K_p$. Moreover,

$$\begin{aligned}\mu(\mathcal{G}_p(\mathcal{O}_p)) =& \mu(k(p))^{-\mathrm{ord}_e(\omega)}\Big(\mu(k(p))^{-\dim G}\mu(G(k(p)))\Big)\\ =& \mu(k(p))^{-\mathrm{ord}_p(\omega)-\dim G}\mu(G(k(p))).\end{aligned}$$

Motivated by this latest calculation, we define a global motivic measure $d\mu$ on $G(\mathbb{A})$ by the motivic Euler product

$$d\mu := \prod_{x\in X}^{\omega} \mathbb{L}_x^{-\mathrm{ord}_{e_p}(\omega_p)-\dim G}\omega_x.$$

Lemma 14. *The global motivic measure $d\mu$ is given by*

$$d\mu = q^{(1-g)\dim G} \prod\nolimits_{x\in X}^{\omega} \omega_x.$$

Proof. Indeed, by the property of the motivic Euler product, we have

$$\begin{aligned} d\mu =& \prod_{x\in X}^{\omega} \mathbb{L}_x^{-\mathrm{ord}_{e_p}(\omega_p)} \prod_{x\in X}^{\omega} \mathbb{L}_x^{-\dim G} \omega_x \\ =& \prod_{x\in X} \mathbb{L}_x^{-\mathrm{ord}_{e_p}(\omega_p)} \prod_{x\in X}^{\omega} \mathbb{L}_x^{-\dim G} \omega_x \\ =& \mathbb{L}^{-\sum_{x\in X} \mathrm{ord}_{e_p}(\omega_p)\deg(x)} \prod_{x\in X}^{\omega} \mathbb{L}_x^{-\dim G} \omega_x \\ =& \mathbb{L}^{(1-g)\dim G} \prod_{x\in X} \mathbb{L}_x^{\omega} \omega_x. \end{aligned}$$

Consequently, for $\mathbb{K} := \prod_{x\in X} \mathcal{G}(\mathcal{O}_x)$, we have

$$\begin{aligned} \mu(\mathbb{K}) =& \prod\nolimits_{x\in X}^{\omega} \mu(k(x))^{-\mathrm{ord}_x(\omega)} \Big(\mu(k(x))^{-\dim G} \mu(\mathcal{G}(k(x))) \Big) \\ =& \prod\nolimits_{x\in X}^{\omega} \mathbb{L}^{-\mathrm{ord}_x(\omega)\deg(x)} \prod_{x\in X} \mu(k(x))^{-\dim G} \mu(\mathcal{G}(k(x)) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{(by the property } (*)) \\ =& \prod\nolimits_{x\in X} \mathbb{L}^{-\mathrm{ord}_x(\omega)\deg(x)} \prod\nolimits_{x\in X}^{\omega} \mu(k(x))^{-\dim G} \mu(\mathcal{G}(k(x)) \\ =& \mathbb{L}^{-(g-1)\dim G} \prod\nolimits_{x\in X}^{\omega} \mu(k(x))^{-\dim G} \mu(\mathcal{G}(k(x)), \end{aligned}$$

by the Riemann-Roch theorem. Here we have used the facts that, being semi-simple, the vector bundle $\mathrm{Lie}\,\mathcal{G}$ is of degree 0, and that for a fixed ω, there are only finitely many zeros and poles on X. So, the motivic product $\prod_{x\in X}^{\omega} \mathbb{L}^{\mathrm{ord}_x(\omega)\deg(x)}$ consists of only finitely many non-trivial terms. Consequently, this part of motivic product coincides with the usual product $\prod_{x\in X} \mathbb{L}^{-\mathrm{ord}_x(\omega)\deg(x)}$.

Theorem 15. *For $G = SL_n$,*

$$\mu(\mathbb{K}) = \mathbb{L}^{(1-g)(n^2-1)} Z_X^{-1}(\mathbb{L}^{-2}) Z_X^{-1}(\mathbb{L}^{-3}) \cdots Z_X^{-1}(\mathbb{L}^{-n}).$$

Proof. Indeed, by definition, if $G = SL_r$, with $\mathbb{L}_x = \mu(k(x)) = \mathbb{L}^{\deg(x)}$, we have

$$\begin{aligned} & \mu(k(x))^{-\dim SL_n} \mu(SL_n(k(x))) \\ =& \mu(k(x))^{-\dim GL_n + 1} \frac{\mu(GL_n(k(x)))}{\mu(k(x)^*)} \\ =& \mathbb{L}_x^{-(n^2-1)} \frac{(\mathbb{L}_x^n - 1)(\mathbb{L}_x^n - \mathbb{L}_x) \cdots (\mathbb{L}_x^n - \mathbb{L}_x^{n-1})}{\mathbb{L}_x - 1} \\ =& (1 - \mathbb{L}_x^{-n})(1 - \mathbb{L}_x^{-(n-1)}) \cdots (1 - \mathbb{L}_x^{-2}) \end{aligned}$$

Consequently, use the compatibility between the motivic product and the ordinary multiplication, namely, the relation $(*)$, we have

$$\begin{aligned}\mu(\mathbb{K}) =&\mathbb{L}^{(g-1)\dim G}\prod\nolimits_{x\in X}^{\omega}\Big((1-\mathbb{L}_x^{-n})(1-\mathbb{L}_x^{-(n-1)})\cdots(1-\mathbb{L}_x^{-2})\Big)\\ =&\mathbb{L}^{(g-1)\dim G}\prod\nolimits_{x\in X}^{\omega}(1-\mathbb{L}_x^{-n})\prod\nolimits_{x\in X}^{\omega}(1-\mathbb{L}_x^{-(n-1)})\cdots\prod\nolimits_{x\in X}^{\omega}(1-\mathbb{L}_x^{-2})\\ =&\mathbb{L}^{(g-1)(n^2-1)}Z_X^{-1}(\mathbb{L}^{-2})Z_X^{-1}(\mathbb{L}^{-3})\cdots Z_X^{-1}(\mathbb{L}^{-n}).\end{aligned}$$

3.3 Canonical Motivic Mass

In particular, our motivic measure on $G(\mathbb{A})$ defined by

$$\mathbb{L}^{(1-g)\dim G}\prod\nolimits_{x\in X}^{\omega}\mu_{\omega_x}$$

does not depend on the choice of ω by the product formula. Furthermore, set

$$\mu_x(\mathcal{G})=\int_{\mathcal{G}_x(\mathcal{O}_x)}\omega_x,\qquad \forall x\in X.$$

Then a set of factors $(\lambda_x)_{x\in X}$ is called a set of good factors for $\mathcal{G}$ if the product $\prod_{x\in X}\lambda_x^{-1}\cdot\mu_x$ is well-defined as an element in $K_0^{\omega}(Sta_k)$. If (λ_x) is a set of good factors, then the motivic measure $\Omega=(\omega,(\lambda_x))$ on $X_{\mathbb{A}}$ derived from ω by means of (λ_x) is defined as the measure on $X_{\mathbb{A}}$ induced in each product $\prod_{v\in S}\mathcal{G}_x(K_x)\times\prod_{p\notin S}\mathcal{G}_p(\mathcal{O}_p)$ the product measure

$$\mu_K^{\dim G}\prod_{x\in X}(\lambda_x^{-1}\omega_v).$$

In particular, when $\lambda_x=1$ for all $x\in X$, we call the associated motivic measure canonical and the volume $\tau(\mathcal{G}):=\int_{G_{\mathbb{A}}/G_F}(\omega,(1))$ the canonical mass.

Theorem 16. *([BD]) Let $G=SL_n$ be the group scheme associated to the special linear group. Then $\tau(SL_n)=1$.*

Proof. For the convenience of the reader, we sketch the proof given in [BD].

