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Ω-admissible theory

II. Deligne pairings over moduli spaces of punctured Riemann surfaces

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Abstract. In Part I, Deligne-Riemann-Roch isometry is generalized for punctured Riemann surfaces equipped with quasi-hyperbolic metrics. This is achieved by proving the Mean Value Lemmas, which explicitly explain how metrized Deligne pairings for ω-admissible metrized line bundles depend on ω. In Part II, we first introduce several line bundles over Knudsen-Deligne-Mumford compactification of the moduli space (or rather the algebraic stack) of stable N-pointed algebraic curves of genus g, which are rather natural and include Weil-Petersson, Takhtajan-Zograf and logarithmic Mumford line bundles. Then we use Deligne-Riemann-Roch isomorphism and its metrized version (proved in Part I) to establish some fundamental relations among these line bundles. Finally, we compute first Chern forms of the metrized Weil-Petersson, Takhtajan-Zograf and logarithmic Mumford line bundles by using results of Wolpert and Takhtajan-Zograf, and show that the so-called Takhtajan-Zograf metric on the moduli space is algebraic.

Introduction

For smooth metrics, Arakelov theory in dimension one may be essentially summarized as follows:

(1) Intersection. If (L,ρ) and (L',ρ') are two metrized line bundles on a compact Riemann surface M of genus g, then we have the so-called Deligne metric $h_D(\rho,\rho')$ on Deligne pairing $\langle L, L'\rangle$;

(2) Cohomology. If τ is a Hermitian metric on K_M induced from a smooth base metric on M, then we have the Quillen metric $h_Q(\rho;\tau)$ on Grothendieck-Mumford determinant $\lambda(L)$;

(3) Deligne-Riemann-Roch Isometry: There exists a canonical isometry

$$\Big(\lambda(L), h_Q(\rho;\tau)\Big)^{\otimes 12} \simeq \Big(\langle L, L\otimes K_M^{\otimes -1}\rangle, h_D(\rho,\rho\otimes\tau^{\otimes -1})\Big)^{\otimes 6} \otimes\Big(\langle K_M, K_M\rangle, h_D(\tau;\tau)\Big)\cdot \mathrm{e}^{a(g)}$$

where $a(g) = (1-g)(24\zeta'_{\mathbf{Q}}(-1)-1)$ denotes the Deligne constant.

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Thus, to develop an Arakelov theory for singular metrics, there are at least two difficulties: (1) for intersections, general singular metrics have too wild singularities; and (2) for cohomology, corresponding Laplacians, if exist, have continuous spectrum.

Clearly, the first is a minor one, as we may use certain growth conditions on singular metrics to overcome it. On the other hand, the second is an essential one. As a matter of fact, we even now have no idea on how to do it in general (along with the line of Ray-Singer-Quillen). However, in this paper, we use a new principal: the so-called Mean Value Lemma, to develop a new cohomology theory.

Key ideas are as follows: First, we start with a metric ds^2 on a Riemann surface M which has at worse hyperbolic growth near some points. Then, we define canonically ω-Arakelov metrics $\rho_{\mathrm{Ar};\omega}$ and $\rho_{\mathrm{Ar};\omega;P}$ on canonical line bundle K_M and $\mathcal{O}_M(P)$ for all $P \in M$, respectively (i.e., Basic Definition I in Sect. 1.2). We know that (a) these metrics are good in the sense of Mumford ([Mu1]) and (b) their first Chern forms are proportional to the normalized volume form ω of ds^2. With this, we define ω-admissible metrics on line bundles by conditions (a) and (b). Clearly, on any line bundle L, ω-admissible metrics exist and are indeed unique up to constant factors. Moreover, if ρ and ρ are admissible (and hence may be singular), Deligne metric $h_D(\rho, \rho')$ is well-defined as well.

Now for any ω-admissible metric ρ on L, we can construct canonically a smooth ω_{can}-admissible metric ρ_{can} on L. (Here ω_{can} denotes the standard canonical volume form of M. See e.g., Sect. 1.1.) In fact, if we write L as $\mathcal{O}_M(\sum a_i R_i)$, there exists a constant c such that $\rho = \otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i} \cdot \mathrm{e}^c$ by admissible condition, and $\rho_{\mathrm{can}} := \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}^{\otimes a_i} \cdot \mathrm{e}^c$ (i.e., Equation (1.2) in Sect. 1.2). Furthermore, ρ_{can} is well-defined, i.e., does not depend on the choice of the divisor $\sum a_i R_i$ used. Similar construction works for τ on K_M, from which we obtain a unique smooth ω_{can}-admissible metric τ_{can} (i.e., Equation (1.1) in Sect. 1.2).

Singular ω-admissible metrics ρ on L and τ on K_M are beautifully related to smooth ω_{can}-admissible metrics ρ_{can} and τ_{can} by the Mean Value Lemma in Sect. 1.3, which claims that

(1) on $\langle L, L'\rangle$; $h_D(\rho, \rho') = h_D(\rho_{\mathrm{can}}, \rho'_{\mathrm{can}})$;

(2) on $\langle L, K_M\rangle$; $h_D(\rho, \tau) = h_D(\rho_{\mathrm{can}}, \tau_{\mathrm{can}})$; and

(3) on $\langle K_M, K_M\rangle$; $h_D(\tau, \tau) = h_D(\tau_{\mathrm{can}}, \tau_{\mathrm{can}})$.

With this, finally, define a determinant metric $h_{\det}(\rho; \tau)$ on $\lambda(L)$ by setting $h_{\det}(\rho; \tau) := h_Q(\rho_{\mathrm{can}}; \tau_{\mathrm{can}})$. We show that if ρ and τ are indeed smooth, then $h_{\det}(\rho; \tau) = h_Q(\rho; \tau)$. That is to say, it coincides with the standard Quillen metric. All this then leads to the Deligne-Riemann-Roch isometry for our admissible metrics in Sect. 1.6, both over arithmetic surfaces and over families of Riemann surfaces, and hence a quite satisfied Arakelov theory for singular metrics is developed.

Naturally, we want to apply our admissible theory to the study of moduli spaces of punctured Riemann surfaces equipped with complete hyperbolic metrics. For doing so, we then meet with another essential difficulty: there exists no geometrically natural admissible metric on the canonical line bundle, without which, it is impossible to apply our general admissible theory. (Complete hyperbolic metric on a punctured Riemann surface is not canonical when we view it as a metric on the canonical line bundle of its smooth compactification.)

To overcome this, we introduce an invariant called Arakelov-Poincaré volume for a (punctured) Riemann surface. (See the Basic Definition II(i) in Sect. 1.5.) Moreover, with the help of the so-called Puncture Democracy in Sect. 1.5, which claims that metrically, all punctures behavior in the same way, we obtain a natural decomposition for the canonical metric on $K_M(P_1+\cdots+P_N)$ induced from the complete hyperbolic metric in terms of these on K_M and $\mathcal{O}_M(P_1+\cdots+P_N)$. (See the Decomposition Rule and Basic Definition II(ii) in Sect. 1.5.) All this is done in Part I.

In Part II, we use Deligne pairing to study moduli spaces of punctured Riemann surfaces, algebraically and metrically. More precisely, for algebraic aspect, we first introduce several line bundles over Knudsen-Deligne-Mumford compactification of the moduli space (or rather the algebraic stack) of stable N-pointed algebraic curves of genus g, which are rather natural and include Weil-Petersson, Takhtajan-Zograf and logarithmic Mumford line bundles. (See Basic Definition III in Sect. 2.1.) Then we use Deligne-Riemann-Roch isomorphism to establish logarithmic Mumford type isomorphisms (i.e., Fundamental Relations I in Sect. 2.2). Moreover, by using a result of Néron and Tate (resp. a result of Cornalba-Harris), we give a comparison between Weil-Petersson line bundles and Takhtajan-Zograf line bundles, (resp. a generalization of Xiao and Cornalba-Harris's inequality). (See Fundamental Relations II and III in Sect. 2.3 and Sect. 2.4, respectively.) All this answers some of open problems concerning line bundles over moduli spaces of marked stable curves.

As for metric aspect, by using decompositions for standard hyperbolic metrics in Sect. 1.5, we are able to introduce natural metrics on the restrictions of Weil-Petersson, Takhtajan-Zograf, and logarithmic Mumford type line bundles to the open part of the moduli space. (See Basic Definition IV in Sect. 2.5.) And, as a direct consequence of our arithmetic Deligne-Riemann-Roch isometry, we obtain logarithmic Mumford type isometries. (See Fundamental Relation IV in Sect. 2.5.) Moreover, our metrized Weil-Petersson and Takhtajan-Zograf line bundles are naturally related with Weil-Petersson metrics, defined by using Petersson norm on spaces of cusp forms, and Takhtajan-Zograf metrics, defined by using Eisenstein series respectively. In fact, as a direct consequence of our logarithmic Mumford type isometries, by using [Wo1] and [TZ1,2], we show that the first Chern form of metrized Weil-Petersson bundle (resp. metrized Takhtajan-Zograf bundle) gives the Kähler form associated to the Weil-Petersson metric (resp. the Takhtajan-Zograf metric, together with Fujiki). (See Fundamental Relations V and VI in Sect. 2.6.) In this way, we answer affirmatively an open problem of Takhtajan and Zograf on whether their newly defined metric on the moduli space is algebraic, and also for the first time clearly point out that the Weil-Petersson metric is in the nature of intersection (rather than that of cohomology). As a by-product, we finally show that the metrics on logarithmic Mumford line bundles introduced in Basic Definition IV can be redefined by using special values of Selberg zeta functions for punctured Riemann surfaces.

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Part I. Ω-admissible theory

In this part, we develop an ω-admissible intersection theory, introduce a new determinant metric and use them to prove a Deligne-Riemann-Roch isometry for ω-admissible metrics which may be singular. Key points are the definition of ω-Arakelov metrics and various versions of the Mean Value Lemma.

1.1. Quasi-hyperbolic metrics and their Green's functions

Throughout this part, we assume that M^0 is a (punctured) Riemann surface of genus q. Denote its smooth compactification by M, and let $M\backslash M^0 =: \{P_1, \dots, P_N\}$. As usual, we call P_i, $i = 1, \dots, N$, *punctures* of M^0.

Recall that a Hermitian metric ds^2 on M^0 is said to be *of hyperbolic growth near punctures*, if for each P_i, $i = 1, \dots, N$, there exists a punctured coordinate disc $\Delta^* := \{z \in \mathbf{C} : 0 < |z| < 1\}$ centered at P_i such that for some constant $C_1 > 0$,

(i) $ds^2 \leq \dfrac{C_1|dz|^2}{|z|^2(\log|z|)^2}$ on Δ^*,

and there exists a local potential function ϕ_i on Δ^* satisfying $ds^2 = \frac{\partial^2\phi_i}{\partial z\partial\bar{z}} dz \otimes d\bar{z}$, and for some constants $C_2,\ C_3 > 0$,

(ii) $|\phi_i(z)| \leq C_2 \max\{1, \log(-\log|z|)\}$, and

(iii) $\left|\dfrac{\partial\phi_i}{\partial z}\right|, \left|\dfrac{\partial\phi_i}{\partial\bar{z}}\right| \leq \dfrac{C_3}{|z|\,|\log|z||}$ on Δ^*.

In this case, we call ds^2 a *quasi-hyperbolic metric*, which is introduced in [TW1].

For a quasi-hyperbolic metric ds^2 over a punctured Riemann surface M^0, it follows easily from (i) that $\mathrm{Vol}(M, ds^2) < \infty$. Denote the normalized volume form of ds^2 by ω so that $\mathrm{Vol}(M, \omega) = 1$. From now on, we always assume that ω is the normalized volume form on M associated to a smooth metric (on M) or a quasi-hyperbolic metric (on M^0).

Proposition ([TW1, Thm 1]). *With respect to the normalized volume form ω associated to a fixed quasi-hyperbolic metric on M^0, there exists a unique ω-Green's function $g_\omega(\cdot,\cdot)$. That is, there exists a function $g_\omega(\cdot,\cdot)$ on $M^0 \times M^0\backslash\mathrm{Diagonal}$ such that the following conditions are satisfied:*
(i) For fixed $P \in M^0$, and $Q \neq P$ near P,

$$g_\omega(P, Q) = -\log|f(Q)|^2 + \alpha(Q),$$

where f is a local holomorphic defining function for P, and α is some smooth function defined near P;
(ii) $d_Q d_Q^c g_\omega(P, Q) = \omega(Q) - \delta_P$. Here $d_Q^c := \frac{\sqrt{-1}}{4\pi}(\bar\partial_Q - \partial_Q)$ is with respect to the second variable (so that $d_Q d_Q^c = \frac{\sqrt{-1}}{2\pi}\partial_Q\bar\partial_Q$), and δ_P is the Dirac delta symbol at P;
(iii) $\displaystyle\int_M g_\omega(P, Q)\omega(Q) = 0$;
(iv) $g_\omega(P, Q) = g_\omega(Q, P)$ for $P \neq Q$;
(v) $g_\omega(P, Q)$ is smooth on $M^0 \times M^0\backslash\mathrm{Diagonal}$;
(vi) Near each puncture P_i of M^0, $i = 1, \ldots, N$, there exists a punctured coordinate neighborhood Δ^* centered at P_i such that for fixed $Q \in \Delta^*$, there exists a constant $C > 0$ such that

$$|g_\omega(Q, z)| \leq C\max\{1, \log(-\log|z|)\} \quad \text{on } \Delta^*.$$

We next sketch a proof. For the details, see e.g., [TW1] or [We2].

First, define the so-called canonical volume form ω_{can} on M as follows:
(a) $q = 0$. Thus $M = \mathbf{P}^1$ is the projective line. Denote its affine coordinate by z. Set

$$\omega_{\mathrm{can}} := \frac{\sqrt{-1}}{2\pi}\frac{dz \wedge d\bar z}{(1 + |z|^2)^2};$$

(b) $q > 0$. Let $\{\phi_i\}$ be an orthonormal basis of the space of global holomorphic differentials $\Gamma(M, K_M)$ of M with respect to the natural pairing $(\phi, \psi) \mapsto \frac{\sqrt{-1}}{2}\int_M \phi \wedge \bar\psi$. Set

$$\omega_{\mathrm{can}} := \frac{\sqrt{-1}}{2q}\sum_{j=1}^{q}\phi_j \wedge \bar\phi_j.$$

Clearly, $\int_M \omega_{\mathrm{can}} = 1$. Hence, we may solve the partial differential equation

$$dd^c\beta_\omega = \omega - \omega_{\mathrm{can}}.$$

Moreover, by using conditions (ii) and (iii) for quasi-hyperbolic metrics, we may further assume that there exist constants $C_4,\ C_5 > 0$ such that near each $P_i, i = 1, \dots, N$,

$$|\beta_\omega(z)| \le C_4 \max\{1, \log(-\log|z|)\},$$
$$\left|\frac{\partial\beta_\omega(z)}{\partial z}\right|, \left|\frac{\partial\beta_\omega(z)}{\partial\bar{z}}\right| \le \frac{C_5}{|z|\,|\log|z||} \qquad \text{on } \Delta^*.$$

Clearly, such β_ω's are unique up to additive constants. So if we normalize it by putting the condition that

$$\int_M \beta_\omega(\omega + \omega_{\text{can}}) = 0,$$

then the locally integrable function β_ω is unique. Now denote by $g(P, Q)$ the Arakelov-Green's function, i.e., the ω_{can}-Green's function. (See e.g., [La2], where the existence of $g(P, Q)$ is proved following Arakelov [Ar].)

Proposition′ ([TW1]). *With the same notation as above, on* $M^0 \times M^0 \backslash \text{Diagonal}$, *the function*

$$g_\omega(P, Q) := g(P, Q) + \beta_\omega(P) + \beta_\omega(Q),$$

satisfies conditions (i)∼(vi) of the Proposition.

Obviously, using properties of Arakelov-Green's functions, see e.g., [La2, Chapter II], we only need to check condition (iii) of the Proposition. But then by the growth conditions for β_ω and $d\beta_\omega$, the arguments in the proof of [La2, Chapter II, Proposition 1.3] involving Stokes' theorem remain valid. This completes the proof of the Proposition′ and hence the Proposition.

1.2. Ω-Arakelov metrics

Our aim here is to introduce canonically metrics on $\mathcal{O}_M(P)$ for any fixed point P on M, and on the canonical line bundle K_M of M associated to ω, the normalized volume form associated to a quasi-hyperbolic metric on M^0. For this purpose, motivated by the work of Arakelov [Ar], we may try simply to use the ω-Green's functions. However in doing so, we meet two main difficulties. These are
(i) ω-Green's function $g_\omega(P, \cdot)$ is not well-defined when P is a puncture; and
(ii) corresponding intersection behaviors very badly.

