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Green's Functions for Quasi-Hyperbolic Metrics on Degenerating Riemann Surfaces with a Separating Node

WING-KEUNG TO
Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260
(E-mail: mattowk@leonis.nus.edu.sg)

LIN WENG
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560, Japan
(E-mail: weng@math.sci.osaka-u.ac.jp)

Abstract. In this article, we consider a family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a Riemann surface with a separating node and many non-separating nodes. We obtain the asymptotic behavior of Green's functions associated to a continuous family of quasi-hyperbolic metrics on such degenerating Riemann surfaces.

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1. Introduction

Let $\{M_t\}$ be a degenerating family of compact Riemann surfaces of genus $q \geq 2$ obtained by shrinking closed loops to form a noded Riemann surface M_0. There are essentially two cases, depending on whether the nodes separate M_0. Throughout this article, we assume that each fiber is stable, so that $q \geq 2$, and each connected component of $M_0^0 := M_0 \backslash \{\text{nodes}\}$ admits a complete hyperbolic metric.

The behavior of Green's functions associated to various canonical metrics on degenerating Riemann surfaces have been widely studied (cf. [4–7, 10]). In particular, Ji [5] obtained the degenerative behaviors of Green's functions for hyperbolic metrics in both cases of separating and non-separating nodes. Ji's approach involved a detailed study of the resolvent kernel of the hyperbolic Laplacians and used Hejhal's results on regular b-group theory [4]. It does not seem to generalize directly to the variable curvature case.

Using a different and more geometric approach, To and Weng [9] recently obtained the degenerative behavior of Green's functions for a 'continuous family of quasi-hyperbolic metrics' on $\{M_t\}$ in the case of a non-separating node (cf. [9, theorem 2]), which, generalized Ji's result in this case [5]. The main idea in [9] is to construct a family of functions (with singularities) using Green's function on M_0^0 to approximate Green's functions on $\{M_t\}$, and then show that the error term goes to 0 as $t \to 0$.

In this article, we are able to obtain the degenerative behavior of Green's functions for a continuous family of quasi-hyperbolic metrics on $\{M_t\}$ with one separating node and finitely many non-separating nodes (cf. Theorem 2.4.1 in Section 2), which generalizes Ji's result in this remaining case [5]. As noted in [9], the main difficulty in adapting the geometric approach in [9] is that in the separating node case, the first non-zero eigenvalue $\lambda_{1,t}$ of the Laplacian on M_t tends to zero as $t \to 0$. We overcome this difficulty by constructing good approximations of the eigenfunctions of the Laplacians on $\{M_t\}$ corresponding to λ_1 using the corresponding eigenfunction on M_0. Then, together with Green's function on M_0, we are able to construct approximations of Green's functions on $\{M_t\}$. To show that the error term goes to 0 as $t \to 0$, we have to make essential use of the fact that there is a positive lower bound for the second non-zero eigenvalues $\lambda_{2,t}$ of the Laplacians as $t \to 0$.

This paper is organized as follows. In Section 2 we introduce some definitions and state our main results. In Section 3 we give the construction of the approximations of the eigenfunctions of the Laplacians on $\{M_t\}$ corresponding to λ_1. The proof of Theorem 2.4.1 is given in Section 4, and finally we deduce Corollaries 2.4.2 and 2.4.3 in Section 5.

2. Notation and Statement of Results

(2.1) Throughout this article, we consider the degeneration of compact Riemann surfaces of fixed genus $q \geq 2$ into a stable singular Riemann surface M with one separating node and m non-separating nodes. Here, we always assume that $0 \leq m < \infty$.