First note that from the definition, as argued in [Weil] using motivic Euler product,

$$\tau(SL_n)\cdot\mathbb{L}^{(g-1)n^2-1}\prod_{i=2}^{n}Z(X,\mathbb{L}^{-i})=(\mathbb{L}-1)\mu(\mathbb{M}_{X,n}(\mathcal{O}_X)).$$

Here $\mathbb{M}_{X,n}(\mathcal{O}_X)$ denotes the moduli stack of rank n bundles on X with determinant $\mathcal{O}_X$, and the factor $\mathbb{L}-1$ comes from the scalar automorphisms.

To go further, consider the finite type open substack $\mathbb{M}_{X,n}^{\leq m}(\mathcal{O}_X)\subset\mathbb{M}_{X,n}(\mathcal{O}_X)$ of bundles of instability $\leq m$. It is well-known that by the boundness, for effective divisors D of sufficient larger degree d, $H^1(X,E^\vee\otimes\mathcal{O}_X(D))=0$ for all E in $\mathbb{M}_{X,n}^{\leq m}(\mathcal{O}_X)$. This in particular implies that $\mathrm{Hom}(E,\mathcal{O}_X(D)^n)$ form a vector bundle of rank $n^2(d-g+1)$. Denote this vector bundle by $W^{\leq m}(D)$. Clearly, as $m\to\infty$, $\mathbb{M}_{X,n}^{\leq m}(\mathcal{O}_X)$ exhausts the space $\mathbb{M}_{X,n}(\mathcal{O}_X)$. So we only need to calculate $\mu(\mathbb{M}_{X,n}^{\leq m}(\mathcal{O}_X)$, or equivalently, the vector bundle $\mu(W^{\leq m}(D)$.

For this latest purpose, first following exactly [BD], introduce the open locus $W_0^{\leq}(D) \subset W^{\leq m}(D)$ defined by the injective maps $E \hookrightarrow \mathcal{O}_X(D)^n$. Denote accordingly the Quot scheme $\mathrm{Div}(D)$ parametrizing subsheaves $E \hookrightarrow \mathcal{O}_X(D)^n$ which are locally free of rank n and degree 0, $\mathrm{Div}_{\mathcal{O}_X}(D)$ the closed subschema of $\mathrm{Div}(D)$ for E's with trivial determinants, and similarly the open subvariety $\mathrm{Div}_{\mathcal{O}_X}^{\leq m}(D)$. By definition, we have

$$W_0^{\leq m}(D) = \mathrm{Div}_{\mathcal{O}_X}^{\leq m}(D).$$

Moreover, by a result of [BGL], we see that the locus of non-injective morphisms has codimension at least d in $\mathrm{Hom}(E, \mathcal{O}_X(D)^n)$. Consequently, ignoring higher order codimension loci, one concludes as in [BD] that $\mu(\mathbb{M}_{X,n}(\mathcal{O}_X))$ can be calculated via first the space $M_{X,n}^{\leq m}(\mathcal{O}_X)$, then the vector bundle $W^{\leq m}(D)$ over it, then its open sub variety $W_0^{\leq m}(D)$ and hence $\mathrm{Div}_{\mathcal{O}_X}^{\leq m}$ and finally $\mathrm{Div}_{\mathcal{O}_X}(D)$. The up-shot is that $\mu(\mathbb{M}_{X,n}(\mathcal{O}_X))$ is nothing but the coefficient of dominant term $\mathbb{L}^{n^2(d+1-g)}$ of $\mu(\mathrm{Div}_{\mathcal{O}_X}(D))$ as $d \to \infty$.

To calculate this leading term, consider then the fixed point of $\mathrm{Div}(X)$ induced by the action of $\mathbb{G}_m$ on $\mathcal{O}_X(D)$. They correspond to inclusions of the form $\oplus_{i=1}^n \mathcal{O}_X(D - E_i)$, where E_i's are effective divisors with $\sum \deg(E_i) = nd$. Hence, the components of the fixed locus in $\mathrm{Div}(D)$ are indexed by the ordered partitions $\mathbf{n'} = (n_1, \ldots, n_k)$ of nd and the component indexed by $\mathbf{n'}$ is isomorphic to $\mathrm{Sym}^{(\mathbf{n'})}X := \prod_{i=1}^k \mathrm{Sym}^{(n_i)}X$. Thus in particular, if $n_1 \geq 2g-2$, the intersection of the fixed component $\mathrm{Sym}^{(\mathbf{n'})}X$ with the sub variety $\mathrm{Div}_{\mathcal{O}_X}(D)$ is then a projective bundle over $\mathrm{Sym}^{(\mathbf{n})}X$ with $\mathbf{n} = (n_2, \ldots, n_k)$. As argued in [BD], in the final deduction concentrating on dominant terms of order $n^2 d$, we may ignore the fixed components with $n_1 \leq 2g-2$, consisting of stratifications of codimensions bounded by nd.

On the other hand, the fixed locus of the $\mathbb{G}_m^n$-action on $\mathrm{Div}(D)$ may be re-catched by a $\mathbb{G}_m$-action given by one-parameter subgroups $\mathbb{G}_m \to \mathbb{G}_m^n$ induced by $t \mapsto (t^{w_1}, t^{w_2}, \ldots, t^{w_n})$ for strictly increasing sequence of integers. Since the tangent space inside $\mathrm{Div}(D)$ to a fixed point $(E_1, \ldots, E_n)$ is equal to $\oplus_{i,j}\mathrm{Hom}(\mathcal{O}_X(D - E_i), \mathcal{O}_{E_j})$, and the torus $\mathbb{G}_m^n$ acts on the summand for (i,j) is given through the character $\chi_i - \chi_j$ with χ_i the i-th projection of $\mathbb{G}_m^n$ to $\mathbb{G}_m$. With the increasing condition on w_i's, we see that $\mathbb{G}_m$-action should with positive weights.

Now to tide all up, notice that the fixed point locus over $\mathrm{Sym}^{(\mathbf{n'})}X$ is a Zariski locally trivial affine space bundle over $\mathrm{Sym}^{(\mathbf{n'})}X$, whose rank can been seen to be $\sum_{i=1}^k (k-i)n_i$ and this works equally well over the subvariety $\mathrm{Sym}^{(\mathbf{n'})}X \cap \mathrm{Div}_{\mathcal{O}_X}(D)$. Consequently, the dominant term $\mathbb{L}^{n^2(d-g+1)}$ has the coefficient

$$\sum_{\mathbf{n}} \frac{\mathbb{L}^{-\sum_{i=2}^k n_i} - \mathbb{L}^{-nd+g-1}}{\mathbb{L}-1} \mu(\mathrm{Sym}^{(\mathbf{n})}X)\mathbb{L}^{(n^2-1)(g-1)+\sum_{i=2}^k (1-i)n_i}$$

where the sum runs over all ordered partitions $\mathbf{n} = (n_2, \ldots, n_k)$ satisfying $\sum_{i=2}^k n_i \leq nd - 2g + 2$. Easily, the above summation is

$$= \frac{\mathbb{L}^{(n^2-1)(g-1)}}{\mathbb{L}-1} \sum_{\mathbf{n}=(n_2,\ldots,n_k)} \mu(\mathrm{Sym}^{(\mathbf{n})}X)\mathbb{L}^{-\sum_{i=2}^k i n_i} = \frac{\mathbb{L}^{(n^2-1)(g-1)}}{\mathbb{L}-1} \prod_{i=2}^n Z(X, \mathbb{L}^{-i}).$$

Conjecture 17. *(***Motivic Tamagawa Number Conjecture***) If G is a connected semi-simple group defined over K, then $\tau(G) = 1$.*

We end this discussion by the following standard, (see e.g. [BD],)

Proposition 18. *For a split connected semi-simple group G over F, we have*

$$\mu(G) = \mathbb{L}^{\dim(G)} \cdot \prod_d (1 - \mathbb{L}^{-d})^{\dim V_d}$$

where

$$\bigoplus_d V_d := \Big(\mathrm{Sym}\, X_*(T)_{\mathbb{Q}}\Big)^W,$$

with T a maximal split sub-torus of G, $X_*(T)$ its group of characters, and W the corresponding Weyl group.