To overcome these difficulties, we make the following modification. First, for any $P \in M^0$, i.e., *for any point but a puncture*, define a Hermitian metric $\rho_{\text{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting

$$\log \|1_P\|^2_{\rho_{\text{Ar};\omega;P}}(Q) := -g_\omega(P, Q) + \beta_\omega(P) \quad \text{for } Q(\neq P) \text{ in } M^0.$$

Here 1_P denotes the canonical defining section of $\mathcal{O}_M(P)$. That is to say, we twist the metric on $\mathcal{O}_M(P)$ corresponding to ω-Green's functions $g_\omega(P,\cdot)$ by a *constant* factor $e^{\beta_\omega(P)}$. (Later on, we will see that such a modification is essential.) Clearly,

$$\begin{aligned} &d_Q d_Q^c(-\log \|1_P\|^2_{\rho_{\mathrm{Ar};\omega;P}}(Q)) \\ &\quad = d_Q d_Q^c(g_\omega(P,Q) - \beta_\omega(P)) \\ &\quad = d_Q d_Q^c g_\omega(P,Q) \\ &\quad = \omega(Q) - \delta_P \\ &\quad = \omega(Q) - \delta_{\mathrm{div}(1_P)}. \end{aligned}$$

Hence $c_1(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P}) = \omega$ for all P which are not punctures, where c_1 denotes the first Chern form.

Now, by Proposition′ in the previous section,

$$-g_\omega(P,Q) + \beta_\omega(P) = -g(P,Q) - \beta_\omega(Q).$$

This leads to the following

Basic definition I(i). *For any point $P \in M$, define the ω-Arakelov metric $\rho_{\mathrm{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting*

$$\log \|1_P\|^2_{\rho_{\mathrm{Ar};\omega;P}}(Q) := -g(P,Q) - \beta_\omega(Q) \quad \text{for } Q(\neq P) \text{ in } M^0.$$

Clearly, now we also have

$$c_1(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P}) = \omega \qquad \text{for all } P \in M.$$

To facilitate ensuing discussion, we next recall the definition of 'good' Hermitian metrics introduced by Mumford [Mu1], in the special case of line bundles over a (punctured) Riemann surface. So let L be a line bundle on M. A smooth Hermitian metric ρ on $L|_{M^0}$ is said to be good on M if there exists a finite set of coordinate discs $\{U_i\}$ covering an open neighborhood of all punctures $\{P_i\}$ such that for each $U_i = \Delta = \{z \in \mathbf{C} : |z| < 1\}$, there exists a non-vanishing holomorphic section $v \in \Gamma(U_i, L\big|_{U_i})$ such that on $U_i \cap M^0 = \Delta^* = \{z \in \Delta : z \neq 0\}$,

(i) $|\rho(v,v)|, 1/|\rho(v,v)| \leq C_1\Big(\log|z|\Big)^{2m}$ for some $C_1 > 0$, $m \geq 1$, and

(ii) $\partial \log \rho(v,v)$ and $\partial\overline{\partial} \log \rho(v,v)$ have Poincaré growth on $U_i - U_i \cap M^0$, i.e., there exist constants $C_2, C_3 > 0$ such that

$$\begin{aligned} |\partial_{t_1} \log \rho(v,v)|^2 &\leq C_2\, \omega_{U_i \cap M^0}(t_1, t_1) \quad \text{and} \\ |\partial_{t_2}\overline{\partial}_{t_3} \log \rho(v,v)|^2 &\leq C_3\, \omega_{U_i \cap M^0}(t_2, t_2) \cdot \omega_{U_i \cap M^0}(t_3, t_3) \end{aligned}$$

for all $t_1, t_2, t_3 \in T_x(U_i \cap M^0)$, $x \in U_i \cap M^0$. Here $\omega_{U\cap M^0}$ denotes the metric on $U \cap M^0$ induced by the Poincaré metric $ds^2 = (|dz|/(|z| \log |z|))^2$ on each Δ^*.

One easily sees that the above definition does not depend on the choice of local coordinate functions and local trivializations of L on each U_i. (cf. [Mu1,Sect. 1])

With this, by definition, a Hermitian line bundle (L, ρ) on M is called ω-*admissible*, if it satisfies the following two conditions:

(1) ρ is a good metric on $L|_{M^0}$; and

(2) $c_1(L, \rho) = d(L) \cdot \omega$. Here $d(L)$ denotes the degree of L.

For example, (from the discussion in the proof of Proposition of Sect. 1.1 on the growth of β_ω and the above computation on first Chern form,) $(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P})$ is ω-admissible. Thus, over any line bundle L on M, we obtain ω-admissible Hermitian metrics on L (by first writing $L = \mathcal{O}_M(\sum a_i R_i)$ as a divisor line bundle, then extending $\rho_{\mathrm{Ar};\omega;P}$ linearly on P's). Clearly, from Conditions (1) and (2), if ρ_1 and ρ_2 are two ω-admissible metrics on L, then there exists a constant c such that $\rho_1 = \rho_2 \cdot \mathrm{e}^c$. Hence, ω-admissible Hermitian metrics over a fixed line bundle are parametrized by $\mathbf{R}^+$.

Thus in particular, on canonical line bundle K_M of M, there exist ω-admissible Hermitian metrics, which are far from being unique. So to get a canonical one, we make the following normalization.

Basic definition I(ii). *On K_M, define the ω-Arakelov metric $\tau_{\mathrm{Ar};\omega}$ by setting*

$$\|h(z)\,dz\|^2_{\tau_{\mathrm{Ar};\omega}}(P) := |h(P)|^2 \cdot \lim_{Q\to P} \frac{|z(P)-z(Q)|^2}{\mathrm{e}^{-g_\omega(P,Q)}} \cdot \mathrm{e}^{-2q\beta_\omega(P)} \text{ for } P \in M^0.$$

Here $h(z)\,dz$ denotes a section of K_M.

So

$$\|h(z)\,dz\|^2_{\tau_{\mathrm{Ar};\omega}}(P) = \|h(z)\,dz\|^2_{\tau_{\mathrm{Ar};\omega_{\mathrm{can}}}}(P) \cdot \mathrm{e}^{(-2q+2)\beta_\omega(P)}.$$

Here $\tau_{\mathrm{Ar};\omega_{\mathrm{can}}}$ denotes the (canonical) Arakelov metric on K_M, which is smooth. Therefore by the growth condition on β_ω, we see that $\tau_{\mathrm{Ar};\omega}$ is good. Moreover, since $\tau_{\mathrm{Ar};\omega_{\mathrm{can}}}$ is ω_{can}-admissible, (see e.g. [La2, Chapter IV, Theorem 5.4],) we have

$$\begin{aligned} &c_1(K_M, \tau_{\mathrm{Ar};\omega}) \\ &\quad = (2q-2)\omega_{\mathrm{can}} + dd^c(-[(-2q+2)\beta_\omega]) \\ &\quad = (2q-2)\omega_{\mathrm{can}} + (2q-2)(\omega - \omega_{\mathrm{can}}) \\ &\quad = (2q-2)\omega. \end{aligned}$$

All in all, what we have just said proves the following

Proposition. *With the same notation as above, $(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega:P})$ and $(K_M, \tau_{\mathrm{Ar};\omega})$ are ω-admissible. Moreover, for any line bundle L over M, there exist ω-admissible metrics, which are parametrized by $\mathbf{R}^+$.*

In particular, if τ is an ω-admissible metric on K_M, then there exists a constant a such that $\tau = \tau_{\mathrm{Ar};\omega} \cdot \mathrm{e}^a$. With this, define a smooth ω_{can}-admissible metric τ_{can} on K_M by setting

$$\tau_{\mathrm{can}} := \tau_{\mathrm{Ar};\omega_{\mathrm{can}}} \cdot \mathrm{e}^a. \tag{1.1}$$

Similarly, for any ω-admissible metrized line bundle (L, ρ) on M, we may introduce a unique smooth ω_{can}-admissible metric ρ_{can} on L as follows.

Write L as $L = \mathcal{O}_M(\sum a_i R_i)$ for a certain divisor $\sum a_i R_i$. Then, by using ω-Arakelov metrics $\rho_{\mathrm{Ar};\omega;R_i}$ on $\mathcal{O}_M(R_i)$, we get another ω-admissible metric $\otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i}$ on L. Therefore, by the Proposition above, there is a constant c such that $\rho = \otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i} \cdot \mathrm{e}^c$. Define a smooth ω_{can}-admissible metric ρ_{can} on L by

$$\rho_{\mathrm{can}} := \otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i} \cdot \mathrm{e}^c. \tag{1.2}$$

Note that in this construction, we use a realization of L as a divisor line bundle $\mathcal{O}_M(\sum a_i R_i)$. Thus we should show that ρ_{can} does not depend on such choices.

Key lemma. *With the same notation as above, ρ_{can} is well-defined. That is to say, if we have $L = \mathcal{O}_M(\sum b_j S_j)$, and $\rho = \otimes\rho_{\mathrm{Ar};\omega;S_j}^{\otimes b_j} \cdot \mathrm{e}^d$ for a certain constant d, then*

$$\otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}^{\otimes a_i} \cdot \mathrm{e}^c = \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};S_j}^{\otimes b_j} \cdot \mathrm{e}^d.$$

Proof. From definition, on L, we have the following equality for ω-admissible metrics

$$\otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i} \cdot \mathrm{e}^c = \rho = \otimes\rho_{\mathrm{Ar};\omega;S_j}^{\otimes b_j} \cdot \mathrm{e}^d.$$

Hence to prove the lemma, it suffices to show that

$$\frac{\otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}^{\otimes a_i}}{\otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i}} = \frac{\otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};S_j}^{\otimes b_j}}{\otimes\rho_{\mathrm{Ar};\omega;S_j}^{\otimes b_j}}. \tag{1.3}$$

Now, by definition, the logarithm of the first ratio (at a fixed point x on M) is

$$\sum a_i\Big(\big(- g_\omega(R_i, x) + \beta_\omega(R_i)\big) + g(R_i, x)\Big),$$

which by Proposition$'$ in Sect. 1.1 is nothing but $-\sum a_i\beta_\omega(x)$. Similarly, the logarithm of the second ratio is $-\sum b_j\beta_\omega(x)$. Clearly, $\sum a_i = \sum b_j$ is the degree of L, so we establish (1.3) and hence show that

$$\otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}^{\otimes a_i} \cdot \mathrm{e}^c = \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};S_j}^{\otimes b_j} \cdot \mathrm{e}^d.$$

This completes the proof of the Lemma.

The equality (1.3) says that if $\sum a_i R_i$ is rationally equivalent to $\sum b_j S_j$, then

$$\frac{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}}{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega;R_i}} = \frac{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega_{\mathrm{can}};S_j}}{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega;S_j}}.$$

That is to say,

$$\frac{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}}{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega_{\mathrm{can}};S_j}} = \frac{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega;R_i}}{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega;S_j}}. \tag{1.4}$$

Clearly, the ratio $\frac{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega;R_i}}{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega;S_j}}$ is a constant by ω-admissible condition and depends only on ω, $\sum a_i R_i$ and $\sum b_j S_j$. Hence if we set

$$C\left(\omega; \sum a_i R_i, \sum b_j S_j\right) := \frac{\otimes \rho^{\otimes a_i}_{\mathrm{Ar};\omega;R_i}}{\otimes \rho^{\otimes b_j}_{\mathrm{Ar};\omega;S_j}},$$

then by (1.4), the constant $C(\omega; \sum a_i R_i, \sum b_j S_j)$ does not really depend on ω. That is to say, we have the following

Mean value lemma I. *With the same notation as above, for any two normalized volume forms ω_1 and ω_2 on M,*

$$C\left(\omega_1; \sum a_i R_i, \sum b_j S_j\right) = C\left(\omega_2; \sum a_i R_i, \sum b_j S_j\right),$$

provided that $\sum a_i R_i$ is rationally equivalent to $\sum b_j S_j$.

1.3. Mean value lemma for ω-admissible intersections

In this section, we define metrized Deligne pairings for line bundles equipped with ω-admissible metrics, which may be singular. More importantly, we study their dependence on ω.

To begin with, let us recall the construction of Deligne pairings and its metrized version when metrics are smooth.

Let $\pi : X \to S$ be a projective flat morphism whose fibers are algebraic curves. Then for any two invertible sheaves L_1, L_2 over X, following Deligne [De2], we may introduce the *Deligne pairing* $\langle L_1, L_2\rangle(X/S)$, which is often written as $\langle L_1, L_2\rangle(\pi)$, or $\langle L_1, L_2\rangle$, by using the following axioms:

(DP1) $\langle L_1, L_2\rangle$ is an invertible sheaf on S, and is symmetric and bi-linear in L_i's;

(DP2) $\langle L_1,\ L_2\rangle$ is locally generated by symbols $\langle l_1, l_2\rangle$ with l_i sections of L_i, $i = 1, 2$, whenever the divisors of l_i's have no common intersection; moreover, if f is a rational function on X, then

$$\langle l_1,\ fl_2\rangle = \otimes_k \mathrm{Norm}_{Y_k/S}(f)^{n_k}\langle l_1, l_2\rangle$$

provided that $\mathrm{div}(l_1) = \sum n_k Y_k$ is finite over S and $\mathrm{div}(f)$ has no intersection with Y_k. Here, as usual, $\mathrm{Norm}_{Y_k/S}$ denotes the standard norm map for the covering Y_k/S.

(DP3) For a section l_2 of L_2 such that all components Y_α of the divisor $\mathrm{div}(l_2) = \sum_\alpha n_\alpha Y_\alpha$ are flat over S, we have a canonical isomorphism

$$\langle L_1, L_2\rangle(X/S) := \otimes_\alpha (\mathrm{Norm}_{Y_\alpha/S}(L_1|_{Y_\alpha}))^{\otimes n_\alpha}.$$

(In practice, Deligne pairings may be constructed by using the above axioms as follows: first, we use (DP3) to reduce to finite flat coverings, by using a certain choice of sections; then we use axiom (DP2) to show that this construction does not really depend on the choice of sections.)

Moreover, if π is defined over $\mathbf{C}$ and L_1, L_2 are with smooth metrics ρ_1 and ρ_2 respectively, we may introduce a natural metric $h_D(\rho_1, \rho_2)$, the so-called Deligne metric, on $\langle L_1, L_2\rangle$ as follows:

$$\log \|\langle l_1, l_2\rangle\|_{h_D(\rho_1,\rho_2)} := \int_\pi dd^c \log \|l_1\|_{\rho_1} \cdot \log \|l_2\|_{\rho_2} + \log(\|l_1\|_{\rho_1})(\mathrm{div}(l_2)) + \log(\|l_2\|_{\rho_2})(\mathrm{div}(l_1)). \tag{1.5}$$

Here, l_1 and l_2 are chosen as in (DP2), and by definition, if $\mathrm{div}(l_2) = \sum a_i R_i$, then $\Big(\|l_1\|_{\rho_1}\Big)(\mathrm{div}(l_2)) := \prod_i \Big(\|l_1\|_{\rho_1}(R_i)\Big)^{a_i}$. Quite often, we write $\Big(\langle L_1, L_2\rangle;\ h_D(\rho_1, \rho_2)\Big)$ also as $\Big\langle (L_1, \rho_1); (L_2, \rho_2)\Big\rangle$.

Particularly, as a consequence of these axioms, metrized Deligne pairing is compactible with base change, and that for any metrized line bundle (H, h) on S,

$$\langle (L_1, \rho_1), \pi^*(H, h)\rangle(\pi) \simeq (H, h)^{\otimes d_\pi(L_1)}.$$

Here $d_\pi(L)$ denote the relative degree of L. (See e.g. [De2].)

Next, we generalize the above metrized version to that for (possibly singular) ω-admissible metrics. We here will only do it for a single Riemann surface, which is enough for our application to arithmetic surfaces, while leave a modification which works for families of Riemann surfaces to Sect. 1.6. (Thus, the Deligne pairing gives a line bundle over a point, i.e., is a one-dimensional vector space.)

Thus let ω_1 and ω_2 be two normalized volume forms on M associated to two, possibly same, quasi-hyperbolic metrics, and let $(L_i, \rho_i), i = 1, 2$ be ω_i-admissible metrized line bundles over M. Then metrics ρ_1 and ρ_2 are good, which implies in particular that each term of (1.5) is well-defined, as we may further assume that supports of $\mathrm{div}(l_i)$'s are away from punctures. Hence we have a metric $h_D(\rho_1, \rho_2)$, which is also called the Deligne metric for ρ_1 and ρ_2, on the Deligne pairing $\langle L_1, L_2\rangle$ for possibly singular ρ_1 and ρ_2. As before, write $\Big(\langle L_1, L_2\rangle; h_D(\rho_1, \rho_2)\Big)$ by $\Big\langle (L_1, \rho_1); (L_2, \rho_2)\Big\rangle$.

To go further, we assume $\omega_1 = \omega_2 =: \omega$ and study the dependence of Deligne metrics on ω. Recall that in the previous section for an ω-admissible metric τ on K_M, the canonical line bundle, we may construct a unique ω_{can}-admissible metric τ_{can} by using ω-Arakelov metric $\tau_{\mathrm{Ar};\omega}$. Similarly, for any ω-admissible metrized line bundle (L, ρ), we construct a unique ω_{can}-admissible metric ρ_{can} by using ω-Arakelov metrics $\rho_{\mathrm{Ar};\omega;P}$ for points $P \in M$.