First, we recall the plumbing construction of a degenerating family of Riemann surfaces starting from M as follows (cf. [2, 12]). Since M has exactly one separating node, the normalization $\tilde{M}$ of M is a disjoint union of two smooth compact Riemann surfaces M_1 and M_2 of genus q_1 and q_2 respectively. Let m_1 and m_2 be the numbers of non-separating nodes in the connected component of $M\backslash\{\text{separating node}\}$ corresponding to M_1 and M_2, respectively (so that $m_1+m_2 = m$). Let $p_1, p_2, \ldots, p_{m+1}$ be all the nodes of M. Rearranging if necessary, we will always assume that p_1 is the separating node, and $p_2, p_3, \ldots, p_{m_1+1}$ (respectively $p_{m_1+2}, p_{m_1+3}, \ldots, p_{m+1}$) are the non-separating nodes in the connected component of $M\backslash\{p_1\}$ corresponding to M_1 (respectively M_2). For $1 \leq i \leq m+1$, the node p_i corresponds to two points $p_{i,1}, p_{i,2}$ in $\tilde{M}$. Moreover, $p_{i,k}$, $1 \leq k \leq 2$, lie in different components or the same component of $\tilde{M}$ depending on whether p_i is a separating or non-separating node. Thus, without loss of generality, we will assume that $\{p_{1,1}\} \cup \{p_{i,k} \mid 2 \le i \le m_1+1, k=1,2\} \subset M_1$ and $\{p_{1,2}\} \cup \{p_{i,k} \mid m_1+2 \le i \le m+1, k=1,2\} \subset M_2$. Let $M^0 := M\backslash\{p_1,\dots,p_{m+1}\}$. Then M^0 is a disjoint union of two punctured Riemann surfaces M_1^0 and M_2^0 with identifications $M_1^0 \simeq M_1\backslash(\{p_{1,1}\} \cup \{p_{i,k} \mid 2 \le i \le m_1+1, k=1,2\})$ and $M_2^0 \simeq M_2\backslash(\{p_{1,2}\} \cup \{p_{i,k} \mid m_1+2 \le i \le m+1, k=1,2\})$. Denote the unit disc in $\mathbf{C}$ by Δ. For each $1 \le i \le m+1$ and $k=1,2$, fix a coordinate function $z_{i,k} : U_{i,k} \to \Delta$ such that $z_{i,k}(p_{i,k}) = 0$, where $U_{i,k}$ is an open neighborhood of $p_{i,k}$ in M_k. Also for each $1 \le i \le m+1$ and $t_i \in \Delta$, let $S_{t_i} := \{(z_{i,1}, z_{i,2}) \in \Delta^2 \mid z_{i,1}z_{i,2} = t_i\}$. Then for each $t = (t_1, t_2, \dots, t_{m+1}) \in \Delta^{m+1}$, we remove the $2m+2$ discs $|z_{i,k}| < |t_i|$, $1 \le i \le m+1$, $k=1,2$, from $\tilde{M}$, and glue the remaining parts of $\tilde{M}$ with $S_{t_1}, S_{t_2}, \dots, S_{t_{m+1}}$ via the identifications $z_{i,1} \sim (z_{i,1}, t_i/z_{i,1})$ and $z_{i,2} \sim (t_i/z_{i,2}, z_{i,2})$, $1 \le i \le m+1$. The resulting surfaces $\{M_t\}_{t\in\Delta^{m+1}}$ form an analytic family $\pi : \mathcal{M} \to \Delta^{m+1}$, where π denotes the holomorphic projection map. Denote the punctured unit disc in $\mathbf{C}$ by $\Delta^* := \Delta\backslash\{0\}$. It is easy to see that each fiber M_t, $t \in (\Delta^*)^{m+1}$, is a smooth compact Riemann surface of genus $q = q_1 + q_2 + m_1 + m_2$. Moreover, M_t is a noded Riemann surface for t along the coordinate hyperplanes of Δ^{m+1}, and at the origin, we have $M_0 = M$. Here and thereafter, by simplification of notation, we simply denote the origin $(0,0,\dots,0) \in \mathbf{C}^{m+1}$ by 0 when no confusion arises. Also, by simplification of notation, we simply say that such $\{M_t\}$ is a family of compact Riemann surfaces of genus q degenerating to M (as $t \in (\Delta^*)^{m+1} \to 0$). Our main concern in this paper is the study of behaviors of analytic objects on the smooth fibers M_t, $t \in (\Delta^*)^{m+1}$, as $t \to 0$.

It is easy to see that for $1 \le i \le m+1$, there is a coordinate neighborhood Δ^{m+2} of p_i in $\mathcal{M}$ centered at p_i and such that for $t = (t_1, t_2, \dots, t_{m+1}) \in \Delta^{m+1}$, $M_t \cap \Delta^{m+2} = \{(t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \in \Delta^{m+2} \mid z_{i,1}z_{i,2} = t_i\}$. Moreover, $\pi\ |_{\Delta^{m+2}}$ is given by $(t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \to (t_1, \dots, t_{i-1}, z_{i,1}z_{i,2}, t_{i+1}, \dots, t_{m+1})$. Also, we remark that the restriction of $\ker(d\pi)$ to $\mathcal{M}\backslash\{\text{nodes}\}$ forms a holomorphic line bundle L over $\mathcal{M}\backslash\{\text{nodes}\}$, which will be called the vertical line bundle, such that for all $t \in \Delta^{m+1}$, $L|_{M_t^0} = TM_t^0$, where M_t^0 denotes the smooth part of M_t.

(2.2) To facilitate ensuing discussion and for convenience of the reader, we recall the following definition in [9]:

DEFINITION 2.2.1. A Hermitian metric ds^2 on a punctured Riemann surface N is said to be *of hyperbolic growth near the punctures* if at each puncture p, there exists a punctured coordinate disc $\Delta^* := \{z \in \mathbf{C} : 0 < |z| < 1\}$ centered at p such that for some constant $C_1 > 0$,

$$\text{(i)}\quad ds^2 \le \frac{C_1|dz|^2}{|z|^2(\log|z|)^2} \quad \text{on } \Delta^*, \tag{2.2.1}$$

and there exists a local potential function ϕ on Δ^* satisfying $ds^2 = ((\partial^2\phi)/(\partial z\partial\bar{z}))dz \otimes d\bar{z}$ on Δ^*, and for some constants $C_2, C_3 > 0$,