Proof. If B is the associated Borel for G defined over F, then, using the Levi decomposition $B = R_u(B) \cdot T$, where $R_u(B)$ denotes the unipotent radical of B, we have

$$\mu(G) = \mu(G/B) \cdot \mu(R_u(B)) \cdot \mu(T).$$

We need to evaluate each terms.

The parts of T and $R_u(B)$ are easy. Since T is a torus, we have $T \simeq \mathcal{G}_m^r$ and hence $\mu(T) = (\mathbb{L} - 1)^r$. Similarly, using the structural filtration for $R_u(B)$, whose graded quotients are isomorphic to $\mathcal{G}_a$, we have $\mu(R_u(B)) = \mathbb{L}^{\dim R_u(B)}$.

To treat the projective variety G/B, we use the Bruhat decomposition $G/B = \sqcup_{w \in W} BwB/B$. Then we find that

$$\mu(G/B) = \sum_{w \in W} \mathbb{L}^{l(w)}$$

where $l(w)$ denotes the length of w. This completes the proof using the Lie theory.

4 Parabolic Reduction, Stability and the Mass

4.1 Motivic Measure of $M_{X,n}^{\mathrm{total}}(d)$

To go further, we fix a vector bundle $\mathcal{E}_0 = \mathcal{O}_X^{(n-1)} \oplus A$ with $A \in \mathrm{Pic}_X(d)$ a line bundle of degree d. Then any vector bundle of rank n with determinant A may be obtained via the following procedure.

First, shift from the vector bundle $\mathcal{E}_0$ on X to its generic fiber, we obtain an n-dimensional K-vector space E_0. With the bases chosen, we may identify $SL_n(\mathbb{A})$ with the adelic group associated to $SL(E_0)$. As such then consider for each $g = (g_x) \in SL_n(\mathbb{A})$ the associated bundle $\mathcal{E}_0^g$ defined by the $\mathcal{O}_x$-lattices $\mathcal{E}_{0,x}^{g_x}$ obtained from $g_x^{-1}\mathcal{E}_{0,x}$, where $\mathcal{E}_{0,x}$ denotes (the completion of) the stack of $\mathcal{E}_0$ at x:

$$\Gamma(U, \mathcal{E}_0^g) := \{v \in E_0 : g_x v \in \mathcal{E}_{0,x} \ \ \forall x \in U\}.$$

Thus we conclude that the isomorphism class of the vector bundle $\mathcal{E}_0^g$ depends only on the double coset $\mathbb{K}\, g\, SL_n(\mathbb{A})$, and the natural morphism

$$\mathbb{K}\backslash SL_n(\mathbb{A})/SL_n(K) \to M_{X,n}^{\text{total}}(A]$$

is surjective. Here $M_{X,n}^{\text{total}}(A]$ denotes the moduli stack of semi-stable bundle of rank n and determinant A. Moreover, if we set $\widetilde{\mathbb{K}} = \prod_{x\in X} GL_n(\mathcal{O}_x)$ be the corresponding subgroup in $GL_n(\mathbb{A})$ determined by $\mathcal{E}_0$, then the group of automorphism of $\mathcal{E}_0^g$ is given by $g^{-1}\widetilde{\mathbb{K}}g \cap GL_n(K)$ and $\mathcal{E}_0^g \simeq \mathcal{E}_0^{g'}$ if and only if $g' \in \widetilde{\mathbb{K}}gGL_n(K)$. Consequently, as argued in [HN, p.230], we see that there is a natural one-to-one correspondence between the quotient space $k^*/\operatorname{Im}\det\operatorname{Aut}\mathcal{E}_0^g$ to the double cosets which map to $\mathcal{E}_0^g$ for a fixed g. Here $\operatorname{Im}\det\operatorname{Aut}\mathcal{E}_0^g$ denotes the image of the determinant of $\operatorname{Aut}\mathcal{E}_0^g$ in k^*. Thus, as in [DR, p.234], for a fixed vector bundle $\mathcal{E}$ on X, from the exact sequence

$$1 \to (\operatorname{Aut}\mathcal{E} \text{ with } \det 1) \to \operatorname{Aut}\mathcal{E} \stackrel{\det}{\to} k^* \to (k^*/\operatorname{Im}\det\operatorname{Aut}\mathcal{E}) \to 1$$

we conclude that

$$\begin{aligned}&(\mathbb{L}-1)\int_{M_{X,n}^{\text{total}}(A]} \frac{1}{\mu(\operatorname{Aut}\mathcal{E})}d\mu\\ =&\int_{[g]\in\mathbb{K}\backslash SL_n(\mathbb{A})/SL_n(K)} \frac{1}{\mu(g^{-1}\mathbb{K}g\cap SL_n(K))}d\mu\\ =&\frac{1}{\mu(\mathbb{K})}\int_{SL_n(\mathbb{A})/SL_n(K)} d\mu\end{aligned}$$

Theorem 19. *We have*

$$\int_{M_{X,n}^{\text{total}}(d)} \frac{1}{\mu(\operatorname{Aut}\mathcal{E})}d\mu = \Big(\widehat{\zeta}_X^\omega(1)\cdot\widehat{\zeta}_X^\omega(2)\cdot\dots\cdot\widehat{\zeta}_X^\omega(n)\Big)\cdot\int_{SL_n(\mathbb{A})/SL_n(K)} d\mu.$$

Here $\widehat{\zeta}_X^\omega(i) := \widehat{Z}_X(L^{-1})$ *for* $i = 1, 2, \dots n$. *In particular, it is independent of* d.

Proof. A direct calculation with the help of the following relations

$$\begin{aligned}\mu(M_{X,n}^{\text{total}}(d)) =&\mu(M_{X,n}^{\text{total}}(A])\cdot\mu(\operatorname{Pic}_X(d)),\\ \beta_{X,q}(d) =&\widehat{\zeta}_X^\omega(\mathbb{L}^{-1}).\end{aligned}$$

For later use, set,

$$\beta_{X,n}^{\text{total}}(d) = \int_{M_{X,n}^{\text{total}}(d)} \frac{1}{\mu(\operatorname{Aut}\mathcal{E})}d\mu.$$

4.2 Invariants α and β

In this section, we give a universal relation between $\beta_{X,n}^\omega(0)$ and $\alpha_{X,n}^\omega(0)$.

Theorem 20. *(Counting Miracle) ([WZ, g=1]; [S, general g])*

$$\alpha_{X,n+1}^\omega(0) = \mathbb{L}^{(g-1)n}\beta_{X,n}^\omega(0)$$

Proof. ([S]) By definition, in order to contribute to the invariant $\alpha^{\omega}_{X,n+1}(0)$, rank $n+1$ semi-stable vector bundle $\mathcal{E}$ of degree zero should have a non-trivial global section. So for such $\mathcal{E}$'s, we have a non-trivial morphism $\mathcal{O}_X \to \mathcal{E}$. But the semi-stability of $\mathcal{E}$ then implies that the morphism $\mathcal{O}_X \to \mathcal{E}$ yields an exact sequence $0 \to \mathcal{O}_X \to \mathcal{E} \to Q \to 0$ with Q a semi-stable vector bundle of rank n and degree 0. Moreover, we know that $H^0(X,\mathcal{E})\backslash\{0\} = \mathrm{Hom}(\mathcal{O}_X,\mathcal{E})\backslash\{0\}$. With these said, consider the action of $\mathrm{Aut}\,\mathcal{O}_X \times \mathrm{Aut}\,Q$ on $\mathrm{Ext}^1(Q,\mathcal{O}_X)$. Its orbits then are parametrized via $[e] \in \Big(\mathrm{Hom}(\mathcal{O}_X,\mathcal{E})\backslash\{0\}\Big)/\mathrm{Aut}(\mathcal{O}_X)$, for which the stabilizer is given by $\mathrm{Aut}\,\mathcal{E}/I + \mathrm{Hom}(Q,\mathcal{O}_X)$. That is to say, we have the decomposition

$$\mathrm{Ext}^1(Q,\mathcal{O}_X) = \cup_{(\mathrm{Hom}(\mathcal{O}_X,\mathcal{E})\backslash\{0\})/\mathrm{Aut}(\mathcal{O}_X)} \mathrm{Aut}\,\mathcal{O}_X \times \mathrm{Aut}\,Q/\mathrm{Aut}\,\mathcal{E}/(\mathrm{Id}+\mathrm{Hom}(Q,\mathcal{O}_X)).$$