Mean value lemma II. *With the same notation as above, for any ω-admissible metrized line bundle (L', ρ') on M, we have*
(1) On $\langle L, L'\rangle$,

$$h_D(\rho, \rho') = h_D(\rho_{\mathrm{can}}, \rho'_{\mathrm{can}});$$

(2) On $\langle K_M, K_M\rangle$,

$$h_D(\tau, \tau) = h_D(\tau_{\mathrm{can}}, \tau_{\mathrm{can}});$$

(3) On $\langle L, K_M\rangle$,

$$h_D(\rho, \tau) = h_D(\rho_{\mathrm{can}}, \tau_{\mathrm{can}}).$$

Proof. Easily from (1.5), we see that, for any constant c and c',

$$h_D(\rho \cdot \mathrm{e}^c, \rho' \cdot \mathrm{e}^{c'}) = h_D(\rho, \rho') \cdot \mathrm{e}^{cd'+c'd} \tag{1.6}$$

Here d and d' denotes the degree of L and L' respectively. Therefore, by definition, or better, the proof of the Key Lemma above, it suffices to prove the following

Mean value lemma II$'$. *For any two normalized volume forms ω_1 and ω_2 on M, we have the following equalities for Deligne metrics*
(i) on $\langle \mathcal{O}_M(\sum_i a_i R_i), \mathcal{O}_M(\sum_i a'_i R'_i)\rangle$,

$$h_D(\otimes\rho_{\mathrm{Ar};\omega_1;R_i}^{\otimes a_i}; \otimes\rho_{\mathrm{Ar};\omega_1;R'_i}^{\otimes a'_i}) = h_D(\otimes\rho_{\mathrm{Ar};\omega_2;R_i}^{\otimes a_i}; \otimes\rho_{\mathrm{Ar};\omega_2;R'_i}^{\otimes a'_i});$$

(ii) on $\langle K_M, K_M\rangle$,

$$h_D(\tau_{\mathrm{Ar};\omega_1}, \tau_{\mathrm{Ar};\omega_1}) = h_D(\tau_{\mathrm{Ar};\omega_2}, \tau_{\mathrm{Ar};\omega_2});$$

(iii) on $\langle \mathcal{O}_M(\sum a_i R_i), K_M\rangle$,

$$h_D(\otimes\rho_{\mathrm{Ar};\omega_1;R_i}^{\otimes a_i}; \tau_{\mathrm{Ar};\omega_1}) = h_D(\otimes\rho_{\mathrm{Ar};\omega_2;R_i}^{\otimes a_i}; \tau_{\mathrm{Ar};\omega_2}).$$

Proof. We only prove (ii) as the proofs for others are the same. Without loss of generality, we may assume that $\omega_2 = \omega_{\text{can}}$. Denote ω_1 simply by ω. Also if (L, ρ) is a metrized line bundle, denote by $(L, \rho) \cdot \mathrm{e}^f$ the line bundle L together with the twisted metric $\rho \cdot \mathrm{e}^f$. With this, for (ii), we have

$$\begin{aligned}
&\Big(\langle K_M, K_M\rangle, h_D(\tau_{\mathrm{Ar};\omega}, \tau_{\mathrm{Ar};\omega})\Big)\\
&\quad = \langle (K_M, \tau_{\mathrm{Ar};\omega}), (K_M, \tau_{\mathrm{Ar};\omega})\rangle\\
&\quad = \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}) \cdot \mathrm{e}^{-(2g-2)\beta_\omega}, (K_M, \tau_{\mathrm{Ar};\omega})\rangle\\
&\qquad\quad \text{(by definition)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega})\rangle \cdot \mathrm{e}^{-(2g-2)\int \beta_\omega \cdot c_1((K_M, \tau_{\mathrm{Ar};\omega}))}\\
&\qquad\quad \text{(by (1.5))}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}) \cdot \mathrm{e}^{-(2g-2)\beta_\omega}\rangle \cdot \mathrm{e}^{-(2g-2)\int \beta_\omega \cdot c_1((K_M, \tau_{\mathrm{Ar};\omega}))}\\
&\qquad\quad \text{(by definition)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}})\rangle \cdot \mathrm{e}^{-(2g-2)\int \beta_\omega \cdot c_1((K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}))} \cdot \mathrm{e}^{-(2g-2)^2 \int \beta_\omega \cdot \omega}\\
&\qquad\quad \text{(by (1.5) and admissible condition)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}})\rangle \cdot \mathrm{e}^{-(2g-2)^2 \int \beta_\omega \cdot \omega_{\text{can}}} \cdot \mathrm{e}^{-(2g-2)^2 \int \beta_\omega \cdot \omega}\\
&\qquad\quad \text{(by admissible condition)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}})\rangle \cdot \mathrm{e}^{-(2g-2)^2 \int \beta_\omega \cdot (\omega_{\text{can}} + \omega)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}})\rangle \cdot \mathrm{e}^{-(2g-2)^2 \cdot 0}\\
&\qquad\quad \text{(by the property of } \beta \text{ in Sect. 1.1)}\\
&\quad \simeq \langle (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}}), (K_M, \tau_{\mathrm{Ar};\omega_{\text{can}}})\rangle\\
&\quad = \Big(\langle K_M, K_M\rangle, h_D(\tau_{\mathrm{Ar};\omega_{\text{can}}}, \tau_{\mathrm{Ar};\omega_{\text{can}}})\Big)
\end{aligned}$$

This completes the proof of the Mean Value Lemma II′ and hence also the Mean Value Lemma II.

As a direct consequence, we have the following

Ω-adjunction isometry. *With the same notation as above, for any point P on M, the natural residue map induces canonically an isometry*

$$\Big(\langle K_M(P), \mathcal{O}_M(P)\rangle, h_D(\tau_{\mathrm{Ar};\omega} \otimes \rho_{\mathrm{Ar};\omega;P}; \rho_{\mathrm{Ar};\omega;P})\Big) \simeq (\mathbf{C}, |\,|).$$

Here $|\,|$ denotes the standard Euclidean measure over **C**.

Proof. This is true when ω is the canonical volume form ω_{can} by the result of Arakelov. See e.g., [La2]. Hence, from (ii) and (iii) above, we complete the proof.

1.4. New determinant metrics

With respect to the normalized volume form ω of a quasi-hyperbolic metric ds^2 on M^0, in the previous section, we define intersections for all ω-admissible line bundles (L, ρ) on M, the smooth compactification of M^0, and study their dependence on ω. In this section, we introduce its counterpart on Grothendieck-Mumford determinant of cohomology $\lambda(L)$.

To begin with, recall that for a fixed smooth metric τ on K_M, if ρ is a smooth metric on a line bundle L, then the corresponding Laplacian $\Delta_{\rho;\tau}$ on L^2-sections $L^2(M, L)$ of L has only discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$. Hence we may define the associated zeta function by $\zeta_{\rho;\tau}(s) := \sum_{i\geq 1} \lambda_i^{-s}$ for $\mathrm{Re}(s) > 1$. It is well-known that $\zeta_{\rho;\tau}(s)$ admits a meromorphic continuation to the whole complex plane which is holomorphic at $s = 0$. Following Ray and Singer, define the regularized determinant $\det^*(\Delta_{\rho;\tau})$ of $\Delta_{\rho;\tau}$ by setting

$$\det{}^*(\Delta_{\rho;\tau}) := \mathrm{e}^{-\zeta'_{\rho;\tau}(0)}.$$

On the other hand, on cohomology spaces $H^i(M, L), i = 0, 1$, we have natural L^2-metrics, which then induces a natural metric $h_{L^2}(\rho; \tau)$ on $\lambda(L) := \det H^0(M, L) \otimes (\det H^1(M, L))^{\otimes -1}$. With this, the Quillen metric $h_Q(\rho; \tau)$ on $\lambda(L)$ is defined to be

$$h_Q(\rho; \tau) := h_{L^2}(\rho; \tau) \cdot \det{}^*(\Delta_{\rho;\tau}).$$

(For details, see e.g.,[Qu], [RS] and [De2].) For example, applying this to K_M and L equipped with ω_{can}-admissible metrics, we obtain corresponding Quillen metrics on $\lambda(L)$.

However such a construction cannot be applied when metrics involved are singular, since, among others, the associated Laplacians, if exist, have continuous spectrum as well. Thus if ω is the normalized volume form of a quasi-hyperbolic metric, to introduce metrics on $\lambda(L)$ for ω-admissible metrized line bundles (L, ρ), we should and will do it very differently, which goes as follows.

First, we fix a metric on M, or better, a metric τ on K_M which is ω-admissible. Then for any ω-admissible metrized line bundle (L, ρ) on M, define the corresponding determinant metric $h_{\det}(\rho; \tau)$ on $\lambda(L)$, which is indeed a one-dimensional vector space, by setting

$$h_{\det}(\rho; \tau) := h_Q(\rho_{\mathrm{can}}; \tau_{\mathrm{can}}). \tag{1.7}$$

Here ρ_{can} and τ_{can} are smooth ω_{can}-admissible metrics on L and K_M corresponding to ρ and τ introduced in (1.1) and (1.2) in Sect. 1.2, respectively. (We reminder the reader that the metric ρ on L is not related to the metric τ on K_M.)

To justify our definition, we give the following

Proposition. *With the same notation as above, if ω is smooth on M, then on $\lambda(L)$*

$$h_{\det}(\rho;\tau) = h_Q(\rho,\tau).$$

That is to say, when metrics on line bundles and base compact Riemann surfaces are smooth, determinant metrics here are the same as the standard Quillen metrics.

Proof. By definition, it suffices to show that

$$h_Q(\rho_{\mathrm{can}};\tau_{\mathrm{can}}) = h_Q(\rho,\tau).$$

But if c is a constant, then

$$h_Q(\rho\cdot \mathrm{e}^c;\tau) = h_Q(\rho,\tau)\cdot \mathrm{e}^{c\cdot\chi(L)}.$$

Here $\chi(L)$ denotes the Euler-Poincaré characteristic of L. Indeed, if we change ρ to $\rho\cdot\mathrm{e}^c$, there is no change for eigen-values of the corresponding Laplacians on L^2 sections, hence regularized determinants remain the same; while the change for L^2 metrics is easily to seen to be $\mathrm{e}^{c\cdot\chi(L)}$. Therefore, we may assume that

$$(L,\rho) = \left(\mathcal{O}_M\left(\sum a_i R_i\right), \otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i}\right).$$

Furthermore, by the fact that Quillen metric $h_Q(\rho;\tau)$ satisfies Deligne-Riemann-Roch isometry, (see e.g., Sect. 1.6,) we have

$$\begin{aligned}&(\lambda(L), h_Q(\rho;\tau))^{\otimes 12}\\ &\simeq \left\langle (K_M,\tau),(K_M,\tau)\right\rangle \otimes \left\langle (L,\rho),(L,\rho)\otimes(K_M,\tau)^{\otimes -1}\right\rangle^{\otimes 6}\cdot \mathrm{e}^{a(q)}\end{aligned}$$

with $a(q) = (1-q)(24\zeta_{\mathbf{Q}}'(-1)-1)$. In particular, if a is a constant, we see that

$$h_Q(\rho;\tau\cdot\mathrm{e}^a) = h_Q(\rho;\tau)\cdot \mathrm{e}^{-\frac{1}{6}(g-1)a-\frac{1}{2}\chi(L)a} \tag{1.8}$$

by (1.6). Therefore, we may further assume that $\tau = \tau_{\mathrm{Ar};\omega}$, the ω-Arakelov metric on K_M.

In this way, finally we are lead to the proof of the following identity of Quillen metrics on $\lambda(\mathcal{O}_M(\sum a_i R_i))$;

$$h_Q(\otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};R_i}^{\otimes a_i};\tau_{\mathrm{Ar};\omega_{\mathrm{can}}}) = h_Q(\otimes\rho_{\mathrm{Ar};\omega;R_i}^{\otimes a_i};\tau_{\mathrm{Ar};\omega}).$$

In this form, the identity is then equivalent to the *Mean Value Lemma III* at p. 489 of [We1], which in fact is the starting point of all our discussions. This completes the proof of the proposition.

We end this section by the following direct consequence of the proof of the Proposition and the definition of determinant metrics.

Mean Value Lemma III. *For any two normalized volume forms ω_1 and ω_2 on M, we have*
(i) on $\lambda(K_M)$,

$$h_{\det}(\tau_{\mathrm{Ar};\omega_1};\tau_{\mathrm{Ar};\omega_1}) = h_{\det}(\tau_{\mathrm{Ar};\omega_2};\tau_{\mathrm{Ar};\omega_2});$$

(ii) on $\lambda(\mathcal{O}_M(\sum a_i R_i))$,

$$h_{\det}(\otimes\rho_{\mathrm{Ar};\omega_1;R_i}^{\otimes a_i};\tau_{\mathrm{Ar};\omega_1}) = h_{\det}(\otimes\rho_{\mathrm{Ar};\omega_2;R_i}^{\otimes a_i};\tau_{\mathrm{Ar};\omega_2}).$$

1.5. Decomposition of hyperbolic metrics: Arakelov-Poincaré volumes

Standard hyperbolic metric ds^2_{hyp} of a punctured Riemann surface M^0 defines a natural metric on $K_M(P_1+\cdots+P_N)$, where M denotes the smooth compactification of M^0 with $P_1,\ldots,P_N$ the corresponding punctures. Such a metric is ω_{hyp}-admissible, where ω_{hyp} denotes the normalized volume form of ds^2_{hyp}. However, for applications, what we need is a natural ω_{hyp}-admissible metric on K_M. In this section, we construct canonical ω_{hyp}-admissible metrics on both K_M and $\mathcal{O}_M(P_1+\cdots+P_N)$, by using ω_{hyp}-Arakelov metrics on K_M and $\mathcal{O}_M(P_i)$'s, $i=1,\ldots,N$. Key points here are the Arakelov-Poincaré volume, a new invariant for M^0, and the Puncture Democracy, which claims that, metrically, all punctures behavior in the same way.

Let M^0 be a punctured Riemann surface of genus q with N punctures $P_1,\ldots,$ P_N. Assume always that $2q-2+N>0$. Then by uniformization theory, there exists a torsion free Fuchsian group Γ such that $M^0\simeq\Gamma\backslash\mathcal{H}$. Moreover, by invariance of the Poincaré metric on $\mathcal{H}$ under ($\mathrm{PSL}_2(\mathbf{R})$ and hence) Γ, we get an induced metric on M^0, which we call the standard hyperbolic metric. Denote by $d\mu_{\mathrm{hyp}}$ its volume form on M. It is well-known that if ω_{hyp} denotes the corresponding normalized volume form, then $2\pi(2q-2+N)\cdot\omega_{\mathrm{hyp}}=d\mu_{\mathrm{hyp}}$.

Recall that if the hyperbolic metric is considered as a singular metric on M, the line bundle naturally attached is the dual of $K_M(P_1+\cdots+P_N)$. Moreover, if we denote the induced Hermitian metric on $K_M(P_1+\cdots+P_N)$ by $\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}$, then

$$c_1\Big(K_M(P_1+\cdots+P_N),\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}\Big)=d\mu_{\mathrm{hyp}}=(2g-2+N)\omega_{\mathrm{hyp}}.$$

That is to say, $\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}$ is an ω_{hyp}-admissible metric on $K_M(P_1+\cdots+P_N)$. (See e.g., [De1], [Mu1] or [Fu]).

However we are not satisfied with this, since the metric $\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}$ is not really an ω_{hyp}-admissible metric on the canonical line bundle K_M, without which we cannot apply our basic constructions such as determinant metrics.

To construct canonical ω_{hyp}-admissible metrics on K_M and $\mathcal{O}_M(P_1),\ldots,$ $\mathcal{O}_M(P_N)$ from the hyperbolic metric on M, we go as follows.

Denote metrics to be constructed on K_M and $\mathcal{O}_M(P_1), \ldots, \mathcal{O}_M(P_N)$ by $\rho_{\mathrm{hyp};K_M}$ and $\rho_{\mathrm{hyp};P_1}, \ldots, \rho_{\mathrm{hyp};P_N}$ respectively. Naturally, we assume that they satisfy the following conditions:

(i) (**Admissibility**) $\rho_{\mathrm{hyp};K_M}$ on K_M and $\rho_{\mathrm{hyp};P_i}$ on $\mathcal{O}_M(P_i)$ are ω_{hyp}-admissible, $i = 1, \ldots, N$;

(ii) (**Decomposition rule**) On $K_M(P_1 + \cdots + P_N)$, the hyperbolic metric has the following decomposition;

$$\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)} = \rho_{\mathrm{hyp};K_M} \otimes \rho_{\mathrm{hyp};P_1} \otimes \cdots \otimes \rho_{\mathrm{hyp};P_N}.$$

Recall that

(1) any two ω_{hyp}-admissible metrics on a fixed line bundle differe only by a constant factor; and

(2) on K_M and $\mathcal{O}_M(P_i)$'s, we have canonical ω_{hyp}-Arakelov metrics $\tau_{\mathrm{Ar};\omega_{\mathrm{hyp}}}$ and $\rho_{\mathrm{Ar},\omega_{\mathrm{hyp}},P_i}$, $i = 1, \ldots, N$, which are ω_{hyp}-admissible. (In the following discussion, we also use $\rho_{\mathrm{Ar};\omega_{\mathrm{hyp}}}$ to denote $\tau_{\mathrm{Ar};\omega_{\mathrm{hyp}}}$.)

