Remark on Langlands' Combinatorial Lemma

Lin WENG

The combinatoric aspect of Arthur's trace formula depends heavily on Langlands' combinatorial lemma. In this note, we supply an easy proof of a special form of it.

Let G be an algebraic reductive group defined over a number field F. Let P be a parabolic subgroup of G over F. Denote by $\mathcal{U}_P$, resp. L_P, the unipotent radical, resp. the Levi quotient of P. Let T_P, resp. T'_P, be a maximal split torus in the center of P over F, resp. a quotient maximal split torus of P. Then the composition of the natural morphisms

$$T_P \hookrightarrow Z_P \hookrightarrow P \twoheadrightarrow L_P \twoheadrightarrow L_P^{\mathrm{ab}} \twoheadrightarrow T'_P \tag{1}$$

is an isogeny, i.e. a morphism with finite kernel and finite cokernels. Set $X^*(P) = \mathrm{Hom}(T_P, \mathbb{G}_m)$ and $\mathfrak{a}_P^* := X^*(P) \otimes \mathbb{R}$. Then for any pair $P \subseteq Q$ of parabolic subgroups of G on F, there exist naturally an inclusion $\mathfrak{a}_Q^* \hookrightarrow \mathfrak{a}_P^*$ and a projection $\mathfrak{a}_P^* \twoheadrightarrow \mathfrak{a}_Q^*$, and hence a natural decomposition

$$\mathfrak{a}_P^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_P^Q. \tag{2}$$

Denote the projections from $\mathfrak{a}_P^*$ to $\mathfrak{a}_Q^*$ and $\mathfrak{a}_P^{Q*}$ by $[\cdot]_Q$ and $[\cdot]_P^Q$ respectively.

Denote by $\Phi_P \subset \mathfrak{a}_P^{G*}$, resp. Φ_P^+ the set of non-trivial characters of T_P which occur in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, resp. $\mathfrak{u} = \mathrm{Lie}(\mathcal{U}_P)$ of $\mathcal{U}_P$. If $P = P_0$ is minimal, then $\Phi_0 := \Phi_{P_0}$ forms a root system with $\Phi_0^+ := \Phi_{P_0}^+$ the set of positive roots. Denote the associated set of simple roots, resp. fundamental weights by $\Delta_0 \subset \mathfrak{a}_0^{G*}$, resp. $\widehat{\Delta}_0$. Then it is well known that Δ_0 and $\widehat{\Delta}_0$ are bases of $\mathfrak{a}_0^{G*}$, and that $\widehat{\Delta}$ is the dual basis of $\mathfrak{a}_0^G := X_*(T_{P_0}) \otimes \mathbb{R}$ to the set of co-roots $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta_0\}$. Here $T_0 := T_{P_0}$ and $X_*(T_P) := \mathrm{Hom}(\mathbb{G}_m, T_P)$ is the set of 1-parameter subgroups in T_P. Accordingly, we may write $\widehat{\Delta}_0 = \{\varpi_\alpha : \alpha \in \Delta\}$ which is known to be contained in $\sum_{\alpha \in \Delta_0} \mathbb{Q}_{\geq 0}\alpha$.

In general (Φ_P, Φ_P^+) does not form a root system. Nevertheless, the constructions above may be similarly carried over here. In particular, we have the subsets

$\Delta_P \subset \Phi_P^+, \Delta_P^\vee$ and $\widehat{\Delta}_P$. Indeed,

$$\Delta_P = \{\alpha|_{\mathfrak{a}_P}\colon a \in \Delta_0\}\smallsetminus\{0\} \quad \text{and} \quad \Delta_P^\vee := \{\alpha^\vee := [\beta]_P^Q\colon \forall\alpha = [\beta]_P^Q \in \Delta_P, \exists\beta \in \Delta_0\}. \tag{3}$$

And they form bases of $\mathfrak{a}_P^{G*}$ and $\mathfrak{a}_P^G$ respectively. Denote the dual basis of $\Delta_P^\vee$ in $\mathfrak{a}_P^{G*}$ by $\widehat{\Delta}_P$. Now, y an abuse of notation, we write $\alpha^\vee$ for $[\alpha^\vee]_P^G$ for $\alpha \in \Delta$. Then

$$\Delta_P^\vee = \{\alpha^\vee : \alpha \in \Delta_P\} \qquad \text{and} \qquad \widehat{\Delta}_P = \{\varpi_\alpha^G : \alpha \in \Delta_P\}. \tag{4}$$

Of course, we have

$$\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta} \qquad \forall\alpha, \beta \in \Delta_P \tag{5}$$