Consequently,

$$\begin{aligned}
\alpha^{\omega}_{X,n+1}(0) =& \int_{M_{X,n+1}(0)} \frac{\mu(\mathrm{Hom}(\mathcal{O}_X,\mathcal{E})\backslash\{0\})}{\mu(\mathrm{Aut}(\mathcal{E})}d\mu \\
=& \int_{M_{X,n}(0)} \mu(\mathrm{Aut}\mathcal{O}_X)\cdot \frac{1}{\mu(\mathrm{Aut}\mathcal{O}_X \times \mathrm{Aut}Q)} \frac{\mu(\mathrm{Ext}^1(Q,O_X)}{\mu(\mathrm{Id}+\mathrm{Hom}(Q,\mathcal{O}_X))}d\mu \\
=& \int_{M_{X,n}(0)} \frac{1}{\mu(\mathrm{Aut}\,Q)} \frac{\mu(\mathrm{Ext}^1(Q,O_X))}{\mu(\mathrm{Hom}(Q,\mathcal{O}_X))}d\mu \\
=& \int_{M_{X,n}(0)} \frac{1}{\mu(\mathrm{Aut}\,Q)} \mathbb{L}^{-\chi(X,Q^\vee)}d\mu = \int_{M_{X,n}(0)} \frac{1}{\mu(\mathrm{Aut}\,Q)} \mathbb{L}^{(g-1)n}d\mu \\
=& \mathbb{L}^{(g-1)n}\cdot \beta^{\omega}_{X,n}(0)
\end{aligned}$$

by the Riemann-Roch theorem.

To properly appreciate this result and understand how the motivic integrations are calculated, we compute the α invariant for rank two vector bundles of degree zero on elliptic curves over $\mathbb{C}$.

By the classification of Atiyah ([A]), we know that $M_{X,2}(0)$ is parametrized by the $\mathbb{P}^1$-bundle over $\mathrm{Pic}_X(0)$. Indeed, recall that for a fixed $A \in \mathrm{Pic}_X(0)$, the semi-stable bundles $\mathcal{E}$ of rank two with A as determinant is classified by $\mathrm{Gr}(\mathcal{E}) = L_1 \oplus L_2$ with $L_1 \times L_2 = A$. Here Gr denotes the associated graded Jordan-Holder bundle. Thus the map $L_1 \mapsto [L_1 \oplus L_2]$ gives a morphism $\mathrm{Pic}_X(0) \to M_{X,2}(A]$ the moduli space of rank two semi-stable bundles with A as determinant. Since L_2 also gives the same bundle, this morphism is 2:1. Moreover, since there exist exactly 4 line bundles L_1 such that $L_1^2 = A$. So this morphism is ramified at 4 points $T_{A,i}$ $(i = 1,2,3,4)$. So the target is $\mathbb{P}^1$.

Thus, motivically, not only we need to calculate

(i) a motivic integration over this $\mathbb{P}^1$ bundles over $\mathrm{Pic}_X(0)$, but

(ii) the 4 additional non-reduced sections $T_{A,i}$ which give the bundles $T_{A,i} \otimes I_2$ $(i = 1,2,3,4)$. Here I_2 is the unique non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$.

However, by noticing that within this $\mathbb{P}^1$-bundle, only the bundles lie on the section corresponding to $[\mathcal{O}_X \oplus A]$ have a nontrivial global sections and at $[\mathcal{O}_X \oplus \mathcal{O}_X]$, which is also the intersection of this section with the section (corresponding to) $T_{1,A}$ at $A=\mathcal{O}_X$, h^0 jumps. Consequently, we have

$$\begin{aligned}\alpha^{\omega}_{X,2}(0)=&\int_{\mathrm{Pic}_X(0)\backslash\{\mathcal{O}_X\}}\frac{H^0(X,\mathcal{O}_X\otimes A)\backslash\{0\})}{\mu(\mathrm{Aut}(O_X\oplus A))}d\mu\\&+\frac{\mu(H^0(X,\mathcal{O}_X^2)\backslash\{0\})}{\mu(\mathrm{Aut}(\mathcal{O}_X^2))}+\frac{\mu(H^0(X,I_2)\backslash\{0\})}{\mu(\mathrm{Aut}(I_2))}\\=&\int_{\mathrm{Pic}_X(0)\backslash\{\mathcal{O}_X\}}\frac{\mathbb{L}-1}{\mu(\mathbb{C}^*\times\mathbb{C}^*)}d\mu+\frac{\mathbb{L}^2-1}{(\mathbb{L}^2-1)(\mathbb{L}^2-\mathbb{L})}+\frac{\mathbb{L}-1}{\mu(\mathbb{C}^*\times\mathbb{C})}\\&=\frac{1}{\mathbb{L}-1}\Big(\int_{\mathrm{Pic}_X(0)\backslash\{\mathcal{O}_X\}}d\mu+1\Big)=\beta^{\omega}_{X,1}(0)\end{aligned}$$

For high ranks, this proves to be quite complicated, even it is a quite fun game. Similarly, when $k=\mathbb{F}_q$, we can use general Atiyah bundles, namely, direct sums of I_r's, where I_r denotes the non-trivial extension of I_{r-1} by $\mathcal{O}_X$, to establish a similar result. However, due to the rationality problems involved, that process is much more complicated. For details, please refer to [WZ].

Consequently, we have

Corollary 21. *For elliptic curves E/k, its associated non-abelian rank n zeta function is given by*

$$\widehat{Z}_{E,n}(u)=\beta^{\omega}_{X,n-1}(0)+\beta^{\omega}_{E,n}(0)\frac{u^n(\mathbb{L}^n-1)}{(1-u^n)(1-u^n\mathbb{L}^n)}.$$

In fact [WZ] gives much more: the Riemann Hypothesis for all $\zeta_{E/\mathbb{F}_q}(s)$ is proved. This is based on the following

Theorem 22. (Multiplicative structure of beta invariants)

$$\sum_n \beta^{\omega}_{E,n}(0)u^n=\prod_m Z_E(u\mathbb{L}^m).$$

For a proof, please refer to [WZ].

4.3 Parabolic Reduction and Geometric Partition

For simplicity, in the following two sections, we will work over algebraic closed k unless otherwise stated. Let $\mathcal{G}$ be a reductive group scheme on X and $\mathcal{E}$ a $\mathcal{G}$-torsor on X. For a fixed parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$, denote by $\mathcal{E}_P$ the induced P-torsor. Following [HN], and more generally, [B] and [R] (see also [AB]), there is a unique canonical parabolic $\mathcal{P}$ such that $\mathcal{E}$ has its Harder-Narasimhan type $\mathcal{P}$. Consequently, if we set $M_{X,G}^{\mathrm{total}}(\nu'_G)$ be the stack of $\mathcal{G}$-torsors of degree ν'_G with $\nu'_G\in X_*(A'_G)$, where A'_G denotes the maximal quotient split torus of G and X_* denotes the collection of the associated one-parameter groups, then $M_{X,G}^{\mathrm{total}}(\nu'_G)$ admits a natural partition by the stacks $M_{X,G,P}(\nu'_P)$ corresponding to these having canonical type $\mathcal{P}$ for $\nu'_P\in X_*(A'_P)$ satisfying $[\nu'_P]_G=\nu'_G$ and $[\nu'_P]^G\in\mathfrak{a}_P^{G+}$ the positive acute Weyl chamber in $\mathfrak{a}_P^G$. Here following [A] (see also [LR]), we write $\mathfrak{a}_P:=X_*(A_P)'_{\mathbb{R}}=\mathfrak{a}_P^Q\oplus\mathfrak{a}_Q^G\oplus\mathfrak{a}_R$ for standard parabolic subgroups $P\subset Q\subset R$ and $[\cdot]^Q,[\cdot]_Q^R$ and $[\cdot]_R$ denote the canonical projections of $\mathfrak{a}_P$ onto $\mathfrak{a}_P^Q,\mathfrak{a}_Q^R$

and $\mathfrak{a}_R$ respectively. In particular, $M_{X,G,G}(\nu'_G) = M_{X,G}(\nu'_G)$, the moduli stack of semi-stable $\mathcal{G}$-torsors of degree ν'_G. For our own use, for a parabolic Q and $\nu'_Q \in X_*(A'_Q)$, set also $M_{X,G,Q}^{\text{total}}(\nu'_Q)$ be the substack of $M_{X,G}^{\text{total}}(\nu'_G)$ whose induced $\mathcal{Q}$-torsors are of degree ν'_Q. Similarly, then we have the substack $M_{X,Q,P}(\nu'_P)$. Moreover, by an abuse of notation, denote the corresponding subspace in the double coset $\mathbb{K}\backslash G(\mathbb{A})/G(K)$ with the same letter. What we have just said then proves the following