Hence, to construct $\rho_{\mathrm{hyp};K_M}$ and $\rho_{\mathrm{hyp};P_i}$'s, the key is to find a canonical way to determine all the constants

$$\frac{\rho_{\mathrm{hyp};K_M}}{\rho_{\mathrm{Ar},\omega_{\mathrm{hyp}}}} \quad \text{and} \quad \frac{\rho_{\mathrm{hyp};P_i}}{\rho_{\mathrm{Ar},\omega_{\mathrm{hyp}},P_i}}, \quad i = 1, \ldots, N.$$

Let us determine the constant ratio $\frac{\rho_{\mathrm{hyp};K_M}}{\rho_{\mathrm{Ar},\omega_{\mathrm{hyp}}}}$ (associated to K_M) first. For this, compare the determinant metric $h_{\det}(\rho_{\mathrm{hyp};K_M}; \rho_{\mathrm{hyp};K_M})$ on $\lambda(K_M)$ introduced in Sect. 1.4 and Takhtajan-Zograf's Quillen metric on $\lambda(K_M)$, whose definition we recall now.

Let $Z_{M^0}(s)$ be the Selberg zeta function of M^0, defined for $\mathrm{Re}(s) > 1$ by the absolutely convergent product

$$Z_{M^0}(s) := \prod_{\{l\}} \prod_{m=0}^{\infty} (1 - \mathrm{e}^{-(s+m)|l|}),$$

where l runs over the set of all simple closed geodesics on M^0 with respect to the hyperbolic metric on M^0, and $|l|$ denotes the length of l. It is known that by using Selberg trace formula for weight zero forms the function $Z_{M^0}(s)$ admits a meromorphic continuation to the whole complex s-plane which has a simple zero at $s = 1$. Thus in particular, it makes sense to talk about $Z'_{M^0}(1)$. (For details, see e.g., [Hej])

Clearly, $\lambda(K_M) =: \lambda_1 := \det H^0(M, K_M) \otimes (\det H^1(M, K_M))^\vee = \det H^0(M, K_M) \otimes \mathbf{C}$. Hence, there is a natural L^2-norm $h_{L^2,1}$ on λ_1. Following Takhtajan-Zograf [TZ1], define the Quillen norm $h_{Q,1}$ on λ_1 by setting

$$h_{Q,1} := h_{L^2,1} \cdot \frac{1}{Z'_{M^0}(1)}. \tag{1.9}$$

With this, we are ready to make the following;

Basic definition II(i). *With the same notation as above, on K_M, define the ω_{hyp}-admissible metric $\rho_{\text{hyp};K_M}$ by the condition that on $\lambda(K_M)$, the determinant metric $h_{\det}(\rho_{\text{hyp};K_M};\rho_{\text{hyp};K_M})$ is equal to Takhtajan-Zograf's Quillen metric $h_{Q,1}$ i.e., by setting*

$$h_{\det}(\rho_{\text{hyp};K_M};\rho_{\text{hyp};K_M}) := h_{Q,1}. \tag{1.10}$$

We claim that this definition determines $\rho_{\text{hyp};K_M}$ uniquely. Indeed, recall that $\rho_{\text{Ar};K_M} = \tau_{\text{Ar};\omega_{\text{hyp}}}$. Thus if we define the *Arakelov-Poincaré volume* for the punctured Riemann surface M^0 by setting

$$A_{\text{Ar;hyp}}(M;M^0) := A_{\text{Ar;hyp}}(M^0) := \frac{\rho_{\text{hyp};K_M}}{\tau_{\text{Ar},\omega_{\text{hyp}}}}, \tag{1.11}$$

which is a constant, then, by definition,

$$\begin{aligned} &h_{\det}(\rho_{\text{hyp};K_M};\rho_{\text{hyp};K_M})\\ &\quad= h_{\det}(\tau_{\text{Ar},\omega_{\text{hyp}}}\cdot A_{\text{Ar;hyp}}(M;M^0);\tau_{\text{Ar},\omega_{\text{hyp}}}\cdot A_{\text{Ar;hyp}}(M;M^0))\\ &\quad= h_Q(\tau_{\text{Ar},\omega_{\text{can}}}\cdot A_{\text{Ar;hyp}}(M;M^0);\tau_{\text{Ar},\omega_{\text{can}}}\cdot A_{\text{Ar;hyp}}(M;M^0)).\end{aligned}$$

But by the Polyakov variation formula for Quillen metrics, (see e.g., [Fay, Formula (3.31)],) we have

$$\begin{aligned} &h_Q(\rho_{\text{Ar},\omega_{\text{can}}}\cdot A_{\text{Ar;hyp}}(M;M^0);\rho_{\text{Ar},\omega_{\text{can}}}\cdot A_{\text{Ar;hyp}}(M;M^0))\\ &\quad= h_Q(\rho_{\text{Ar},\omega_{\text{can}}},\rho_{\text{Ar},\omega_{\text{can}}})\cdot A_{\text{Ar;hyp}}(M;M^0)^{\frac{1}{6}(2q-2)}.\end{aligned}$$

Therefore, we finally arrive at, by the definition (1.10),

$$h_{L^2,1}\cdot\frac{1}{Z'_{M^0}(1)} = h_Q(\rho_{\text{Ar},\omega_{\text{can}}},\rho_{\text{Ar},\omega_{\text{can}}})\cdot A_{\text{Ar;hyp}}(M;M^0)^{\frac{1}{6}(2q-2)}.$$

This certainly uniquely defines the Arakelov-Poincaré volume $A_{\text{Ar;hyp}}(M;M^0)$, and hence the metric

$$\rho_{\text{hyp};K_M} = \tau_{\text{Ar},\omega_{\text{hyp}}}\cdot A_{\text{Ar;hyp}}(M;M^0).$$

Remark. The name of the Arakelov-Poincaré volume is suggested by the following;

Proposition ([We1]). *With the same notation as above, if $M^0 = M$ is compact, then*

$$A_{\text{Ar;hyp}}(M) = \frac{\text{Vol(Ar; hyp)}}{2\pi(2g-2)},$$

where Vol(Ar; hyp) *denotes the volume of M with respect to the $\omega_{\rm hyp}$-Arakelov metric $\rho_{\mathrm{Ar};\omega_{\rm hyp}}$. Moreover,*

$$\log A_{\mathrm{Ar},\mathrm{hyp}}(M^0) = 12 \cdot \frac{1}{2g-2} \cdot \Big(\log \frac{\det^* \Delta_{\rm Ar}}{\mathrm{Vol(Ar)}} - \log \frac{\det^* \Delta_{\rm hyp}}{\mathrm{Vol(Hyp)}}\Big).$$

Here $\Delta_{\rm Ar}$ (resp. $\Delta_{\rm hyp}$) denotes the Laplacian for the Arakelov metric (resp. standard hyperbolic metric) on M, $\det^*$ *denotes the regularized determinant of Ray-Singer, and* Vol(Ar) *(resp.* Vol(Hyp)*) denotes the volume of M with respect to the Arakelov metric (resp. the standard hyperbolic metric, i.e., $2\pi(2g-2)$).*

Obviously, the Arakelov-Poincaré volume is a very natural invariant for the punctured Riemann surface M^0, hence can be viewed as a certain interesting function on the Teichmüller space $T_{g,N}$ of N-punctured Riemann surfaces of genus g. The reader may consult [We1] for the degeneration behavior of this invariant when $N = 0$.

Once the canonical $\omega_{\rm hyp}$-admissible metric $\rho_{\mathrm{hyp};K_M}$ is introduced on K_M, we are left only with the problem to define canonical $\omega_{\rm hyp}$-admissible metrics $\rho_{\mathrm{hyp};P_i}$ on $\mathcal{O}_M(P_i)$, $i = 1, \ldots, N$. Or equivalently, we are left to determine constant factors

$$\frac{\rho_{\mathrm{hyp};P_i}}{\rho_{\mathrm{Ar};\omega_{\rm hyp};P_i}}, \qquad i = 1, \ldots, N.$$

For this, we introduce the following

Puncture democracy. *The (constant) ratio $C^i_{\rm hyp} := \mathrm{e}^{c^i_{\rm hyp}} := \dfrac{\rho_{\mathrm{hyp};P_i}}{\rho_{\mathrm{Ar};\omega_{\rm hyp};P_i}}$ does not depend on i.*

Clearly, together with the Decomposition Rule (ii), the Puncture Democracy determines all $\rho_{\mathrm{hyp};P_i}$'s. Indeed, by the Decomposition Rule, as metrics on $K_M(P_1 + \cdots + P_N)$, we have

$$\Big(\rho_{\mathrm{Ar};\omega_{\rm hyp}} \otimes \otimes_{i=1}^N \rho_{\mathrm{Ar};\omega_{\rm hyp};P_i}\Big) \cdot \mathrm{e}^{a_{\rm hyp} + c^1_{\rm hyp} + \cdots + c^N_{\rm hyp}} = \rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}.$$

Here for simplicity, we set $a_{\rm hyp} := A_{\mathrm{Ar;hyp}}(M)$. But by the Puncture Democracy, $c^i_{\rm hyp} := c^j_{\rm hyp} =: c_{\rm hyp}$, for $i, j = 1, \ldots, N$.

Basic definition II(ii). *We define the canonical metric $\rho_{\mathrm{hyp};P_i}$ by setting*

$$\rho_{\mathrm{hyp};P_i} = \rho_{\mathrm{Ar};\omega_{\rm hyp};P_i} \cdot \mathrm{e}^{c_{\rm hyp}}$$

where $c_{\rm hyp}$ is a constant defined by

$$\mathrm{e}^{N \cdot c_{\rm hyp}} := \frac{\rho_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}}{\rho_{\mathrm{Ar};\omega_{\rm hyp}} \otimes \otimes_{i=1}^N \rho_{\mathrm{Ar};\omega_{\rm hyp};P_i}} \cdot \frac{1}{A_{\mathrm{Ar;hyp}}(M; M^0)}.$$

1.6. Arithmetic Deligne-Riemann-Roch isometry for singular metrics

We in this section show that our Deligne metrics and determinant metrics satisfy a Deligne-Riemann-Roch isometry as well. We work over both arithmetic surfaces and families of Riemann surfaces.

Arithmetic surfaces: Let F be a number field, by which we mean a finite extension field of $\mathbf{Q}$, the field of rational numbers. Denote its ring of integers by $\mathcal{O}_F$ and $S = \mathrm{Spec}\mathcal{O}$ the associated scheme. Then by an arithmetic surface over S we mean a two dimensional regular scheme X together with a projective flat morphism $\pi : X \to S$. Also we assume that the generic fiber X_F of π is geometrically irreducible.

An arithmetic surface $\pi : X \to S$ is called semi-stable, if all geometric fibers X_v over $v \in S$ are reduced, have at most ordinary double points as singularities, and all rational components intersect with others at least at two points. Denote by δ_v the number of double point on X_v, and call the divisor over S defined by $\Delta_\pi := \sum_v \delta_v[v]$ the discriminant divisor of π. Denote the relative dualizing sheaf of π by K_π which is in particular invertible. (See e.g., [La2].)

Algebraic Deligne-Riemann-Roch isomorphism ([Mu2,3] and [De2]). *Let $\pi : X \to S$ be a semi-stable arithmetic surface. Then for any line bundle L over X, we have the following canonical algebraic isomorphism of line bundles over S*

$$\lambda(L)^{\otimes 12} \simeq \langle L, L \otimes K_M^{\otimes -1}\rangle^{\otimes 6} \otimes \langle K_\pi, K_\pi\rangle \otimes \mathcal{O}_S(\Delta_\pi).$$

Furthermore, in [De2], for smooth metrics on L and K_π, by using Deligne metrics on Deligne pairing and Quillen metrics on determinants of cohomology, Deligne shows that this algebraic isomorphism is indeed an isometry. (See e.g., Arithmetic Deligne-Riemann-Roch Isometry stated below.) We will generalize this metrized version to the case when metrics on L and K_π's are admissible and hence may be singular. For this let fix some notation.

Let S_∞ be the collection of all Archimedean places of F. Denote by X_∞ the collection of all infinite fibers of π. That is, $X_\infty = \{X_\sigma\}_{\sigma\in S_\infty}$ with X_σ a Riemann surface of genus q associated to X_F corresponding to the natural inclusion $F \hookrightarrow F_\sigma \hookrightarrow \mathbf{C}$. Here as usual F_σ denote the σ-completion of F.

Let ds^2 be a quasi-hyperbolic metric on X_∞, by which we mean $ds^2 = \{ds_\sigma^2\}_{\sigma\in S_\infty}$ is a collection of quasi-hyperbolic metrics on $\{X_\sigma\}_{\sigma\in S_\infty}$. Denote associated normalized volume forms by $\omega := \{\omega_\sigma\}_{\sigma\in S_\infty}$. By definition, an ω-admissible Hermitian line bundle (L, ρ) on X is a line bundle L on X together with a Hermitian metric $\rho = \{\rho_\sigma\}_{\sigma\in S_\infty}$ on the line bundle $\{L_\sigma\}_{\sigma\in S_\infty}$ over X_∞ induced from L such that (L_σ, ρ_σ) is ω_σ-admissible metric on X_σ for all $\sigma \in S_\infty$.

Let $\tau = \{\tau_\sigma\}_{\sigma\in S_\infty}$ be an ω-admissible metric on K_π. Then by applying constructions in Sect. 1.3 and Sect. 1.4 for determinant metrics and Deligne metrics for $\{\rho_\sigma\}_{\sigma\in S_\infty}$ and $\{\tau_\sigma\}_{\sigma\in S_\infty}$, we get the following metrized line bundles on S: $(\lambda(L), h_{\det}(\rho;\tau))$, $\langle (L,\rho), (L,\rho)\otimes (K_\pi,\tau)^{\otimes -1}\rangle$, $\langle (K_\pi,\tau), (K_\pi,\tau)\rangle$. (That is to say, for each $\sigma\in S_\infty$, apply the constructions of Deligne metrics and determinant metrics for ω_σ-admissible metrics ρ_σ and τ_σ.) Put the trivial metric 1 on $\mathcal{O}_S(\Delta_\pi)$, i.e., in terms of [La2], the metrized line bundle $(\mathcal{O}_S(\Delta_\pi), 1)$ on S corresponding to the Arakelov divisor Δ_π.

With this, we may state our main result in this Part.

Arithmetic Deligne-Riemann-Roch isometry for singular metrics. *Let $\pi : X\to S$ be a semi-stable regular arithmetic surface. Let ω be the normalized volume form for quasi-hyperbolic metrics on X_∞. Let (L,ρ) be an ω-admissible metrized line bundle on X, and τ be an ω-admissible metric on the relative dualizing bundle K_π of π. Then we have a canonical isometry*

$$\Big(\lambda(L), h_{\det}(\rho,\tau)\Big)^{\otimes 12} \simeq \Big\langle (L,\rho), (L,\rho)\otimes (\mathcal{K}_\pi,\tau)^{\otimes -1}\Big\rangle^{\otimes 6} \otimes\Big\langle (\mathcal{K}_\pi,\tau), (\mathcal{K}_\pi,\tau)\Big\rangle \otimes (\mathcal{O}_S(\Delta_\pi),1)\cdot \mathrm{e}^{a(q)}.$$

Here $\Delta_\pi := \sum_{v\in S}\delta_v[v]$ (with δ_v the number of double points on the fiber X_v of X at v) denotes the discriminant divisor on S associated to π and $a(q) := (1-q)(24\zeta'_{\mathbf{Q}}(-1)-1)$ denotes the Deligne constant.

Proof. First use (1.1) and (1.2), from singular ω-admissible metrics τ and ρ on K_π and L, we obtain smooth ω_{can}-admissible metrics τ_{can} and ρ_{can}. (That is to say, we first do it for all infinite fibers X_σ, then put them together over X_∞.) With this, by definition, $h_{\det}(\rho;\tau) = h_Q(\rho_{\mathrm{can}};\tau_{\mathrm{can}})$. In particular, now we may apply the original Deligne-Riemann-Roch isometry for smooth metrics to get the isometry

$$\Big(\lambda(L), h_Q(\rho_{\mathrm{can}},\tau_{\mathrm{can}})\Big)^{\otimes 12} \simeq \Big\langle (L,\rho_{\mathrm{can}}), (L,\rho_{\mathrm{can}})\otimes (\mathcal{K}_\pi,\tau_{\mathrm{can}})^{\otimes -1}\Big\rangle^{\otimes 6} \otimes\Big\langle (\mathcal{K}_\pi,\tau_{\mathrm{can}}), (\mathcal{K}_\pi,\tau_{\mathrm{can}})\Big\rangle \otimes (\mathcal{O}_S(\Delta_\pi),1)\cdot \mathrm{e}^{a(q)}.$$

On the other hand, by the Mean Value Lemma II, we have (first fiberwise at infinity then globally) the isometries

$$\Big\langle (L,\rho_{\mathrm{can}}), (L,\rho_{\mathrm{can}})\otimes (\mathcal{K}_\pi,\tau_{\mathrm{can}})^{\otimes -1}\Big\rangle \simeq \Big\langle (L,\rho), (L,\rho)\otimes (\mathcal{K}_\pi,\tau)^{\otimes -1}\Big\rangle$$

and

$$\Big\langle (\mathcal{K}_\pi,\tau_{\mathrm{can}}), (\mathcal{K}_\pi,\tau_{\mathrm{can}})\Big\rangle \simeq \Big\langle (\mathcal{K}_\pi,\tau), (\mathcal{K}_\pi,\tau)\Big\rangle.$$

Therefore,

$$\Big(\lambda(L), h_{\det}(\rho,\tau)\Big)^{\otimes 12} \simeq \Big(\lambda(L), h_Q(\rho_{\mathrm{can}}, \tau_{\mathrm{can}})\Big)^{\otimes 12}$$
$$\simeq \Big\langle (L,\rho), (L,\rho)\otimes(\mathcal{K}_\pi,\tau)^{\otimes -1}\Big\rangle^{\otimes 6}$$
$$\otimes\Big\langle (\mathcal{K}_\pi,\tau), (\mathcal{K}_\pi,\tau)\Big\rangle \otimes (\mathcal{O}_S(\Delta_\pi),1)\cdot \mathrm{e}^{a(q)}.$$

This completes the proof.