In addition, standard parabolic subgroups of G are parametrized by subsets of Δ_0: there is an order reversing bijection $P \longrightarrow \Delta_0^P$ between standard parabolic subgroups P of G and subsets Δ_0^P of Δ_0 such that

$$\mathfrak{a}_P = \big\{H \in \mathfrak{a}_0 : \langle \alpha, H \rangle = 0 \quad \forall\alpha \in \Delta_0^P\big\}. \tag{6}$$

Obviously, for any P, Δ_0^P forms a basis of $\mathfrak{a}_0^P$, and there is a bijection between $\Delta_0 \smallsetminus \Delta_0^P$ and Δ_P induced by taking restrictions. More generally, for any wto standard parabolic subgroups $P \subseteq Q$, we set Δ_P^Q to be the set of linear forms on the subspace $\mathfrak{a}_P^Q$ of $\mathfrak{a}_P$ obtained by restricting element in $\Delta_0^Q \smallsetminus \Delta_0^P$ and set $\widehat{\Delta}_P^Q$ to be the set of linear forms on $\mathfrak{a}_P^Q$ obtained by restricting element in $\widehat{\Delta}_P \smallsetminus \widehat{\Delta}_Q$. Then both Δ_P^Q and $\widehat{\Delta}_P^Q$ form bases of $\mathfrak{a}_P^{Q*}$ denote their dual bases in $\mathfrak{a}_P^Q$ by $\widehat{\Delta}_P^{Q\vee}$ and $\Delta_P^{Q\vee}$. We have

$$\Delta_P^{Q\vee} = \{\alpha^\vee : \alpha \in \Delta_P^Q\}, \quad \widehat{\Delta}_P^Q = \{\varpi_\alpha : \alpha \in \Delta_P^Q\} \quad \text{and} \quad \widehat{\Delta}_P^{Q\vee} = \{\varpi_\alpha^\vee : \alpha \in \Delta_P^Q\} \tag{7}$$

Note that for any parabolic subgroup P of G its Levi quotient can be lifted to a subgroup of P, denoted also by L_P, by an abuse of notation, and this lift is unique if we assume that its is θ-stable. Here θ is a Borel-Serre Cartan involution whose existence can be guaranteed by a choice of maximal compact subgroup K_∞ of $G(F_\infty)$. It is well known that L_Q is a reductive group of Q and $P \cap L_Q$ is a standard parabolic subgroup of L_Q (with respect to the fixed minimal parabolic subgroup $P_0 \cap L_Q$) and that

$$\mathfrak{a}_{P\cap L_Q} = \mathfrak{a}_P, \quad \mathfrak{a}_{P\cap L_Q}^{L_Q} = \mathfrak{a}_P^Q, \quad \Delta_{P\cap L_Q} = \Delta_P^Q \quad \text{and} \quad \widehat{\Delta}_{P\mathcal{L}_\mathcal{Q}} = \widehat{\Delta}_P^Q \tag{8}$$

Definition 1. *For a pair parabolic subgroups $P \subseteq Q$ of a reductive group G,, define the positive chamber $\mathfrak{a}_P^{Q+}$ and the positive cone ${}^+\mathfrak{a}_P^Q$ to be the subsets of $\mathfrak{a}_P^Q$ by*

$$\begin{aligned} \mathfrak{a}_P^{Q+} :=& \{H \in \mathfrak{a}_P^Q : \langle \alpha, H \rangle > 0 \ \forall\alpha \in \Delta_P^Q\}, \\ {}^+\mathfrak{a}_P^Q :=& \{H \in \mathfrak{a}_P^Q : \langle \varpi_\alpha^G, H \rangle > 0 \ \forall\alpha \in \Delta_P^Q\} \end{aligned}$$

respectively. Denote their characteristic functions (as subsets of $\mathfrak{a}_P^Q$ or $\mathfrak{a}_P^G$ or $\mathfrak{a}_0^G$ or even $\mathfrak{a}_0$) by τ_P^Q and $\widehat{\tau}_P^Q$ respectively.

Lemma 1 ((A Special Form of) Langlands' Combinatorial Lemma [1, 2]). *For each pair of parabolic subgroups $P \subseteq Q$ of a reductive group G, we have*

$$\sum_{R:P\subseteq R\subseteq Q} (-1)^{d(R)-d(Q)} \tau_P^R \widehat{\tau}_R^Q = \delta_{PQ} \tag{4.a}$$

$$\sum_{R:P\subseteq R\subseteq Q} (-1)^{d(P)-d(R)} \widehat{\tau}_P^R \tau_R^Q = \delta_{PQ} \tag{4.b}$$

Proof. We prove the two relations simultaneously with an induction on $d(P) - d(Q)$.