Theorem 23. (Parabolic Reduction and Stability) *Determined by the parabolic reduction using stability, within the space* $M_{X,G,Q}^{\text{total}}(\nu'_Q)$*, we have a natural partition*

$$M_{X,G,Q}^{\text{total}}(\nu'_Q) = \bigcup_{P\subset Q} \bigcup_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q,[\nu'_P]^Q\in\mathfrak{a}_P^{Q+}}} M_{X,Q,P}(\nu'_P).$$

Consequently, if for a subset $\Sigma \subset \mathbb{K}\backslash G(\mathbb{A})/G(K)$, we write $\mathbf{1}_\Sigma$ for its characteristic functions, then we may restate the partition of the theorem quantitively as

$$\mathbf{1}_{M_{X,G,Q}^{\text{total}}(\nu'_Q)} = \sum_{P\subset Q} \sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \tau_P^Q([\nu'_P]^Q)\cdot \mathbf{1}_{M_{X,Q,P}(\nu'_P)}.$$

Here τ_P^Q denotes the characteristic function of the acute positive Weyl chamber in $\mathfrak{a}_P^Q$ ([A]). Thus if we set $\widehat{\tau}_P^Q$ to be the characteristic function of the obcute positive Weyl chamber in $\mathfrak{a}_P^Q$ (as in [A]), then we have

Theorem 24. (Parabolic Reduction and Stability) *We have*

$$\mathbf{1}_{M_{X,G,Q}^{\text{total}}(\nu'_Q)} = \sum_{P\subset Q} \sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \tau_P^Q([\nu'_P]^Q)\cdot \mathbf{1}_{M_{X,Q,P}(\nu'_P)};$$

$$\mathbf{1}_{M_{X,G,Q}(\nu'_Q)} = \sum_{P\subset Q} (-1)^{\dim \mathfrak{a}_P^Q} \sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \widehat{\tau}_P^Q([\nu'_P]^Q)\cdot \mathbf{1}_{M_{X,Q,P}^{\text{total}}(\nu'_P)}.$$

Proof. The first is a reformulation of the previous theorem and the second is a direct consequence of the Langlands' (combinatorial) lemma.

4.4 Reduction to Levi Factors

For a standard parabolic subgroup P of G, we have then the Levi decomposition $P = M_P N_P$ with N_P its unipotent radical and M_P its Levi factor. With M_P reductive, it makes sense to talk about the moduli stack $M_{X,M_P}(\nu'_P)$ of semi-stable M_P-torsors of degree $\nu'_P \in X_*(A'_P)$ since $A'_{M_P} = A'_P$. Moreover, by a result of [B], the natural morphism $M_{X,G,P}(\nu'_P) \to M_{X,M_P}(\nu'_P)$ defined by $\mathcal{E} \mapsto \mathcal{E}_P/N_P$ is a stack fibration of affine spaces, whose dimension is known to be $2\langle\rho_P^G, \nu'_P\rangle + \dim(N_P)(g-1)$. Here $\rho_P^G \in \mathfrak{a}_P^{G*}$ denotes the associated Weyl vector for P. For details, please refer to [AB], [LR], and in particular, [DR]. In fact, we have the following

Example 2. (Motivic Hall Algebra) *For the time being assume that k is general. Then for a vector bundle $\mathcal{E}$ over X, denote its associated Harder-Narasimhan filtration by*

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\mathcal{E}_2\subset\cdots\subset\mathcal{E}_k=\mathcal{E}.$$

From the uniqueness, $\mathcal{E}_i$ are also defined over k. To go further, consider then the exact sequence

$$0\to\mathcal{E}_1\to\mathcal{E}\to\mathcal{Q}_1\to 0.$$

Then it is well known that there is no non-trivial morphism from $\mathcal{E}_1$ to $\mathcal{Q}_1$. Consequently, for the natural action of $\mathrm{Aut}\,\mathcal{E}_1\times\mathrm{Aut}\,\mathcal{Q}_1$ on $\mathrm{Ext}^1(\mathcal{Q}_1,\mathcal{E}_1)$, the stabilizer corresponding to $[\mathcal{E}]$ is given by $\mathrm{Aut}\,\mathcal{E}/(\mathrm{Id}+\mathrm{Hom}(\mathcal{Q}_1,\mathcal{E}_1))$. Here we have used the fact that being canonical, any automorphism of $\mathcal{E}$ fixes $\mathcal{E}_1$ and $\mathcal{Q}_1$ and that $(\mathrm{Id}+\mathrm{Hom}(\mathcal{Q}_1,\mathcal{E}_1))$ is a normal subgroup of $\mathrm{Aut}\,\mathcal{E}$. That is to say, we have the following natural decomposition:

$$\mathrm{Ext}^1(\mathcal{Q}_1,\mathcal{E}_1)=\cup_{[E]\in\mathrm{Ext}^1(\mathcal{Q}_1,\mathcal{E}_1)}(\mathrm{Aut}\,\mathcal{E}_1\times\mathrm{Aut}\,\mathcal{Q}_1)\big/(\mathrm{Aut}\,\mathcal{E}/(\mathrm{Id}+\mathrm{Hom}(\mathcal{Q}_1,\mathcal{E}_1)).$$

Consequently, we arrive at the following generating relation for the associated motivic Hall algebra

$$\mathbb{L}^{\chi(X,\mathcal{Q}_1^\vee\otimes\mathcal{E}_1)}\int\frac{1}{\mu(\mathrm{Aut}\,\mathcal{E})}\,d\mu(\mathcal{E})=\frac{1}{\mu(\mathrm{Aut}\,\mathcal{E}_1)\times\mu(\mathrm{Aut}\,\mathcal{Q}_1)}$$

which has its root from [DR, p.235].

Consequently, we have

$$\mu(M_{X,G,Q}^{\mathrm{total}}(\nu'_Q))=\sum_{P\subset Q}\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\tau_P^Q([\nu'_P]^Q)\cdot\mathbb{L}^{2\langle\rho_P^G,\nu'_P\rangle+\dim(N_P)(g-1)}\cdot\mu(M_{X,M_P}(\nu'_P));$$

Hence, by the Langlands' combinatorial lemma,

$$\mathbb{L}^{2\langle\rho_Q^G,\nu'_Q\rangle+\dim(N_Q)(g-1)}\cdot\mu(M_{X,M_Q}(\nu'_Q))=\sum_{P\subset Q}(-1)^{\dim\mathfrak{a}_P^Q}\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\widehat{\tau}_P^Q([\nu'_P]^Q)\cdot\mu(M_{X,Q,P}^{\mathrm{total}}(\nu'_P)).$$

These formulas while very close to the original geometric picture involve infinite summations due to the terms involving τ and $\widehat{\tau}$. To get finite closed formulas, we introduce the following spaces

$$\Lambda_P^Q=X_*(A'_P)\Big/\sum_{\alpha\in\Delta_P^Q}\mathbb{Z}\alpha^\vee,\qquad \Pi_P^Q=X_*(A'_P)\Big/\sum_{\alpha\in\Delta_P^Q}\mathbb{Z}\varpi_\alpha^\vee.$$

Also, as usual, for each $\lambda\in\mathbb{R}/\mathbb{Z}$, $\langle\lambda\rangle\in\mathbb{R}$ is the representative of the class λ such that $0<\langle\lambda\rangle\le 1$. And we write $\{\lambda\}=1-\langle\lambda\rangle$. Set then

$$\mathbb{L}^{\dim(N_P)(g-1)}\cdot\mu(M_{X,M_P}(\nu'_P))=\widetilde{\mu}(M_{X,M_P}(\nu'_P))$$

$$\mathbb{L}^{\dim(N_P)(g-1)}\cdot\mu^{\mathrm{total}}(M_{X,M_P}(\nu'_P))=\widetilde{\mu}(M_{X,M_P}^{\mathrm{total}}(\nu'_P))$$

And denote by A_P the maximal split subtorus in (the center of) P.