Family of Riemann surfaces: Next, we indicate a necessary modification in order to get a Deligne-Riemann-Roch isometry for singular metrics over families of Riemann surfaces.

Let $\pi : X \to S$ be a flat family of compact Riemann surfaces of genus q. Clearly π is also projective. Denote by K_π the relative canonical line bundle of π. Let $\mathbf{P}_1,\dots,\mathbf{P}_N$ be C^∞-sections of π. Let κ be a smooth metric on $K_\pi|_{X\setminus\cup_{i=1}^N \mathbf{P}_i}$. Moreover we assume that for any point $m\in S$, on $X_m := \pi^{-1}(m)$, the restriction $(K_\pi,\kappa)|_{X_m\setminus X_m\cap\cup_{i=1}^N\mathbf{P}_i}$ induces a quasi-hyperbolic ds_m^2 on $X_m^0 := X_m\setminus X_m\cap\cup_{i=1}^N\mathbf{P}_i$. Denote the corresponding normalized volume form by ω_m. By definition, an ω-admissible metrized line bundle (L,ρ) on X consists of a line bundle L on X and a Hermitian metric ρ on $L|_{X\setminus\cup_{i=1}^N\mathbf{P}_i}$ such that $(L,\rho)|_{X_m}$ is ω_m-admissible.

From the definition, by the Proposition of Sect. 1.2, if ρ_1 and ρ_2 are two ω-admissible metrics on the same line bundle L, then there exists a smooth function f on S such that $\rho_1=\rho_2\cdot \mathrm{e}^{\pi^* f}$. Moreover, it is easy to see that the gluing of ω_m-Arakelov metrics $\tau_{\mathrm{Ar};\omega_m}$ on K_{X_m} gives an ω-admissible metric $\rho_{\mathrm{Ar};\omega}$ on K_π. And, if we have a holomorphic section $\mathbf{R}$ of π, the gluing of ω_m-Arakelov metrics $\rho_{\mathrm{Ar};\omega_m;R:=\mathbf{R}\cap X_m}$ on $\mathcal{O}_{X_m}(R)$ gives an ω-admissible metric $\rho_{\mathrm{Ar};\omega;\mathbf{R}}$ on $\mathcal{O}_X(\mathbf{R})$.

To facilitate the ensuing discussion, let us recall the Deligne-Riemann-Roch isometry for smooth metrics.

So let (L,ρ) be a metrized line bundle on X with ρ smooth. Then for any smooth metric τ on K_π, we have the corresponding Quillen metric $h_Q(\rho;\tau)$ on $\lambda(L)$. (See Sect. 1.4 for details.) Also, if (L',ρ') is another metrized line bundle on X with ρ_2 smooth, by (1.5), we have the metrized Deligne pairing $\langle(L,\rho),(L',\rho')\rangle$ on S, which is usually denoted as $(\langle L,L'\rangle, h_D(\rho,\rho'))$ as well. As above, we call $h_D(\rho,\rho')$ the Deligne metric.

Deligne-Riemann-Roch isometry for smooth metrics ([De2]). *Let $\pi : X\to S$ be a flat family of compact Riemann surfaces with K_π the relative canonical line bundle. Then for any smooth metrized line bundle (L,ρ) on X, and any smooth metric τ on K_π, we have the following canonical isometry*

$$(\lambda(L), h_Q(\rho;\tau))^{\otimes 12} \simeq \langle (L,\rho), (L,\rho)\otimes(K_\pi,\tau)^{\otimes -1}\rangle^{\otimes 6}$$
$$\otimes\langle (K_\pi,\tau),(K_\pi,\tau)\rangle\cdot \mathrm{e}^{a(q)}.$$

To give a corresponding isometry for ω-admissible metrics, we do as before by using the projection formula for metrized Deligne pairing. More precisely, this goes as follows:

First, note that for any point $m \in S$, we may choose a small neighborhood U_m such that $L|_{\pi^{-1}(U_m)}$ may be written as a divisor line bundle $\mathcal{O}_{\pi^{-1}(U_m)}(\sum a_i \mathbf{R}_i)$ with $\mathbf{R}_i$ holomorphic sections of $\pi^{-1}(U_m) \to U_m$, disjoint from $\mathbf{P}_i$'s, as a direct consequence of the fact that π is projective. Thus, in particular, there exists a smooth function f_m on U_m such that over $\pi^{-1}(U_m)$, $\rho = \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f_m)}$. Hence, in particular, we get a natural smooth metric $\rho_{\mathrm{can}} := \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f_m)}$. Here ω_{can} corresponds to the standard canonical volume forms on fibers. Moreover, as in the Key Lemma of Sect. 1.2, ρ_{can} depends only on ρ and in particular does not depend on the choice of the divisors used in the definition. By moving $m \in S$, we then get a unique smooth ω_{can}-admissible metric, also denote by ρ_{can}, of L on the whole X. Similarly, from ω-admissible metric τ on K_π, by using ω-Arakelov metric $\tau_{\mathrm{Ar};\omega}$, as above, we get a smooth ω_{can}-admissible metric τ_{can} on K_π. With this, we define the determinant metric $h_{\det}(\rho;\tau)$ on $\lambda(L)$ for (L,ρ) with respect to (K_π,τ) by setting

$$h_{\det}(\rho;\tau) := h_Q(\rho_{\mathrm{can}};\tau_{\mathrm{can}}) \tag{1.12}$$

which is compactible with (1.7).

Secondly, by using the same proof of the Mean Value Lemma II, we obtain also the family version of Mean Value Lemma II for Deligne metrics. More precisely, we have the following

Mean value lemma II$''$. *With the same notation as above,*

(1) on $\langle L, L'\rangle$,

$$h_D(\rho,\rho') = h_D(\rho_{\mathrm{can}};\rho'_{\mathrm{can}});$$

(2) on $\langle K_\pi, K_\pi\rangle$,

$$h_D(\tau,\tau) = h_D(\tau_{\mathrm{can}};\tau_{\mathrm{can}});$$

(3) on $\langle L, K_\pi\rangle$,

$$h_D(\rho,\tau) = h_D(\rho_{\mathrm{can}};\tau_{\mathrm{can}}).$$

Proof. We only prove (1) as the proof for others are similar. That is to say, on $\langle L, L'\rangle$ over S, we should check that two metrics are the same. Hence, it suffices to do it locally. With the same notation as above, over $\pi^{-1}(U_m)$, we have $\rho = \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f)}$ on $L|_{\pi^{-1}(U_m)} = \mathcal{O}_{\pi^{-1}(U_m)}(\sum a_i\mathbf{R}_i)$ for a certain smooth function f on U_m. Moreover, if necessary, by shrinking U_M, we may assume that $L'|_{\pi^{-1}(U_m)} = \mathcal{O}_{\pi^{-1}(U_m)}(\sum a_i'\mathbf{R}_i')$ for some sections $\mathbf{R}_i'$, disjoint from $\mathbf{P}_i$'s and $\mathbf{R}_j$'s. Thus $\rho' = \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}'_i}^{\otimes a'_i} \cdot \mathrm{e}^{\pi^*(f')}$ for a certain smooth function f' on U_m. Clearly, from the definition of Deligne metrics (1.5), we have over U_m,

$$\begin{aligned}
&\Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f)}\Big),\\
&\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}'_i}^{\otimes a'_i} \cdot \mathrm{e}^{\pi^*(f')}\Big)\Big\rangle\\
&\quad= \Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big),\\
&\qquad \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i}, \Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}'_i}^{\otimes a'_i}\Big)\Big\rangle \cdot \mathrm{e}^{f\cdot d_\pi(L')+f'\cdot d_\pi(f)}\Big),
\end{aligned}$$

which is what we usually would call a projection formula for metrized Deligne pairings. But, by using the Mean Value Lemma II$'$, we have first pointwise then over U_m

$$\begin{aligned}
&\Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i}\Big), \Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}'_i}^{\otimes a'_i}\Big)\Big\rangle\\
&= \Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big), \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};\mathbf{R}_i}^{\otimes a_i}\Big), \Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};\mathbf{R}'_i}^{\otimes a'_i}\Big)\Big\rangle.
\end{aligned}$$

Therefore, by using the standard projection formula for metrized Deligne pairing again, we have finally

$$\begin{aligned}
&\Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f)}\Big),\\
&\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega;\mathbf{R}'_i}^{\otimes a'_i} \cdot \mathrm{e}^{\pi^*(f')}\Big)\Big\rangle\\
&\quad= \Big\langle\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a_i\mathbf{R}_i\Big), \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};\mathbf{R}_i}^{\otimes a_i} \cdot \mathrm{e}^{\pi^*(f)}\Big),\\
&\qquad\Big(\mathcal{O}_{\pi^{-1}(U_m)}\Big(\sum a'_i\mathbf{R}'_i\Big), \otimes\rho_{\mathrm{Ar};\omega_{\mathrm{can}};\mathbf{R}'_i}^{\otimes a'_i} \cdot \mathrm{e}^{\pi^*(f')}\Big)\Big\rangle.
\end{aligned}$$

That is to say,

$$h_D(\rho,\rho') = h_D(\rho_{\mathrm{can}};\rho'_{\mathrm{can}}).$$

This completes the proof.

Now by definition and the above Deligne-Riemann-Roch isometry for smooth metrics, we have

$$\begin{aligned}
&(\lambda(L), h_{\det}(\rho;\tau))^{\otimes 12} = (\lambda(L), h_Q(\rho_{\mathrm{can}};\tau_{\mathrm{can}}))^{\otimes 12}\\
&\quad\simeq \langle(L,\rho_{\mathrm{can}}), (L,\rho_{\mathrm{can}})\otimes(K_\pi,\tau_{\mathrm{can}})^{\otimes -1}\rangle^{\otimes 6}\otimes\langle(K_\pi,\tau_{\mathrm{can}}),(K_\pi,\tau_{\mathrm{can}})\rangle\cdot\mathrm{e}^{a(q)}.
\end{aligned}$$

Therefore, by applying the above Mean Value Lemma II$''$, we finally obtain the following generalization of the fundamental Deligne-Riemann-Roch isometry.

Deligne-Riemann-Roch isometry for singular metrics. *Let $\pi : X \to S$ be a flat family of compact Riemann surfaces with K_π the relative canonical line* *bundle. Then for any ω-admissible metrized line bundle (L, ρ) on X, with respect to a fixed ω-admissible metric τ on K_π, we have the following canonical isometry*

$$\Big(\lambda(L), h_Q(\rho; \tau)\Big)^{\otimes 12} \simeq \Big\langle (L, \rho), (L, \rho) \otimes (K_\pi, \tau)^{\otimes -1}\Big\rangle^{\otimes 6} \otimes \Big\langle (K_\pi, \tau), (K_\pi, \tau)\Big\rangle \cdot \mathrm{e}^{a(q)}.$$

Part II. Deligne pairings over moduli spaces of marked stable curves

In this part, we first introduce several line bundles over Knudsen-Deligne-Mumford compactification of the moduli space (or rather the algebraic stack) of stable N-pointed algebraic curves of genus g, which are rather natural and include Weil-Petersson, Takhtajan-Zograf and logarithmic Mumford line bundles. Then we use Deligne-Riemann-Roch isomorphism and its metrized version (proved in Part I) to establish some fundamental relations among these line bundles. Finally, we compute first Chern forms of the metrized Weil-Petersson, Takhtajan-Zograf and logarithmic Mumford line bundles by using results of Wolpert and Takhtajan-Zograf, and show that the so-called Takhtajan-Zograf metric on the module space is algebraic.

As for the language, we have the following remarks. Of course, I am working with moduli stacks rather than with moduli spaces. For the reader who is not familiar with stacks, this means that I am allowed to pretend that moduli spaces are smooth and that there are universal families over them. Thus, it is more economic to simply use ordinary language rather than those in stacks.

2.1. Weil-Petersson line bundles and Takhtajan-Zograf line bundles

We start with some general facts about moduli spaces of marked stable curves. For details, please consult [DM], [Kn] and [KM].

Denote by $\mathcal{M}_{g,N}$ the moduli space of smooth projective irreducible curves M of genus g together with N ordered marked points $P_1, \ldots, P_N$. It is well-known that $\mathcal{M}_{g,N}$ is not compact and has a natural compactification $\overline{\mathcal{M}_{g,N}}$ constructed by Knudsen, Deligne and Mumford by adding the so-called stable marked curves.

As algebraic stack language is assumed here, we may assume that there exists a universal curve

$$\pi := \pi_{g,N} : \overline{\mathcal{C}_{g,N}} \to \overline{\mathcal{M}_{g,N}},$$

which has N-sections $\mathbf{P}_1, \ldots \mathbf{P}_N$ such that $\mathbf{P}_i \cap \mathbf{P}_j = \emptyset$ for all $i \neq j$ with $1 \leq i, j \leq N$. So for any $x = [(M; P_1, \ldots, P_N)] \in \overline{\mathcal{M}_{g,N}}$, $\pi^{-1}(x) = M$ and $\mathbf{P}_i \cap M = P_i$, $i = 1, \ldots, N$, which are not only ordered but distinct. In fact,

$\overline{\mathcal{C}_{g,N}} = \overline{\mathcal{M}_{g,N+1}}$, and π is essentially the map of dropping the last marked points P_{N+1}'s. (More correctly, if by dropping P_{N+1}, we get a rational curve together with only two marked points, then we have to contract this component.)

The boundary of $\mathcal{M}_{g,N}$ has a natural algebraic structure, from which we may obtain naturally a normal crossing divisor Δ_{bdy} of Knudsen-Deligne-Mumford, which may be described roughly as follows.

As a divisor on $\overline{\mathcal{M}_{g,N}}$,

$$\Delta_{\text{bdy}} =:= \sum_{i=0}^{[\frac{q}{2}]} \Delta_i + \sum_{S\subset\{1,\dots,N\},\#S\geq 2} \Delta_S.$$

Here irreducible divisors Δ_i and Δ_S may be understood via the universal curve as follows: (See e.g., [Kn].)

(1) Δ_i's come from degenerations of compact Riemann surfaces. In particular, for a general point of Δ_0, the corresponding fiber of π is a genus q curve with one non-separating node, together with N-punctures $P_1, \dots, P_N$; while for a general point of Δ_i, $i = 1, \dots, [\frac{q}{2}]$, the corresponding fiber is a genus q curve with one separating node, together with N-marked points $P_1, \dots, P_N$, so that the only two irreducible components are smooth and of genera i and $q - i$ respectively.

(2) Δ_S's come from degenerations of punctures. In particular, for any subset S of $\{1, \dots, N\}$ with cardinal number $\#S$ at least two, the fiber of π over a general point in Δ_S consists of two irreducible components, one is the original curve M together with $N - \#S$ marked points, and the other is the projective line $\mathbf{P}^1$ together with remaining $\#S > 2$ marked points.

Moreover, we know that π is flat. (See e.g., [Kn].) Hence, the relative dualizing sheaf of π is indeed invertible. Denote the coresponding line bundle on $\overline{\mathcal{C}_{g,N}}$ by K_π, and call it the relative canonical line bundle of π.

With this, we may state a fundamental result of Deligne-Mumford as follows:

Deligne-Riemann-Roch isomorphism for stable curves ([De2], [Mu3], see also [We2]). *With the same notation as above, for any line bundle L on $\overline{\mathcal{C}_{g,N}}$, there exists a canonical isomorphism*

$$\lambda(L)^{\otimes 12} \simeq \langle L, L\otimes K_\pi^{\otimes -1}\rangle^{\otimes 6} \otimes \langle K_\pi, K_\pi\rangle \otimes \Delta_{\text{bdy}}.$$

Before ending the discussion on $\overline{\mathcal{M}_{g,N}}$, we recall a few standard relations between relative canonical line bundle K_π and line bundles $\mathcal{O}_{\overline{\mathcal{C}_{g,N}}}(\mathbf{P}_i)$'s of sections $\mathbf{P}_i$'s from [Kn]. For this, we need the following commutative diagram, which may be checked from the definition:

$$\begin{array}{ccccc} \overline{\mathcal{M}_{g,N+1}} = & \overline{\mathcal{C}_{g,N}} & \stackrel{\phi_{g,N}}{\to} & \overline{\mathcal{C}_{g,N-1}} & = \overline{\mathcal{M}_{g,N}} \\ & \pi_{g,N} \downarrow & & \downarrow \pi_{g,N-1} & \\ & \overline{\mathcal{M}_{g,N}} & \stackrel{\pi_{g,N-1}}{\to} & \overline{\mathcal{M}_{g,N-1}} & . \end{array}$$

Here $\phi_{g,N}$ viewed as a morphism from $\overline{\mathcal{M}_{g,N+1}}$ to $\overline{\mathcal{M}_{g,N}}$ is essentially the morphism defined by dropping the second to the last marking. Moreover, for simplicity, we often use $\mathbf{P}_i$ to denote $\mathcal{O}_{\overline{\mathcal{C}_{g,N}}}(\mathbf{P}_i)$. Also, if we need to emphasis the fact that the number of marked points is N, we write K_π as $K_{\pi_{g,N}}$ and $\mathbf{P}_i$ as $\mathbf{P}_{i,N}$.