We start with the case $d(P)-d(Q) = 0$. Obviously, since $d(P) = d(Q)$, $P = Q$. Hence in this case, it suffices to notice that

$$\tau_P^P \equiv 1 \qquad \text{and} \qquad \widehat{\tau}_P^P \equiv 1. \tag{9}$$

Now assume that two relations holds for all pairs $P \subseteq Q$ of parabolic subgroups satisfying $d(P) - d(Q) \leq r$. We treat the case when $d(P) - d(Q) = r + 1 > 0$ (in particular, $P \neq Q$).

$$\begin{aligned}
\sum_{R:P\subseteq R\subseteq Q} (-1)^{d(P)-d(R)} \widehat{\tau}_P^R \tau_R^Q &= \sum_{\substack{R,R':\\ P\subseteq R\subseteq R'\subseteq Q\\ d(R)-d(R')\leq r}} (-1)^{d(P)-d(R)} \widehat{\tau}_P^R \delta_{RR'} \tau_{R'}^Q \\
&= \sum_{\substack{R,R',S:\\ P\subseteq R\subseteq S\subseteq R'\subseteq Q\\ d(R)-d(R')\leq r}} (-1)^{d(P)-d(R)} \widehat{\tau}_P^R \Big((-1)^{d(S)-d(R')} \tau_R^S \widehat{\tau}_S^{R'} \Big) \tau_{R'}^Q \\
&= \sum_{\substack{S:\\ P\subseteq S\subseteq Q}} \sum_{R:P\subseteq R\subseteq S} (-1)^{d(P)-d(R)} \widehat{\tau}_P^R \tau_R^S \sum_{R':S\subseteq R'\subseteq Q} (-1)^{d(S)-d(R')} \widehat{\tau}_S^{R'} \tau_{R'}^Q \\
&= \sum_{\substack{S:\\ P\subseteq S\subseteq Q}} \delta_{PS} \delta_{SQ} = 0
\end{aligned}$$

Here in the 2ed equality, we have used the induction hypothesis on (4.a), and in the 4th equality, we have used first the fact that $d(R) - d(R') \leq r$ which implies that both $d(P) - d(S)$ and $d(S) - d(Q) \leq r$ and then the induction hypothesis on (4.b). Similarly, if instead, we apply the induction hypothesis on (4.b) in the 2ed equality first and apply the induction hypothesis on (4.a) in the 4th equalitysecondly, then we have

$$\begin{aligned}\sum_{R:P\subseteq R\subseteq Q}(-1)^{d(R)-d(Q)}\tau_P^R\widehat{\tau}_R^Q &= \sum_{\substack{R,R':\\ P\subseteq R\subseteq R'\subseteq Q\\ d(R)-d(R')\leq r}}(-1)^{d(R')-d(Q)}\tau_P^R\delta_{RR'}\widehat{\tau}_{R'}^Q\\
&= \sum_{\substack{R,R',S:\\ P\subseteq R\subseteq S\subseteq R'\subseteq Q\\ d(R)-d(R')\leq r}}(-1)^{d(R')-d(Q)}\tau_P^R\Big((-1)^{d(R)-d(S)}\widehat{\tau}_R^S\tau_S^{R'}\Big)\widehat{\tau}_{R'}^Q\\
&= \sum_{\substack{S:\\ P\subseteq S\subseteq Q}}\sum_{R:P\subseteq R\subseteq S}(-1)^{d(R)-d(S)}\tau_P^R\widehat{\tau}_R^S\sum_{R':S\subseteq R'\subseteq Q}(-1)^{d(R')-d(Q)}\tau_S^{R'}\widehat{\tau}_{R'}^Q\\
&= \sum_{\substack{S:\\ P\subseteq S\subseteq Q}}\delta_{PS}\delta_{SQ}=0\end{aligned}$$

This completes the proof. □

This proof is motivated by the discussions in § III.2 of [2].

References

[1] J. Arthur, A trace formula for reductive groups, *Duke Math. J*, 45 (1978) 911-952

[2] H. Jacquet, E. Lapid, J. Rogawski, Periods of Automorphic Forms, *Journal of AMS* 12.1 (1999) 137-240

Lin WENG
Graduate School of Mathematics,
Kyushu University,
Fukuoka 819-0395,
JAPAN
E-Mail: weng@math.kyushu-u.ac.jp