Theorem 25. (Parabolic Reduction : Quantitive Version) *We have*

$$\widetilde{\mu}(M^{\text{total}}_{X,M_Q}(\nu'_Q)) = \sum_{P\subset Q} \sum_{\overline{\nu}'_P\in(\Lambda_P^Q)^\perp} \widetilde{\mu}(M_{X,M_P}(\overline{\nu}'_P)) \sum_{\substack{\pi\in\Pi_P^Q\\ [\pi]_Q=\nu'_Q}} \prod_{\alpha\in\Delta_P^Q} \frac{\mathbb{L}^{2\langle\rho_P^Q,\varpi_\alpha^{\vee Q}\rangle\cdot\{\alpha^{\vee Q}(\pi)\}}}{\mathbb{L}^{2\langle\rho_P^Q,\varpi_\alpha^{\vee Q}\rangle}-1},$$

$$\widetilde{\mu}(M_{X,M_Q}(\nu'_Q)) = \sum_{P\subset Q} (-1)^{\dim \mathfrak{a}_P^Q}\ \widetilde{\mu}(M^{\text{total}}_{X,M_P}(0)) \sum_{\substack{\lambda\in\Lambda_P^Q\\ [\lambda]_Q=\nu'_Q}} \prod_{\alpha\in\Delta_P^Q} \frac{\mathbb{L}^{2\langle\rho_P^Q,\alpha^\vee\rangle\cdot\{\varpi_\alpha^Q(\lambda)\}}}{\mathbb{L}^{2\langle\rho_P^Q,\alpha^\vee\rangle}-1}.$$

Proof. ([LR]) Note that for each $x\in\mathbb{R}$ we have

$$\sum_{n\in\mathbb{Z},n+x>0} t^n = \frac{t^{\langle x+\mathbb{Z}\rangle - x}}{1-t}.$$

Then the proof may be given as in [LR, Thm 2.4] for the second equality by noticing that $\widetilde{\mu}(M^{\text{total}}_{X,M_Q}(\nu'_Q)) = \widetilde{\mu}(M^{\text{total}}_{X,M_Q}(0))$ is in fact independent of ν'_P as proved in Theorem 2. As for the first, we use the fact that $\widetilde{\mu}(M_{X,M_P}(\overline{\nu}'_P))$ depends only on the class $\bar{\nu}'_P\in X_*(A'_P)/X_*(A_P)$.

By comparing with the results on Poincare series for the moduli stacks of $\mathcal{G}$-torsors on Riemann surfaces in [LR], our motivic class formluas should be viewed as the roots of them. Hence our motivic measures answer the existence question of such measures raised by Atiyah and Bott in [AB][2].

To end this section, with a similar discussion, we have the following generalization of a result of Lafforgue for GL_n, which will not be used in the sequel:

Corollary 26. *The Arthur's analytic truncation*

$$\Lambda^T\mathbf{1}(g) = \sum_P (-1)^{\dim\mathfrak{a}_P} \sum_{\delta\in P(K)\backslash G(K)} \widehat{\tau}_P(H_P(\delta g))$$

defines a characteristic function of a certain compact subset $\Sigma^T_{X,G}$ for all $T\geq 0$. In particular, when $T=0$, $\Sigma^0_{X,G} = M_{X,G}(\nu'_G)$. Here $H: P(\mathbb{A})\to\mathfrak{a}_P$ the usual logarithmic map.

4.5 Semi-Simple Group Schemes

Back to general base field. Let $\mathcal{G}$ be a connected semi-simple group scheme and $\mathcal{G}_0$ be the constant reductive group scheme over X having the same type as $\mathcal{G}$. Then the scheme $\text{Isomext}(\mathcal{G},\mathcal{G}_0)$ is quasi-isotrivial, since $\mathcal{G}$ is by [SGA3, XXIV, Thm 1.3 and XXIV, Cor 2.3]. Thus, from [SGA3, X, Cor 5.4], we know that $\text{Isomext}(\mathcal{G},\mathcal{G}_0)$ is etale and finite over X. Consequently, by taking one of the components Y, we may assume that over the finite Galois etale cover $f: Y\to X$, $f^*\mathcal{G}$ is an inner form, since then $\text{Isomext}(f^*\mathcal{G}, f^*\mathcal{G}_0)$ has a

[2]This and other aspects of the comparison suggest that the basic relation between numbers of points and Betti numbers for algebraic varieties may have some extension to infinite dimensions in which counting of points is replaced by a suitable measure. ... Comparison with the number theory suggests that there might be a natural measure, ... so that what we have been computing as Poincaré series actually turn out to be measures. – From [AB, p.598]

tautological section. Denote by L/K the finite Galois extension corresponding to the Galois etale covering Y/X. Let then B be a Borel subgroup and $T \subset B$ a maximal torus. Then the Weyl group $W = (N_G(T)/T)(L)$ acts naturally on the symmetric algebra of $X_*(G \otimes_K L)_{\mathbb{Q}}$. By a theorem of Chevelley, the invariants of this action is isomorphic to an algebra of polynomials $\mathbb{Q}[I_1, \dots, I_{\dim \mathfrak{a}_0^G}]$, where $I_1, \dots, I_{\dim \mathfrak{a}_0^G}$ are algebraically independent homogeneous polynomials on $\mathfrak{a}_0^G$ of degrees $d_i (\geq 2)$ given by $-\#\{\alpha > 0 : \langle \rho_G, \alpha^\vee \rangle = i\} + \#\{\alpha > 0 : \langle \rho_G, \alpha^\vee \rangle = i - 1\}$. Clearly, this works also locally for each $x \in X$. Thus by a result of Steinberg [S, 11.16], we may assume that

$$\nu(G(k_x)) = \mathbb{L}_x^{\dim G} \cdot \prod_{i \geq 2} (1 - \mathbb{L}_x^i)^{n_i}$$

with $\mathbb{L}_x = \mathbb{L}^{\deg(x)}$. Thus, if $\mathbb{K} = \prod_{x \in X} \mathcal{G}(\mathcal{O}_x)$, with the formula that

$$\mu(\mathbb{K}) = \mathbb{L}^{(1-g)\dim G} \prod_{x \in X} \mu(k(x))^{-\dim G} \mu(\mathcal{G}(k(x))),$$

we, using *the motivic Euler product*, conclude that

$$\mu(\mathbb{K}) = \prod_{i \geq 2} \widehat{\zeta}_X^\omega(i)^{n_i}.$$

Consequently, if we set

$$\beta_{X,G}^\omega(\nu'_G) := \int_{M_{X,G}(\nu'_G)} \frac{1}{\nu(\mathrm{Aut}_G(\mathcal{E}))} d\mu,$$

or the same, in terms of semi-simple group G,

$$\beta_{X,G}^\omega(\nu'_G) = \int_{\mathbb{K} \backslash G(\mathbb{A})/G(K)} \frac{1}{\mu(g\mathbb{K}g^{-1} \cap G(K))} d\mu,$$

we have the following

Theorem 27. *The total mass of $\mathcal{G}$-torsors of degree ν'_G for a semi-simple group scheme $\mathcal{G}$ is given by*

$$\beta_{X,G}(\nu'_G) = \prod_{i \geq 2} \widehat{\zeta}_X^\omega(i)^{-n_i}.$$

In particular, it is independent of the degree.