Standard facts ([Kn II]). *With the same notation as above, over $\overline{\mathcal{M}_{g,N}}$,*
(a) $\langle \mathbf{P}_i, \mathbf{P}_j \rangle \simeq \mathcal{O}$, if $i, j = 1, \dots, N$ and $i \neq j$;
(b) $\langle K_\pi(\mathbf{P}_i), \mathbf{P}_i \rangle \simeq \mathcal{O}$, if $i = 1, \dots, N$;
(c) $\langle K_{\pi_{g,N}}, \mathbf{P}_{i,N} \rangle \simeq \Big(\pi_{g,N-1}^* \langle K_{\pi_{g,N-1}}, \mathbf{P}_{i,N-1} \rangle\Big)(\mathbf{P}_{i,N-1})$, if $i = 1, \dots, N-1$;
(d) $\langle K_{\pi_{g,N}}, \mathbf{P}_{N,N} \rangle \simeq K_{\pi_{g,N-1}}(\mathbf{P}_{1,N-1} + \cdots + \mathbf{P}_{N-1,N-1})$.
As usual, we call (b) the relative adjunction isomorphism. (Deligne pairings are not used in Knudsen's original papers [Kn]. But the verbatim change is rather trivial.)

Now we are ready to use Deligne-pairing formalism and Grothendieck-Mumford determinant formalism to construct the following new line bundles over $\overline{\mathcal{M}_{g,N}}$.

Basic definition II. (i) *The Weil-Petersson line bundle Δ_{WP} over $\overline{\mathcal{M}_{g,N}}$ is defined by setting*

$$\Delta_{\mathrm{WP}} := \Big\langle K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N),\, K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)\Big\rangle;$$

(ii) *The (total) Takhtajan-Zograf line bundle Δ_{TZ} over $\overline{\mathcal{M}_{g,N}}$ is defined by setting*

$$\Delta_{\mathrm{TZ}} := \Big\langle K_\pi, \mathcal{O}_{\overline{\mathcal{C}_{g,N}}}(\mathbf{P}_1 + \cdots + \mathbf{P}_N)\Big\rangle;$$

(iii) *The m-th logarithmic Mumford type line bundle λ_m over $\overline{\mathcal{M}_{g,N}}$ is defined by setting*

$$\lambda_m := \begin{cases} \lambda\Big(K_\pi^{\otimes m}((m-1)\mathbf{P}_1 + \cdots + (m-1)\mathbf{P}_N)\Big), & \text{if } m \geq 1; \\ \lambda\Big(\big((K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N))^\vee\big)^{\otimes -m}\Big), & \text{if } m \leq 0. \end{cases}$$

Proposition. *With the same notation as above, there exists the following canonical isomorphism*

$$\lambda_m \simeq \lambda_{1-m} \qquad \text{for } m \leq 0.$$

Proof. This is a direct consequence of the Serre duality for the cohomology, by definition.

2.2. Logarithmic Mumford type isomorphisms

In this section, we prove the following

Fundamental relation I. (Logarithmic Mumford Type Isomorphisms) *Over the moduli space $\overline{\mathcal{M}_{g,N}}$ of N-punctured Riemann surfaces of genus g, there exist the following canonical isomorphisms:*

$$\lambda_m^{\otimes 12} \simeq \Delta_{\mathrm{WP}}^{\otimes(6m^2-6m+1)} \otimes \Delta_{\mathrm{TZ}}^{\otimes -1} \otimes \Delta_{\mathrm{bdy}} \qquad \text{for } m \geq 0.$$

Proof. There are three ingredients in this proof.
(1) The algebraic Deligne-Riemann-Roch isomorphism

$$\lambda(L)^{\otimes 12} \simeq \langle L, L \otimes K_\pi^{-1} \rangle^{\otimes 6} \otimes \langle K_\pi, K_\pi \rangle \otimes \Delta_{\mathrm{bdy}};$$

(2) Standard Fact (a), which comes from the fact that two sections $\mathbf{P}_i$ and $\mathbf{P}_j$ never meet in $\overline{\mathcal{C}_{g,N}}$, i.e.,

$$\langle \mathbf{P}_i, \mathbf{P}_j \rangle \simeq \mathcal{O}$$

if $i \neq j$. Here for simplicity, we use $\mathbf{P}_i$ to denote the line bundle $\mathcal{O}(\mathbf{P}_i)$. (This convention applies to all calculations.)
(3) Standard Fact (b), the Relative Adjunction Isomorphism, i.e.,

$$\langle K_\pi(\mathbf{P}_i), \mathbf{P}_j \rangle \simeq \mathcal{O}.$$

Indeed, if $m = 0$, by (1), we have

$$\lambda_0^{\otimes 12} \simeq \langle \mathcal{O}, \mathcal{O} \otimes K_\pi^{-1} \rangle^{\otimes 6} \otimes \langle K_\pi, K_\pi \rangle \otimes \Delta_{\mathrm{bdy}} \simeq \langle K_\pi, K_\pi \rangle \otimes \Delta_{\mathrm{bdy}}.$$

So it suffices to prove the following

Lemma. *With the same notation as above, $\langle K_\pi, K_\pi \rangle \simeq \Delta_{\mathrm{WP}} \otimes \Delta_{\mathrm{TZ}}^{\otimes -1}$.*

Proof of the lemma. By definition,

$$\begin{aligned}\Delta_{\mathrm{WP}} \otimes \Delta_{\mathrm{TZ}}^{\otimes -1} &= \langle K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N), K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N) \rangle \\ &\quad \otimes \langle K_\pi, \mathbf{P}_1 + \cdots + \mathbf{P}_N \rangle^{\otimes -1} \\ &= \langle K_\pi, K_\pi \rangle \otimes \langle K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N), \mathbf{P}_1 + \cdots + \mathbf{P}_N \rangle.\end{aligned}$$

Hence, we only need to prove the following

Lemma$'$. *With the same notation as above,*

$$\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle \simeq \mathcal{O}.$$

Proof of the lemma$'$. Use an induction on N. If $N=0$, there is nothing to prove. If $N=1$, the result is given by (3), the relative adjunction isomorphism. Hence we may assume that the latest isomorphism holds for N and try to show that it also holds for $N+1$, i.e.,

$$\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N+\mathbf{P}_{N+1}), \mathbf{P}_1+\cdots+\mathbf{P}_N+\mathbf{P}_{N+1}\rangle \simeq \mathcal{O}.$$

Clearly, the left hand side is simply

$$\begin{aligned}&\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle\\ &\quad\otimes\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_{N+1}\rangle\otimes\langle \mathbf{P}_{N+1}, \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle\otimes\langle \mathbf{P}_{N+1}, \mathbf{P}_{N+1}\rangle.\end{aligned}$$

Moreover, by the induction hypothesis and (2) above, both the first and the third factors are isomorphic to $\mathcal{O}$. Hence we only need to show that

$$\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_{N+1}\rangle\otimes\langle \mathbf{P}_{N+1}, \mathbf{P}_{N+1}\rangle \simeq \mathcal{O}.$$

But by (2) again, the left hand side is simply $\langle K_\pi, \mathbf{P}_{N+1}\rangle\otimes\langle \mathbf{P}_{N+1}, \mathbf{P}_{N+1}\rangle$, or better, $\langle K_\pi(\mathbf{P}_{N+1}), \mathbf{P}_{N+1}\rangle$, which is indeed trivial from (3), the relative adjunction isomorphism. This completes the proof of the lemma$'$, the lemma, and hence the Fundamental Relation I when $m=0$.

Now for $m\geq 1$, by definition, and (1), we have

$$\begin{aligned}\lambda_m^{\otimes 12} &\simeq \Big\langle mK_\pi+((m-1)\mathbf{P}_1+\cdots+(m-1)\mathbf{P}_N), mK_\pi\\ &\qquad +((m-1)\mathbf{P}_1+\cdots+(m-1)\mathbf{P}_N)-K_\pi\Big\rangle^{\otimes 6}\otimes\langle K_\pi, K_\pi\rangle\otimes\Delta_{\text{bdy}}\\ &=\Big\langle mK_\pi+((m-1)\mathbf{P}_1+\cdots+(m-1)\mathbf{P}_N),\\ &\qquad K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N))\rangle^{\otimes 6(m-1)}\otimes\langle K_\pi, K_\pi\Big\rangle\otimes\Delta_{\text{bdy}}.\end{aligned}$$

Thus by the lemma above, it suffices to show that

$$\langle mK_\pi+((m-1)\mathbf{P}_1+\cdots+(m-1)\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N))\rangle \simeq \Delta_{\text{WP}}^{\otimes m}.$$

Clearly, by the linearity, the left hand side is isomorphic to

$$\begin{aligned}&\Big\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N))\rangle^{\otimes m}\\ &\qquad\otimes\langle \mathbf{P}_1+\cdots+\mathbf{P}_N, K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N))\Big\rangle^{\otimes -1}.\end{aligned}$$

Thus by the lemma$'$ above, we completes the proof of the Fundamental Relation I.

2.3. Comparision between Weil-Petersson and Takhtajan-Zograf line bundles

Weil-Petersson metrics have been studied for many years. By contrast, very little is known about Takhtajan-Zograf metrics on moduli spaces defined by using Eisenstein series. In this section, we prove a result which compares Weil-Petersson line bundles with Takhtajan-Zograf line bundles.

We begin with the following definition: A line bundle L on $\overline{\mathcal{M}_{g,N}}$, for our own convinence, is called *generically positive in dimension one* and denoted by $L \geq 0$ if for any irreducible curve C with support not all in the boundary of $\mathcal{M}_{g,N}$, $\deg(L\big|_C) \geq 0$.

Fundamental relation II. (Comparison between Weil-Petersson and Takhtajan-Zograf) *Over the moduli space* $\overline{\mathcal{M}_{g,N}}$,

$$\Delta_{\mathrm{WP}}^{\otimes N^2} \leq \Delta_{\mathrm{TZ}}^{\otimes (2g-2+N)^2}.$$

Before proving this relation, we would like to recall a result in a recent book of Harris and Morrison on: Moduli of Curves. At pages 308 and 309, they show the following

Basic inequality ([HM]). *Over* $\overline{\mathcal{M}_{g,1}}$, *i.e., on the universal curve over moduli space of compact Riemann surfaces* $\overline{\mathcal{M}_g}$,

$$4g(g-1)K_{\pi_{g,0}} \geq 12\lambda_1 - \Delta_{\mathrm{bdy}}.$$

Harris and Morrison emphasize the importance of this inequality by calling it *the basic inequality* and ask in general how to find such an inequality for all g, N, which I learned at the beginning of 1999, after the first version of this paper was written.

Claim. *The Basic Inequality is equivalent to the special case when* $N = 1$ *of our Fundamental Relation II.*

Indeed, since Harris and Morrison work over $\mathcal{M}_{g,1}$, so the total Takhtajan-Zograf line bundle Δ_{TZ} is simply $\langle K_\pi, \mathbf{P}_1\rangle$, which by standard fact (d) in Sect. 2.1 is nothing but the relative canonical line bundle $K = K_{\pi_{g,0}}$ of the morphism $\pi_{g,0} : \overline{\mathcal{M}_{g,1}} \to \mathcal{M}_g$. Hence, the Basic Inequality may be rewritten as

$$4g(g-1)\Delta_{\mathrm{TZ}} \geq 12\lambda_1 - \Delta_{\mathrm{bdy}}.$$

On the other hand, by our Fundamental Relation I with $m = 1$,

$$12\lambda_1 - \Delta_{\mathrm{bdy}} = \Delta_{\mathrm{WP}} - \Delta_{\mathrm{TZ}}.$$

Thus, the Basic Inequality is equivalent to

$$4g(g-1)\Delta_{\mathrm{TZ}} \geq \Delta_{\mathrm{WP}} - \Delta_{\mathrm{TZ}}.$$

That is, $(2g-1)^2\Delta_{\mathrm{TZ}} \geq \Delta_{\mathrm{WP}}$, or better,

$$\Delta_{\mathrm{WP}}^{\otimes 1^2} \leq \Delta_{\mathrm{TZ}}^{\otimes (2g-2+1)^2},$$

which certainly is the special case of our Fundamental Relation II when $N = 1$.

Proof of Fundamental relation II. We start with the following

Standard fact (e). *Let $f : S \to C$ be a fibration of curves from a regular projective surface S to a regular projective curve C. Assume that L is a line bundle on S which has relative degree zero, i.e., for any fiber F of f,* $\deg(L\big|_F) = 0$. *Then the self-intersection $(L, L) \leq 0$.*

Proof. By a result of Néron-Tate (see e.g. [La]), we know that there exists a unique vertical divisor $\sum_j a_j F_j$ such that
(i) the self-intersection of $L(\sum_j a_j F_j)$ is negative, possibly zero;
(ii) for any irreducible vertical prime divisor F', $\deg\Big(\big(L(\sum_j a_j F_j)\big)\Big|_{F'}\Big) = 0$.

Therefore, if we denote by (X, Y) the intersection number of the line bundles or divisors X and Y on S, then, by (ii),

$$\Big(\sum_k a_k F_k, \sum_k a_k F_k\Big) = \sum_{j,k} a_j a_k \Big(F_j, F_k\Big) = -\sum_k a_k \Big(L, \mathcal{O}_S(F_k)\Big).$$

But, by (i), we have

$$\begin{aligned} 0 &\geq (L, L) + 2\sum_k a_k\Big(L, \mathcal{O}_S(F_k)\Big) + \Big(\sum_k a_k F_k, \sum_k a_k F_k\Big) \\ &= (L, L) - \Big(\sum_k a_k F_k, \sum_k a_k F_k\Big) \\ &\geq (L, L), \end{aligned}$$

since the matrix $\Big((F_j, F_k)\Big)$ is negative definite. This completes the proof of the standard fact (e).

Now on the universal curve $\overline{\mathcal{C}_{g,N}}$, consider the difference $NK_\pi - (2g-2)(\mathbf{P}_1 + \cdots + \mathbf{P}_N)$. Obviously, the vertical degree of this difference is simply zero. Thus by using the standard fact (e), and putting it in the form of Deligne pairing, we see that

$$\langle NK_\pi - (2g-2)(\mathbf{P}_1 + \cdots + \mathbf{P}_N), NK_\pi - (2g-2)(\mathbf{P}_1 + \cdots + \mathbf{P}_N)\rangle \leq 0.$$

That is to say,

$$\begin{aligned}0 \geq\ & \langle N\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)-(2g-2+N)(\mathbf{P}_1+\cdots+\mathbf{P}_N),\\ & N\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)-(2g-2+N)(\mathbf{P}_1+\cdots+\mathbf{P}_N)\rangle\\ =\ & N^2\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\rangle\\ & -2N(2g-2+N)\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle\\ & +(2g-2+N)^2\langle \mathbf{P}_1+\cdots+\mathbf{P}_N, \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle.\end{aligned}$$

But by the lemma′ in the previous section, we conclude that

$$\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle = 0$$

and that

$$\langle \mathbf{P}_1+\cdots+\mathbf{P}_N, \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle = -\langle K, \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle.$$

So, by definition, we finally have

$$N^2\Delta_{\mathrm{WP}} \leq (2g-2+N)^2\Delta_{\mathrm{TZ}}.$$

This completes the proof of the Fundamental Relation II.

We suggest the reader to compare our Fundamental Relation II with the Basic Inequality of Harris-Morrison: While the basic inequality does give an exact relation between various line bundles over universal curves, our Fundamental Relation II exposes an intrinsic relation between Δ_{WP} and Δ_{TZ}. It is in this sense we prefer the Fundamental Relation II. Indeed, with our Fundamental Relation II, we may use Weil-Petersson metric to guide the study of Takhtajan-Zograf metric. This is in fact very fruitful. For examples,

(1) Recently, K. Obitsu [Ob] shows that Takhtajan-Zograf metric is incomplete, motivated by a result of Wolpert for Weil-Petersson metric.
(2) Motivated by our Fundamental Relation II and the fact that holomorphic sectional curvature of Weil-Petersson metric on the Teichmüller space $\mathcal{T}_{g,N}$ is bounded from above by $-\frac{1}{\pi(2g-2+N)}$, as proved in the appendix, we make the following

Conjecture. *The holomorphic sectional curvature of the Takhtajan-Zograf metric is bounded from above by* $-\frac{1}{\pi N}$.

2.4. Xiao and Cornalba-Harris type inequalities

In this section, we prove Xiao and Cornalba-Harris type inequalities over $\overline{\mathcal{M}_{g,N}}$, $N \geq 1$, which hence answer a question asked in [CH]. For simplicity, we here assume that $N \geq 3$. ($N = 1, 2$ cases pave a similar way.)

Clearly, then the Chow point for any regular projective curve C corresponding to the map given by the complete linear system $|K_C(P_1+\cdots+P_N)|$ is automatically stable. (See e.g., [Mu2, Thm 4.15].) In addition, $h^0(C, K_C(P_1+\cdots+P_N)) = g-1+N$. Hence, by applying a fundamental result of Cornalba-Harris, i.e., Theorem 1.1, or better, Proposition 2.9 of [CH], we know that the line bundle

$$\langle (g-1+N)\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big) - \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big), \\ (g-1+N)\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big) - \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)\rangle$$

is generically positive in dimension one. That is to say, in our notation introduced in Sect. 2.3,

$$\begin{aligned} 0 \le &\Big\langle (g-1+N)\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big) + \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big), \\ &(g-1+N)\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big) + \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)\Big\rangle \\ = &(g-1+N)^2\Big\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big\rangle \\ &-2(g-1+N)\Big\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)\Big\rangle \\ &+\Big\langle \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big), \pi^*\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)\Big\rangle \\ = &(g-1+N)^2\Big\langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big\rangle \\ &-2(g-1+N)(2g-2+N)\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big). \end{aligned}$$

Therefore,

$$(g-1+N)\Delta_{\mathrm{WP}} \ge 2(2g-2+N)\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big).$$

Next we compare $\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)$ with λ_1.

Lemma. *With the same notation as above, up to torsion,* $\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big) = \lambda_1$.

Proof. One actually can prove this relation without modulo torsions. But, as our final goal in this section is to show a certain generic positivity, so we pay no attention to torsion bundles. Indeed, by the Deligne-Riemann-Roch isomorphism recalled in Sect. 2.1, we see that

$$\begin{aligned} &\lambda\Big(K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)\Big)^{\otimes 2} \\ &= \lambda_1^{\otimes 2} \otimes \langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N) - K_\pi\rangle \\ &= \lambda_1^{\otimes 2} \otimes \langle K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N), \mathbf{P}_1+\cdots+\mathbf{P}_N\rangle \\ &= \lambda_1^{\otimes 2} \qquad \text{(by the lemma}' \text{ in Sect. 2.3).} \end{aligned}$$

This completes the proof of the lemma.

As a direct consequence, we have the following

Corollary. *With the same notation as above,* $(g-1+N)\Delta_{\mathrm{WP}} \geq 2(2g-2+N)\lambda_1$.

To go further, we first recall the Xiao and Cornalba-Harris inequality. As usual, set $\lambda := \lambda_1$.

Fundamental relation III(i). ([Xi] and [CH]) *Over the moduli space* $\overline{\mathcal{M}_g}$,

$$\left(8+\frac{4}{g}\right)\lambda \geq \Delta_{\mathrm{bdy}}.$$

Thus to get a generalization of the Xiao and Cornalba-Harris inequality from $N=0$ to general N, in the corollary above, we should remove Δ_{WP}. This can be done, since by the Fundamental Relation I, we have

$$12\lambda_1 = \Delta_{\mathrm{WP}} - \Delta_{\mathrm{TZ}} + \Delta_{\mathrm{bdy}}.$$

Therefore, from Corollary, we have

$$12\lambda_1 \geq -\Delta_{\mathrm{TZ}} + \Delta_{\mathrm{bdy}} + \frac{2(2g-2+N)}{g-1+N}\lambda_1.$$

This then implies the following

Fundamental relation III(ii). (Xiao and Cornalba-Harris Type Inequality) *Over* $\overline{\mathcal{M}_{g,N}},\ N \geq 1$, *we have*

$$\left(8+\frac{2N}{g-1+N}\right)\lambda + \Delta_{\mathrm{TZ}} \geq \Delta_{\mathrm{bdy}}.$$

Recently, Moriwaki brings to our attention a result of R. Hain [H], in which a result similar to the Fundamental Relation III(ii) in the case $N=1$ is established, by using intermediate Jacobian, Morita fundamental cycles, and Moriwaki's sharp result in [Mo] for line bundles over $\overline{\mathcal{M}_g}$.

We end our study on algebiarc aspect of Deligne pairings with the following comment. In this paper, we only study Deligne pairings associated to universal curves over moduli spaces. Similarly, we may use Deligne pairings to study the tower of moduli spaces. For details, see e.g., [WZ].

2.5. *Logarithmetic Mumford type isometries*

From now on, we study the corresponding metric aspect of Deligne pairings. Hence, we only work over $\mathcal{M}_{g,N}$, view it as the moduli space of punctured Riemann surfaces of genus g with N-punctures. Thus V-manifold language is assumed here.

Let $\pi : \mathcal{C}_{g,N} \to \mathcal{M}_{g,N}$ denote the universal curve over the (open) moduli space $\mathcal{M}_{g,N}$ corresponding punctured Riemann surfaces M^0 of genus g with N punctures. Denote by $\mathbf{P}_1, \ldots, \mathbf{P}_N$ the sections corresponding to N punctures. Naturally, we then obtain on $\mathcal{C}_{g,N}$ the following line bundles: the relative canonical line bundle K_π, $\mathcal{O}_{\mathcal{C}_{g,N}}(\mathbf{P}_1 + \cdots + \mathbf{P}_N)$, and $K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)$.

To metrize these line bundles, we use ω_{hyp}-admissible metrics introduced in Sect. 1.5. So assume that $2g - 2 + N > 0$. Then by uniformization theory, the fibers M^0 of π naturally admit standard hyperbolic metrics (induced from the Poincaré metric on the upper half plane). Moreover, by the decomposition introduced in Sect. 1.5, we get the fiberwise canonical ω_{hyp}-admissible metrics $\rho_{\text{hyp};K_M}$, $\rho_{\text{hyp};P_i}$ and $\rho_{\text{hyp};K_M(P_1+\cdots+P_N)}$ on K_M, $\mathcal{O}_M(P_i)$ and $K_M(P_1 + \cdots + P_N)$ respectively. Here M denotes the smooth compactification of M^0 and P_i denotes $\mathbf{P}_i \cap M$, i.e., the punctures of M^0, $i = 1, \ldots, N$. Hence, gluing them along with $\mathcal{M}_{g,N}$, we finally obtain natural ω_{hyp}-admissible metrics $\rho_{\text{hyp};K_\pi}$, $\rho_{\text{hyp};\mathbf{P}_i}$ and $\rho_{\text{hyp};K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}$ on line bundles K_π, $\mathcal{O}_{\mathcal{C}_{g,N}}(\mathbf{P}_i)$, and $K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)$ on $\mathcal{C}_{g,N}$, $i = 1, \ldots, N$. For simplicity, denote these resulting metrized line bundles by $\underline{K_\pi}_{\text{hyp}}$, $\underline{\mathbf{P}_1 + \cdots + \mathbf{P}_N}_{\text{hyp}}$, and $\underline{K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)}_{\text{hyp}}$ respectively.

Now for $m \geq 1$, set $\bar{L} = \underline{K_\pi^{\otimes m}\big((m-1)\mathbf{P}_1 + \cdots + (m-1)\mathbf{P}_N\big)}_{\text{hyp}}$, i.e., the tensor of the admissibly metrized line bundle $\underline{K_\pi}_{\text{hyp}}^{\otimes m}$ with $\Big(\underline{\mathbf{P}_1 + \cdots + \mathbf{P}_N}_{\text{hyp}}\Big)^{\otimes m-1}$. Moreover, assume that the base metric is given by the metrized line bundle $\underline{K_\pi}_{\text{hyp}}$. Clearly all these metrized line bundles are ω_{hyp}-admissible in the sense of Sect. 1.6, hence we may apply our Deligne-Riemann-Roch isometry for singular metrics proved in Sect. 1.6, which says that the determinant metric $h_{\det}(\rho_{\text{hyp};K_\pi}^{\otimes m} \otimes \big(\otimes_{i=1}^N \rho_{\text{hyp};\mathbf{P}_i}\big)^{\otimes m-1}; \rho_{\text{hyp};K_\pi})$ on λ_m satisfies the Deligne-Riemann-Roch isometry. That is to say, we have the canonical isometry

$$
\begin{aligned}
&\Big(\lambda_m, h_{\det}(\rho_{\text{hyp};K_\pi}^{\otimes m} \otimes \big(\otimes_{i=1}^N \rho_{\text{hyp};\mathbf{P}_i}\big)^{\otimes m-1}; \rho_{\text{hyp};K_\pi})\Big)^{\otimes 12} \\
&\simeq \Big\langle \underline{K_\pi^{\otimes m}\big((m-1)\mathbf{P}_1 + \cdots + (m-1)\mathbf{P}_N\big)}_{\text{hyp}}, \\
&\quad \underline{K_\pi^{\otimes m}\big((m-1)\mathbf{P}_1 + \cdots + (m-1)\mathbf{P}_N\big)}_{\text{hyp}} \otimes \underline{K_\pi}_{\text{hyp}}^{\otimes -1}\Big\rangle^{\otimes 6} \\
&\quad \otimes \Big\langle \underline{K_\pi}_{\text{hyp}}, \underline{K_\pi}_{\text{hyp}} \Big\rangle \cdot e^{a(g)}.
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\Big(\lambda_m, h_{\det}(\rho_{\mathrm{hyp};K_\pi}^{\otimes m} \otimes \big(\otimes_{i=1}^N \rho_{\mathrm{hyp};\mathbf{P}_i}\big)^{\otimes m-1}; \rho_{\mathrm{hyp};K_\pi})\Big)^{\otimes 12}\\
&\simeq \Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes 6m(m-1)}\\
&\quad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes -6(m-1)}\\
&\quad\otimes\Big\langle \underline{K_\pi}_{\mathrm{hyp}}, \underline{K_\pi}_{\mathrm{hyp}}\Big\rangle\cdot \mathrm{e}^{a(g)}\\
&= \Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes 6m(m-1)}\\
&\quad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes -6(m-1)}\\
&\quad\otimes\Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle\\
&\quad\otimes\Big\langle \underline{K_\pi}_{\mathrm{hyp}}, \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}\Big\rangle^{\otimes -1}\\
&\quad\otimes\Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}\Big\rangle^{\otimes -1}\cdot \mathrm{e}^{a(g)}\\
&= \Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes 6m^2-6m+1}\\
&\quad\otimes\Big\langle \underline{K_\pi}_{\mathrm{hyp}}, \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}\Big\rangle^{\otimes -1}\\
&\quad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes -6m+5}\cdot \mathrm{e}^{a(g)}.
\end{aligned}$$

In other words, we have the canonical isometry

$$\begin{aligned}
&\Big(\lambda_m, h_{\det}(\rho_{\mathrm{hyp};K_\pi}^{\otimes m} \otimes \big(\otimes_{i=1}^N \rho_{\mathrm{hyp};\mathbf{P}_i}\big)^{\otimes m-1}; \rho_{\mathrm{hyp};K_\pi})\Big)^{\otimes 12}\\
&\quad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes -6(m-1)}\\
&\simeq \Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes 6m^2-6m+1)}\\
&\quad\otimes\Big\langle \underline{K_\pi}_{\mathrm{hyp}}, \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}\Big\rangle^{\otimes -1}\\
&\quad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle\cdot \mathrm{e}^{a(g)}.
\end{aligned}$$

On the other hand, by the Lemma′ in Sect. 2.1,

$$\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle$$

is indeed the trivial line bundle equipped with the metric by using the Deligne pairing formalism developed in Part 1. Hence, we may indeed take the square root for such a metrized line bundle: for the line bundle, it is still the trivial one, while for the metric, we simply take the positive square root pointwise. Denote the resulting metrized line bundle simply by

$$\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes\frac{1}{2}}$$

by abuse of notation.

Basic definition IV. *With the same notation as above, on $\mathcal{M}_{g,N}$, define*
(i) the metrized logarithmic Mumford type line bundle $\underline{\lambda_m}_{\mathrm{hyp}}$ with respect to hyperbolic metrics by setting

$$\begin{aligned}&\underline{\lambda_m}_{\mathrm{hyp}}\\ &\quad:=\Big(\lambda_m, h_{\det}\big(\rho_{\mathrm{hyp};K_\pi}^{\otimes m}\otimes\big(\otimes_{i=1}^N\rho_{\mathrm{hyp};\mathbf{P}_i}\big)^{\otimes m-1};\rho_{\mathrm{hyp};K_\pi}\big)\Big)\\ &\qquad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes-\frac{m-1}{2}};\end{aligned}$$

(ii) the metrized Weil-Petersson line bundle $\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}$ with respect to hyperbolic metrics by setting

$$\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}:=\Big\langle \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle;$$

(iii) the metrized Takhtajan-Zograf line bundle $\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}$ with respect to hyperbolic metrics by setting

$$\begin{aligned}&\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}\\ &\quad:=\Big\langle \underline{K_\pi}_{\mathrm{hyp}}, \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}\Big\rangle\\ &\qquad\otimes\Big\langle \underline{\mathbf{P}_1+\cdots+\mathbf{P}_N}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle^{\otimes-1}.\end{aligned}$$

We here in particular reminder the reader that the base line bundle of $\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}$ (resp. $\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}$, $\underline{\lambda_m}_{\mathrm{hyp}}$) is indeed the restrictions of Takhtajan-Zograf line bundle Δ_{TZ} (resp. the Weil-Petersson line bundle Δ_{WP}, the logarithmic Mumford type line bundle λ_m) introduced in Sect. 2.1 to $\mathcal{M}_{g,N}$. Moreover, if $m=1$, we have

$$\begin{aligned}\underline{\lambda_1}_{\mathrm{hyp}}&=\Big(\lambda_m, h_{\det}\big(\rho_{\mathrm{hyp};K_\pi}^{\otimes m}\otimes\big(\otimes_{i=1}^N\rho_{\mathrm{hyp};\mathbf{P}_i}\big)^{\otimes m-1};\rho_{\mathrm{hyp};K_\pi}\big)\Big)\\ &=\Big(\lambda_m, h_{\det}\big(\rho_{\mathrm{hyp};K_\pi};\rho_{\mathrm{hyp};K_\pi}\big)\Big)\end{aligned}$$

is simply the line bundle λ_1 together with the determinant metric introduced in Part 1; while for $m \geq 2$, $\underline{\lambda_m}_{\text{hyp}}$ is not simply the line bundle λ_m together with the determinant metric introduced in Part 1 – we should multiple the determinant metric introduced in Part 1 by the metric induced from

$$\langle \underline{\mathbf{P}_1 + \cdots + \mathbf{P}_N}_{\text{hyp}}, \underline{K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)}_{\text{hyp}} \rangle^{\otimes - \frac{m-1}{2}}$$

which is indeed a smooth positive function on $\mathcal{M}_{g,N}$.

With this basic definition, all in all, what we have just proved may be restated as the following

Fundamental relation IV. *Over $\mathcal{M}_{g,N}$, there exist the canonical isometries*

$$\underline{\lambda_m}_{\text{hyp}}^{\otimes 12} \simeq \underline{\Delta_{\text{WP}}}_{\text{hyp}}^{\otimes 6m^2-6m+1} \otimes \underline{\Delta_{\text{TZ}}}_{\text{hyp}}^{\otimes -1} \cdot e^{a(g)}, \qquad m \geq 0.$$

2.6. Weil-Petersson and Takhtajan-Zograf metrics in terms of intersections

With the Basic Definition IV and the Fundamental Relation IV established in the previous section, next we show that indeed, the metrized line bundles $\underline{\Delta_{\text{WP}}}_{\text{hyp}}$ and $\underline{\Delta_{\text{TZ}}}_{\text{hyp}}$ are naturally associated to the so-called Weil-Petersson metric and the Takhtajan-Zograf metric over $\mathcal{M}_{g,N}$, definition of which we recall next.

For an N-punctured Riemann surface M^0 of genus g (with $2g + N \geq 3$), let Γ be a torsion free Fuchsian group uniformizing M^0, i.e., $M^0 \simeq \Gamma \backslash \mathcal{H}$, where $\mathcal{H}$ denotes the complex upper-half plane. Denote by $\Gamma_1, \ldots, \Gamma_N$ the set of non-conjugate parabolic subgroups in Γ, and for every $i = 1, \ldots, N$, fix an element $\sigma_i \in PSL(2, \mathbf{R})$ such that $\sigma_i^{-1} \Gamma_i \sigma_i = \Gamma_\infty$, where the group Γ_∞ is generated by the parabolic transformation $z \mapsto z + 1$. As usual, define the Eisenstein series $E_i(s, z)$ corresponding to the i-th cusp of the group Γ for $\text{Re}(s) > 1$ by

$$E_i(s, z) := \Sigma_{\gamma \in \Gamma_i \backslash \Gamma} \text{Im}(\sigma_i^{-1} \gamma z)^s, \quad i = 1, \ldots, N.$$

Denote the Teichmüller space of N-punctured Riemann surfaces of genus g by $T_{g,N}$. Then at the point $[M^0]$ corresponding to a punctured Riemann surface M^0, the tangent space $T_{[M^0]} T_{g,N}$ can be naturally identified with the space $\Omega^{-1,1}(M^0)$ of harmonic L^2-tensors on M^0 of type (-1,1), harmonic with respect the hyperbolic metric τ_{hyp} on $\Gamma \backslash \mathcal{H}$. By definition, the *Weil-Petersson metric* on $T_{g,N}$ is given by

$$\langle \phi, \psi \rangle_{\text{WP}} := \int_{\Gamma \backslash \mathcal{H}} \phi \bar{\psi} \cdot d\mu_{\text{hyp}},$$

where $\phi, \psi \in \Omega^{-1,1}(M^0)$ are considered as tangent vectors of $T_{g,N}$ at $[M^0]$ via the deformation theory, and $d\mu_{\text{hyp}} = 2\pi(2g - 2 + N)\, \omega_{\text{hyp}}$ is the Kähler form corresponding to the metric τ_{hyp}. It is well-known that the Weil-Petersson metric is Kähler. For later use, denote its corresponding Kähler form on $\mathcal{M}_{g,N}$ by ω_{WP}.