4.6 Simple Factors

Even we start with a semi-simple groups, with parabolic reductions, we are led to the study of the associated Levi factors, which are reductive but may not be semi-simple. So to finally close the ring, we need to investigate the relation between reductive groups and its associated semi-simple counter part, its derived group.

Recall that we have the following decomposition theorem for reductive groups. If G is a connected semi-simple group over a field K, then the set $\{G_i\}_{i \in I}$ of minimal non-trivial normal smooth connected K-subgroups of G is finite, each

G_i is K-simple, the G_i' pairwise commute, and the multiplication homomorphism $\prod_{i\in I} G_i \to G$ is a cantral isogeny. More generally, if G is a connected reductive group over a field K, Z its maximal central K-torus, and $G' = \mathcal{D}(G)$ its semi-simple derived group. Let $\{G_i\}$ be the K-simple factors of G'. Then G_i are precisely the minimal non-trivial normal smooth connected non-central K-subgroups of G and the multiplication $Z \times \prod_{i\in I} G_i \to G$ is a central isogeny. (See e.g., [C].) Thus up to a central isogeny, reductive groups may be viewed as direct product of the central torus and its simple semi- simple factors. But for $Z \times \prod_i G_i$ it is clear that

$$\mu(M^{\mathrm{total}}_{X, Z\times\prod_{i\in I} G_i}(\nu_G')) = \prod_{i\in I}\prod_{j\geq 2} \widehat{\zeta}_X^\omega(1)^{n_{j1}} \widehat{\zeta}_X^\omega(2)^{n_2} \cdots \widehat{\zeta}_X^\omega(j_i)^{n_{j_i}}$$

since $\mu(M^{\mathrm{total}}_{X,Z}(\nu_Z')) = \widehat{\zeta}_X^{\dim Z}(1)$. So it is left to consider how the total motivic mass β^ω changes under central isogeny. For this, we use Ono's relative theory of Tamagawa numbers ([O1,2]). Thus, the factors related to the Tamagawa measure of the universal cover $\tilde{G}$ of G and the motivic classes associated to Sh and H^0 of the kernel of the natural covering $\tilde{G} \to G$ should be added. With the works of Ono ([O1,2]) and of Lang on the lifting ([L]), this theory is now becoming quite standard. We leave the details to the reader.

4.7 Analytic Zetas are Motivic

To end this paper, let us ask whether our analytic zeta functions constructed here is motivic. In particular, whether the quantitive version of the parabolic reduction and stability stated in Theorem 9 holds. Note that, under the motivic assumption, the coefficients in the parabolic reduction and stability relation is totally independent of the complex structures of the Riemann surfaces used in the definition. In this sense, we may say that being motivic is some intrinsic property which is compatible with the conformal fields theory claiming the existence of projective flat connections on conformal blocks.

Acknowledgements: Part of the works was done during our March visit to UCLA, kindly arranged by Hida. Our motivic interests were stimulated by several discussions with Lafforgue during his recent visit to Fukuoka. I would like to express my thanks to both of them. Special thanks also due to Deninger for introducing us the works of Kapranov.

This work is partially supported by JSPS.

Appendix: **Invariants $\alpha_{X,n}(0)$ and $\beta_{X,n-1}(0)$**

K. Sugahara

Abstract

In this appendix, we establish the following intrinsic relation between alpha and beta invariants for genus g curves X:

$$\alpha_{X,n+1}(0) = q^{n(g-1)}\beta_{X,n}(0).$$

Let X be an irreducible reduced regular projective curve of genus g over $\mathbb{F}_q$, the finite field with q elements. For any coherent sheaves $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ on X, for our own use, we introduce the auxiliary spaces

$$\begin{aligned}
\mathrm{Fil}(\mathcal{A},\mathcal{B};\mathcal{E}) &:= \{0 \subset \mathcal{F} \subset \mathcal{E} : \mathcal{F} \simeq \mathcal{A},\ \mathcal{E}/\mathcal{F} \simeq \mathcal{B}\},\\
\mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) &:= \{(f,\ g) \in \mathrm{Hom}(\mathcal{A},\mathcal{E}) \times \mathrm{Hom}(\mathcal{E},\mathcal{B}) : f \textit{ injective},\ g \textit{ surjective},\ g \circ f = 0\},\\
\mathrm{U}(\mathcal{A},\mathcal{B};\mathcal{E}) &:= \{f \in \mathrm{Hom}(\mathcal{A},\mathcal{E}) : f \textit{ injective},\ \exists \text{surjective}\, g \in \mathrm{Hom}(\mathcal{E},\mathcal{B})\ \textit{s.t.}\ g \circ f = 0\},
\end{aligned}$$

the associated morphisms

$$\begin{aligned}
&\varphi : \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) \to \mathrm{Fil}(\mathcal{A},\mathcal{B};\mathcal{E}); && (f,\ g) \mapsto (0 \subset \mathrm{Im}\,(\mathrm{f}) \subset \mathcal{E})\\
&\psi : \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{B},\mathcal{A}); && (f,\ g) \mapsto (0 \to \mathcal{A} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{B} \to 0)\\
&\pi : \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) \to \mathrm{U}(\mathcal{A},\mathcal{B};\mathcal{E}); && (f,\ g) \mapsto f
\end{aligned}$$

and the following natural actions:

$$\begin{aligned}
&\chi : \{(\mathrm{Aut}\,\mathcal{A})^{\mathrm{op}} \times \mathrm{Aut}\,\mathcal{B}\} \times \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) \to \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}); && (\rho,\sigma,(f,\ g)) \mapsto (f \circ \rho, \sigma \circ g)\\
&\mu : \mathrm{Aut}\,\mathcal{E} \times \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}) \to \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E}); && (\tau,(f,\ g)) \mapsto (\tau \circ f,\ g \circ \tau^{-1})
\end{aligned}$$

We know that
(i) φ is surjective, and
(ii) there is a natural bijection between the fiber of φ and $(\mathrm{Aut}\,\mathcal{A})^{\mathrm{op}} \times \mathrm{Aut}\,\mathcal{B}$.

This is a direct consequence of the five lemma. Indeed, this follows from the facts that $\varphi(f,g) = \varphi(f',g')$ if and only if there exists $(\rho,\sigma) \in (\mathrm{Aut}\,\mathcal{A})^{\mathrm{op}} \times \mathrm{Aut}\,\mathcal{B}$ such that $(f',g') = \chi(\rho,\sigma,(f,g))$, and that the action χ is free. Hence, we have

$$\#\mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E})\big/\big(\#(\mathrm{Aut}\,\mathcal{A}) \cdot \#\mathrm{Aut}\,\mathcal{B}\big) = \#\mathrm{Fil}(\mathcal{A},\mathcal{B};\mathcal{E}). \tag{1}$$

(iii) The image of ψ is exactly the set $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{B},\mathcal{A})$ of isomorphism classes of extensions of $\mathcal{B}$ by $\mathcal{A}$ the middle term of which is isomorphic to $\mathcal{E}$. Moreover, any fiber of ψ is an orbit of $\mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E})$ under the action μ.

Indeed, this follows from the fact that $\psi(f,g) = \psi(f',g')$ if and only if there exists $\tau \in \mathrm{Aut}\,\mathcal{E}$ such that $(f',g') = \mu(\tau,(f,g))$. Besides,
(iv) The stabilizer group of $(f,g) \in \mathrm{W}(\mathcal{A},\mathcal{B};\mathcal{E})$ under the action μ is isomorphic to $\mathrm{Hom}(\mathcal{B},\mathcal{A})$.