Proposition ([Wo1]). *Over* $\mathcal{M}_{g,N}$,

$$\int_\pi \left(c_1(\underline{K_\pi(\mathbf{P}_1+\cdots+\mathbf{P}_N)}_{\text{hyp}}\right)^2 = \frac{\omega_{\text{WP}}}{\pi^2}.$$

Here as usual, c_1 *denotes the first Chern form of a metrized line bundle.*

In a certain sense, the Weil-Petersson metric reflects the deformation of complex structures for $\pi : \mathcal{C}_{g,N} \to \mathcal{M}_{g,N}$. But for π, there exists another deformation, i.e., the deformation for punctures. For this, we have then the so-called Takhtajan-Zograf metric on $\mathcal{M}_{g,N}$.

By definition, for $i = 1, \dots, N$, define the i-th *Takhtajan-Zograf metric* $\langle\, ,\, \rangle_i$ on $T_{g,N}$ by setting

$$\langle \phi, \psi\rangle_i := \int_{\Gamma\backslash\mathcal{H}} \phi\bar{\psi}\cdot E_i(\cdot,2)\cdot d\mu_{\text{hyp}},\quad \phi,\ \psi\in\Omega^{-1,1}(M^0).$$

More globally, we define the (total) *Takhtajan-Zograf metric* on $\mathcal{T}_{g,N}$ by setting

$$\langle \phi, \psi\rangle_{\text{TZ}} := \sum_{i=1}^N \int_{\Gamma\backslash\mathcal{H}} \phi\bar{\psi}\cdot E_i(z,2)\cdot d\mu_{\text{hyp}},\quad \phi,\ \psi\in\Omega^{-1,1}(M^0).$$

In [TZ2], it is proved that $\langle\, ,\, \rangle_i$, $i = 1, \dots, N$, are Kähler metrics on $T_{g,N}$. Moreover, $\Sigma_{i=1}^N\langle\, ,\, \rangle_i$ is invariant under the action of the Teichmüller modular group. And hence, we get an induced new Kähler metric on $\mathcal{M}_{g,N}$. Often, we also call it the Takhtajan-Zograf metric on $\mathcal{M}_{g,N}$, and denote the corresponding Kähler form by ω_{TZ}. It is an open question whether such a metric is algebraic. (See e.g., [TZ2].) We next want to solve this problem. For this purpose, let us recall the fundamental work of Takhtajan-Zograf on a local family index theorem for punctured Riemann surfaces ([TZ1,2]).

Note that for *compact Riemann surfaces M*, a work of D'Hoker-Phong [D'HP] shows that the so-called regularized determinant $\det^*\Delta_m$ associated to $K_M^{\otimes m}$ with respect to hyperbolic metrics defined via the zeta function formalism of Ray-Singer, is equal, up to a constant multiplier depending only on g and m, to $Z_M'(1)$ for $m=1$, and $Z_M(m)$ for $m\geq 2$ respectively. Here $Z_M(s)$ denotes the Selberg zeta function associated to M. Motivated by this and the Quillen metric on determinant of cohomology, for *punctured Riemann surfaces,* Takhtajan and Zograf ([TZ1,2]) define $\det_{\text{TZ}}^*\Delta_m$ with respect to hyperbolic metrics by simply setting

$$\det{}_{\text{TZ}}^*\Delta_m := \begin{cases} Z_{M^0}'(1), & \text{if m=1;}\\ Z_{M^0}(m), & \text{if } m\geq 2.\end{cases}$$

Here $Z_{M^0}(s)$ denotes the Selberg zeta function of M^0. (See e.g., Sect. 1.5.) With this, for any $m \geq 1$, on $\lambda_m := \lambda(K_M^{\otimes m} \otimes \mathcal{O}_M(P_1 + \cdots + P_N)^{\otimes(m-1)})$, following Takhtajan-Zograf, the corresponding Quillen *norm* $h_{Q,m}$ is defined by setting

$$h_{Q;m} := h_{L^2,m} \cdot \det{}^*_{\mathrm{TZ}} \Delta_m^{-1},$$

where $h_{L^2,m}$ is defined as follows:
(1) If $m \geq 2$, then λ_m is simply the determinant of $\Gamma_m := \Gamma(M, K_M^{\otimes m} \otimes \mathcal{O}_M(P_1 + \cdots + P_N)^{\otimes(m-1)})$, i.e., the determinant of the space Γ_m of cusp forms of weight 2m. By definition, $h_{L^2,m} := \det h_{P,m}$, where $h_{P,m}$ denotes the standard Petersson norm on Γ_m. (See e.g., [Sh].)
(2) If $m = 1$, then $\lambda_1 = \det\Gamma(M, K_M) \otimes \Gamma(M, \mathcal{O}_M)^\vee = \det\Gamma(M, K_M) \otimes \mathbf{C}$. We define $h_{L^2,1}$ to be the determinant of the natural pairing on $\Gamma(M, K_M)$. (Note that our base manifold is of dimension one. Hence the canonical pairing may also be understood as the one introduce by using the singular volume form ω_{hyp}.)

Fundamental theorem. (Local Family Index Theorem [TZ1,2]) *With the same notation as above, for $m \geq 1$, as (1,1) forms on $T_{g,N}$ and hence on $\mathcal{M}_{g,N}$,*

$$12c_1(\lambda_m, h_{Q;m}) = (6m^2 - 6m + 1) \cdot \frac{\omega_{\mathrm{WP}}}{\pi^2} - \frac{4}{3}\omega_{\mathrm{TZ}}. \tag{2.1}$$

Now let us go back to the discussion on Takhtajan-Zograf metrics. Recall that from our Fundamental Relation IV proved in Sect. 2.5,

$$12\, c_1(\underline{\lambda_m}_{\mathrm{hyp}}) = (6m^2 - 6m + 1) \cdot c_1(\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}) - c_1(\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}). \tag{2.2}$$

Thus by comparing with the local family index theorem of Takhtajan-Zograf above, we may expect the follows:
(i) $c_1(\underline{\lambda_m}_{\mathrm{hyp}}) = c_1(\lambda_m, h_{Q;m})$;
(ii) $c_1(\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}) = \frac{\omega_{\mathrm{WP}}}{\pi^2}$; and
(iii) $c_1(\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}) = \frac{4}{3}\omega_{\mathrm{TZ}}$.

We claim that all these are correct. Roughly, the proof is given as follows. First, we use the result of Wolpert recalled above to show that (ii) holds. Then we compare the above two relations of (1,1) forms, i.e., Takhtajan-Zograf's fundamental result (2.1) and our fundamental relation (2.2) above, but only with $m = 1$, based on the fact that on λ_1, by our Basic Definition II(i), the metric on $\underline{\lambda_1}_{\mathrm{hyp}}$ used by us and the metric on $(\lambda_1, h_{Q,1})$ used by Takhtajan-Zograf are exactly the same. Hence, clearly, (iii) holds as well. Finally, by applying Takhtajan-Zograf's fundamental result (2.1) and our fundamental relation (2.2) above again but this time for all $m \geq 2$, we conclude that (i) holds for $m \geq 2$ as well.

That is to say, we have the following

Theorem. *With the same notation as above,*
(i) (**Fundamental relation V**) $\quad c_1(\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}) = \frac{\omega_{\mathrm{WP}}}{\pi^2};$
(ii) (**Fundamental relation VI**) (Together with Fujiki) $\quad c_1(\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}) = \frac{4}{3}\omega_{\mathrm{TZ}};$
(iii) (**Determinant metrics in terms of Selberg zeta functions**) *For a fixed $m \geq 1$, up to a constant factor depending only on (g, N), there exists an isometry*

$$(\lambda_m, h_{Q,m}) \simeq \underline{\lambda_m}_{\mathrm{hyp}}.$$

Proof. By the definition of metrized Deligne pairing, we have

$$c_1\Big(\Big\langle \underline{K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)}_{\mathrm{hyp}}, \underline{K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)}_{\mathrm{hyp}}\Big\rangle\Big) = \int_\pi c_1\Big(\underline{K_\pi(\mathbf{P}_1 + \cdots + \mathbf{P}_N)}_{\mathrm{hyp}}\Big)^2.$$

On the other hand, by the result of Wolpert recalled above as the Proposition at the beginning of this section, this latest (1,1) form is simply $\frac{\omega_{\mathrm{WP}}}{\pi^2}$. Hence we get the Fundamental Relation V.

Here we should reminder the reader that essentially the Fundamental Relation V is due to Wolpert. Our contribution, if any, is that our Fundamental Relation V for the first time points out clearly that indeed the Weil-Petersson metric is in the nature of intersection, rather than in the nature of cohomology. (See e.g., [Wo2] and the fundamental work done by Fujiki and Schumacher [FS].)

Now note that by Basic Definition II(i), we have the isometry

$$\underline{\lambda_1}_{\mathrm{hyp}} \simeq (\lambda_1, h_{Q;1}).$$

Thus from Takhtajan-Zograf's fundamental result (2.1) and our fundamental relation (2.2), we see that

$$c_1(\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}) - c_1(\underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}) = \frac{\omega_{\mathrm{WP}}}{\pi^2} - \frac{4}{3}\omega_{\mathrm{TZ}}.$$

Therefore,

$$c_1(\underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}) = \frac{4}{3}\omega_{\mathrm{TZ}}$$

by the Fundamental Relation V. This then gives the Fundamental Relation VI.

Now, clearly, with the help of our Fundamental Relations V and VI and Takhtajan-Zograf's fundamental result (2.1), we see that up to some universal constant depending only on (g, N), there exists an isometry

$$(\lambda_m, h_{Q,m}) \simeq \underline{\lambda_m}_{\mathrm{hyp}},$$

which in particular gives an interpretation of our new determinant metric in terms of the Selberg zeta function. This completes the proof of the Theorem.

As a direct consequence, we have the following

Corollary. *The Takhtajan-Zograf metric on moduli space of punctured Riemann surfaces is algebraic.*

Fundamental relation IV′. *With the same notation as above, on $\mathcal{M}_{g,N}$, for a fixed $m \geq 1$, up to some universal constant depending only on g, N, such that there exists canonical isometry*

$$(\lambda_m, h_{Q,m})^{\otimes 12} \simeq \underline{\Delta_{\mathrm{WP}}}_{\mathrm{hyp}}^{\otimes 6m^2-6m+1} \otimes \underline{\Delta_{\mathrm{TZ}}}_{\mathrm{hyp}}^{\otimes -1}.$$

We end this paper by noticing that the Weil-Petersson and Takhtajan-Zograf line bundles are well-defined even over $\overline{\mathcal{M}_{g,N}}$. Thus we naturally expect to get factorizations for Weil-Petersson and Takhtajan-Zograf metrics and hence degenerations of Selberg zeta functions by using our Fundamental Relations. For details, please see [We3].

Appendix: Holomorphic sectional curvature of Weil-Petersson metric on $\mathcal{M}_{g,N}$

Many of Wolpert's results on Weil-Petersson metrics for compact Riemann surfaces may be generalized to these for punctured Riemann surfaces. As an example, we in this appendix, prove the following

Proposition. *The holomorphic sectional curvature of Petersson-Weil metric on the Teichmüller space $\mathcal{T}_{g,N}$ is bounded from above by $-\frac{1}{\pi(2g-2+N)}$.*

Proof. We will mainly follow Wolpert, and hence without any further explanation use parallel notation as in [Wo1]. In particular, by a verbatim change of Wolpert's computations of Riemann curvature tensor of Weil-Petersson metrics, we see that Riemann tensor for Weil-Petersson metrics on the Teichmüller space of punctured Riemann surfaces has the same form as in compact case. That is to say, we have the following

Lemma. ([Wo1]) *With the same notation as in [Wo1], the Riemannian tensor of Weil-Petersson norm is given by*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) = \frac{\partial^2}{\partial t_\gamma \partial \bar{t}_\delta} g_{\alpha\bar{\beta}}(0) = \langle \Delta(\mu_\alpha \bar{\mu}_\beta), \bar{\mu}_\gamma \mu_\delta \rangle + \langle \Delta(\mu_\alpha \bar{\mu}_\delta), \bar{\mu}_\gamma \mu_\beta \rangle.$$

To estimate the holomorphic sectional curvature, we choose $\mu_\alpha \in \mathcal{B}(\Gamma)$ such that $\langle \mu_\alpha, \mu_\alpha \rangle = 1$. Then, by the lemma, the holomorphic sectional curvature is given by

$$-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -2\langle \Delta|\mu_\alpha|^2, |\mu_\alpha|^2 \rangle.$$

Now as $\mu_\alpha \in \mathcal{B}(\Gamma)$, $|\mu_\alpha|^2 \in L^2(\Gamma\backslash\mathcal{H}, d\mu_{\mathrm{hup}})$, i.e., it is L^2 with respect to the natural (singular) hyperbolic metric on the punctured Riemann surface $\Gamma\backslash\mathcal{H}$. Therefore, by spectral decomposition,

$$|\mu_\alpha|^2 = \Sigma_{j\geq 0} c_{\alpha,j}\psi_j + \Sigma_{a=1}^N \int_0^\infty \left(\frac{1}{2\pi} \int |\mu_\alpha|^2 E_a \left(\frac{1}{2} - \sqrt{-1}t \right) dA \right) \times E_a \left(\frac{1}{2} + \sqrt{-1}t \right) dt.$$

Here E_a' are Eisenstein series, ψ_j are orthonormal discrete spectrum eigen-functions of D_0 with the eigen-values λ_j and $c_{\alpha,j} =< |\mu_\alpha|^2, \psi_j >$. (See e.g. [Hej, Ch. 6, Sect. 9].)

In particular, $\lambda_0 = 0$ and $\psi_0 = \frac{1}{\sqrt{2\pi(2g-2+N)}}$, so that $< \mu_\alpha, \mu_\alpha >= 1$ implies

$$c_{\alpha,0} =< |\mu_\alpha|^2, \psi_0 >= \frac{1}{\sqrt{2\pi(2g-2+N)}}.$$

Moreover,

$$(D_0 - 2)^{-1} E_a \left(\frac{1}{2} + \sqrt{-1}t \right) = -\frac{4}{9+4t^2} E_a \left(\frac{1}{2} + \sqrt{-1}t \right),$$

so

$$\Delta|\mu_\alpha|^2 = \Sigma_{j\geq 0} \frac{-2}{\lambda_j - 2} \cdot c_{\alpha,j} \cdot \psi_j + \Sigma_{a=1}^N \int_0^\infty \left(\frac{1}{2\pi} \int |\mu_\alpha|^2 \times E_a \left(\frac{1}{2} - \sqrt{-1}t \right) dA \right) \frac{8}{9+4t^2} E_a \left(\frac{1}{2} + \sqrt{-1}t \right) dt.$$

Using Parseval formula, we have

$$\begin{aligned} -R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = {} & \Sigma_{j\geq 0} \frac{4}{\lambda_j - 2} \cdot c_{\alpha,j}^2 \cdot \left\langle \psi_j, \psi_j \right\rangle - 2\Sigma_{a=1}^N \\ & \Big\langle \int_0^\infty \Big(\frac{1}{2\pi} \int |\mu_\alpha|^2 E_a \Big(\frac{1}{2} - \sqrt{-1}t \Big) dA \Big) E_a \left(\frac{1}{2} + \sqrt{-1}t \right) dt, \\ & \times \int_0^\infty \Big(\frac{1}{2\pi} \int |\mu_\alpha|^2 E_a \left(\frac{1}{2} - \sqrt{-1}t \right) dA \Big) \frac{4}{9+4t^2} \\ & \times E_a \left(\frac{1}{2} + \sqrt{-1}t \right) dt \Big\rangle \end{aligned}$$

$$= \Sigma_{j\geq 0}\frac{4}{\lambda_j-2}\cdot c_{\alpha,j}^2\cdot\Big\langle\psi_j,\psi_j\Big\rangle - 2\Sigma_{a=1}^N\Big\langle\int_0^\infty\Big|\frac{1}{2\pi}\int|\mu_\alpha|^2$$
$$\times E_a\Big(\frac{1}{2}-\sqrt{-1}t\Big)dA\Big|^2\frac{4}{9+4t^2}dt\Big\rangle$$
(by [Hej., Ch. 6, (9.33)])

$$\leq \Sigma_{j\geq 0}\frac{4}{\lambda_j-2}\cdot c_{\alpha,j}^2\cdot\Big\langle\psi_j,\psi_j\Big\rangle$$
(by $\lambda_j\leq 0$ and ψ_j are not constants)
$$< -2c_{\alpha,0}^2 = -\frac{1}{\pi(2g-2+N)},$$

which proves the proposition.

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