To prove, for any element $\tau \in \mathrm{Aut}\,\mathcal{E}$, we write it as

$$\tau = \begin{pmatrix} \tau_{\mathcal{A}} & \tau_{\mathcal{A}\mathcal{B}} \\ \tau_{\mathcal{B}\mathcal{A}} & \tau_{\mathcal{B}} \end{pmatrix}$$

with

$$\tau_{\mathcal{A}} \in \operatorname{Aut} \mathcal{A},\ \tau_{\mathcal{B}} \in \operatorname{Aut} \mathcal{B},\ \tau_{\mathcal{A}\mathcal{B}} \in \operatorname{Hom}(\mathcal{A}, \mathcal{B}),\ \tau_{\mathcal{B}\mathcal{A}} \in \operatorname{Hom}(\mathcal{B}, \mathcal{A}).$$

A direct calculation shows that $f = \tau \circ f$ if and only if

$$\tau = \begin{pmatrix} \mathrm{Id}_{\mathcal{A}} & 0 \\ \tau_{\mathcal{B}\mathcal{A}} & \tau_{\mathcal{B}} \end{pmatrix}.$$

Here we have used the claim that $\operatorname{Hom}(\mathcal{A}, \mathcal{B})$ part $\tau_{\mathcal{A}\mathcal{B}}$ must be 0 since we have the inclusion $\tau(f(\mathcal{A})) \subset f(\mathcal{A})$ by condition. Similarly, $g = g \circ \tau^{-1}$ if and only if $g \circ \tau = g$ if and only if

$$\tau = \begin{pmatrix} \tau_{\mathcal{A}} & 0 \\ \tau_{\mathcal{B}\mathcal{A}} & \mathrm{Id}_{\mathcal{B}} \end{pmatrix}.$$

Thus, $(f, g) = \mu(\tau, (f, g))$ if and only if

$$\tau = \begin{pmatrix} \mathrm{Id}_{\mathcal{A}} & 0 \\ \tau_{\mathcal{B}\mathcal{A}} & \mathrm{Id}_{\mathcal{B}} \end{pmatrix} \in \begin{pmatrix} \mathrm{Id}_{\mathcal{A}} & 0 \\ \operatorname{Hom}(\mathcal{B}, \mathcal{A}) & \mathrm{Id}_{\mathcal{B}} \end{pmatrix}.$$

Consequently, the stabilizer group of $(f, g) \in \mathrm{W}(\mathcal{A}, \mathcal{B}; \mathcal{E})$ under the action μ is isomorphic to $\operatorname{Hom}(\mathcal{B}, \mathcal{A})$.

Therefore, we obtain the following relation

$$\#\mathrm{W}(\mathcal{A}, \mathcal{B}; \mathcal{E}) \big/ \big(\#\mathrm{Aut}\mathcal{E} / \#(\operatorname{Hom}(\mathcal{B}, \mathcal{A}))\big) = \#\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{B}, \mathcal{A}). \tag{2}$$

By (1) and (2), we have

$$\#\mathrm{Fil}(\mathcal{A}, \mathcal{B}; \mathcal{E}) = \frac{\#\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{B}, \mathcal{A}) \cdot \#\mathrm{Aut}\mathcal{E}}{\#\mathrm{Aut}\,\mathcal{A} \cdot \#\mathrm{Aut}\mathcal{B} \cdot \#(\operatorname{Hom}(\mathcal{B}, \mathcal{A}))}. \tag{3}$$

This is an analogue of the formula given by [Ri].

(v) π is surjective, and any fiber of π is isomorphic to $\operatorname{Aut} \mathcal{B}$.

This follows from the fact that $\pi(f, g) = \pi(f', g')$ if and only if $f = f'$ and there exists $\sigma \in \operatorname{Aut} \mathcal{B}$, by the five lemma. Consequently, we have

$$\#\mathrm{W}(\mathcal{A}, \mathcal{B}; \mathcal{E}) \big/ \#\mathrm{Aut}\, \mathcal{B} = \#\mathrm{U}(\mathcal{A}, \mathcal{B}; \mathcal{E}). \tag{4}$$

Thorem. *Let $\mathcal{E}_0$ be a stable vector bundle of rank m $(< n)$ and degree 0. Then*

$$\sum_{\mathcal{E} \in \mathcal{M}_{X,n}(0)} \frac{q^{\#\mathrm{Hom}(\mathcal{E}_0, \mathcal{E})} - 1}{\#\mathrm{Aut}\, \mathcal{E}} = q^{m(n-m)(g-1)} \sum_{\mathcal{F} \in \mathcal{M}_{X,n-m}(0)} \frac{1}{\#\mathrm{Aut}\, \mathcal{F}}.$$

In particular,

$$\alpha_{X,n+1}(0) = q^{n(g-1)} \beta_{X,n}(0).$$

Proof. We have the following calculation:

$$
\begin{aligned}
& \sum_{\mathcal{E}\in\mathcal{M}_{X,n}(0)} \frac{q^{\#\mathrm{Hom}(\mathcal{E}_0,\mathcal{E})}-1}{\#\mathrm{Aut}\,\mathcal{E}} \\
= & \sum_{\mathcal{E}\in\mathcal{M}_{X,n}(0)} \frac{\#\{f\in\mathrm{Hom}(\mathcal{E}_0,\mathcal{E}) : f \text{ injective}\}}{\#\mathrm{Aut}\,\mathcal{E}} \\
& \qquad\qquad\qquad \text{(by the stability of } \mathcal{E}_0\text{)} \\
= & \sum_{\mathcal{E}\in\mathcal{M}_{X,n}(0)} \frac{1}{\#\mathrm{Aut}\,\mathcal{E}}\,\#\mathrm{U}(\mathcal{E}_0,\mathcal{E}/\mathcal{E}_0;\mathcal{E}) \\
& \text{(since the category of semistable bundles} \\
& \qquad\qquad \text{of degree 0 \textit{is an abelian category})} \\
= & \sum_{\mathcal{E}\in\mathcal{M}_{X,n}(0)} \frac{1}{\#\mathrm{Aut}\,\mathcal{E}}\cdot\frac{\#\mathrm{W}(\mathcal{E}_0,\mathcal{E}/\mathcal{E}_0;\mathcal{E})}{\#\mathrm{Aut}(\mathcal{E}/\mathcal{E}_0)} \\
& \qquad\qquad\qquad \text{(by (4))} \\
= & \sum_{\mathcal{E}\in\mathcal{M}_{X,n}(0)} \frac{1}{\#\mathrm{Aut}\,\mathcal{E}}\cdot\frac{\#\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{E}/\mathcal{E}_0,\mathcal{E}_0)\cdot\#\mathrm{Aut}\mathcal{E}}{\#\mathrm{Aut}\,(\mathcal{E}/\mathcal{E}_0)\cdot\#\mathrm{Hom}(\mathcal{E}/\mathcal{E}_0,\mathcal{E}_0)} \\
& \qquad\qquad\qquad \text{(by (3))} \\
= & \sum_{\mathcal{F}\in\mathcal{M}_{X,n-m}(0)} \frac{\#\mathrm{Ext}^1(\mathcal{F},\mathcal{E}_0)}{\#\mathrm{Aut}\,\mathcal{F}\cdot\#\mathrm{Hom}(\mathcal{F},\mathcal{E}_0)} \\
& \text{(since the category of semistable bundles} \\
& \qquad\qquad \text{of degree 0 \textit{is an abelian category})} \\
= & \; q^{m(n-m)(g-1)} \sum_{\mathcal{F}\in\mathcal{M}_{X,n-m}(0)} \frac{1}{\#\mathrm{Aut}\,\mathcal{F}} \\
& \qquad\qquad \text{(by the Riemann-Roch Thm).}
\end{aligned}
$$

□

Kotaro Sugahara
Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, JAPAN
E-Mail: k-sugahara@math.kyushu-u.ac.jp

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Lin WENG
Institute for Fundamental Research, The L Academy *and*
Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, JAPAN
E-Mail: weng@math.kyushu-u.ac.jp