(ii) $|\phi(z)| \le C_2 \max\{1, \log(-\log|z|)\}$, and (2.2.2)

(iii) $\left|\frac{\partial\phi}{\partial z}\right|, \left|\frac{\partial\phi}{\partial\bar{z}}\right| \le \frac{C_3}{|z|\,|\log|z||}$ on Δ^*. (2.2.3)

DEFINITION 2.2.2. Let $N = \sqcup_{1\le k\le l} N_k$ be a disjoint union of punctured Riemann surfaces N_k, $1 \le k \le l$. A Hermitian metric ds^2 on N is simply defined to be an ordered l-tuple $(ds_1^2, \ldots, ds_l^2)$, where ds_k^2 is a Hermitian metric on N_k. ds^2 is said to be of *hyperbolic growth near the punctures* if each ds_k^2 is of hyperbolic growth near the punctures in the sense of Definition 2.2.1, $k = 1, 2, \ldots, l$.

Now let $\pi : \{M_t\} \to \Delta^{m+1}$, $M^0 = M\backslash\{p_1, p_2, \ldots, p_{m+1}\} = M_1^0 \sqcup M_2^0$, q, q_1, q_2, m_1, m_2, be as in (2.1). At the origin $t = 0 \in \Delta^{m+1}$, the stable condition on M implies that $q_1 + m_1 > 0$ and $q_2 + m_2 > 0$, or equivalently, M^0 admits the complete hyperbolic metric $ds^2_{\mathrm{hyp},0}$ of constant sectional curvature -1. Also, it is easy to see that the two inequalities $q_1 + m_1 > 0$ and $q_2 + m_2 > 0$ actually imply that M_t is stable for each $t \in \Delta^{m+1}$, and thus the smooth part M_t^0 of each M_t admits the complete hyperbolic metric, which we denote by $ds^2_{\mathrm{hyp},t}$. Now for each $t \in \Delta^{m+1}$, let ds_t^2 be a Hermitian metric on the smooth part M_t^0 of M_t.

DEFINITION 2.2.3. $\{ds_t^2\}$ is said to be a continuous family of quasi-hyperbolic metrics on $\{M_t\}$ if

(i) $\{ds_t^2\}$ form a continuous section of $L \otimes \bar{L}^*$, where L is as in (2.1);
(ii) there exist constants $C_1, C_2 > 0$ such that

$$C_1 ds^2_{\mathrm{hyp},t} \le ds_t^2 \le C_2 ds^2_{\mathrm{hyp},t} \quad \text{for all } t \in \Delta^{m+1}; \tag{2.2.4}$$

and
(iii) for each $t \in \Delta^{m+1}$, ds_t^2 is of hyperbolic growth near the punctures on M_t^0 (cf. Definition 2.2.2).

Remark 2.2.4. (i) By [12, theorem 5.8], $\{ds^2_{\mathrm{hyp},t}\}$ form a continuous family of quasi-hyperbolic metrics on $\{M_t\}$. Also, one can easily construct non-trivial families of quasi-hyperbolic metrics on $\{M_t\}$ by the grafting procedure in [12, §3, §4].

(ii) One easily sees from Equation (2.2.1) that $\mathrm{Vol}(M_t^0, ds_t^2) < \infty$ for each $t \in \Delta^{m+1}$.

(2.3) Let $\{M_t\}$ be as in (2.1), and $\{ds_t^2\}$ be a continuous family of quasi-hyperbolic metrics on $\{M_t\}$. For $t \in (\Delta^*)^{m+1}$, denote the Kähler form on the smooth compact fiber M_t associated to ds_t^2 by ω_t, and denote the associated normalized Kähler form by $\hat{\omega}_t := (1/\mathrm{Vol}(M_t, \omega_t))\omega_t$. It is well known that there exists a unique Green's function $g_t(\cdot, \cdot)$ on $M_t \times M_t \backslash\{\text{diagonal}\}$ satisfying the following conditions:

(a) For fixed $x \in M_t$, and $y \neq x$ near x,

$$g_t(x, y) = -\log|f(y)|^2 + \alpha(y), \tag{2.3.1}$$

where f is a local holomorphic defining function for x, and α is some smooth function defined near x;

(b) $d_y d_y^c g_t(x, y) = \hat{\omega}_t(y) - \delta_x;$ (2.3.2)

(c) $\displaystyle\int_{M_t} g_t(x, y)\hat{\omega}_t(y) = 0;$ (2.3.3)

(d) $g_t(x, y) = g_t(y, x)$ for $x \neq y;$ (2.3.4)

(e) $g_t(x, y)$ is smooth on $M_t \times M_t \backslash\{\text{diagonal}\}$. (2.3.5)

Here $d_x^c := (i/4\pi)(\bar{\partial} - \partial)$, and δ_x is the Dirac delta function at x.

At the origin $t = 0$, we write $(M^0, ds_0^2) = (M_1^0, ds_{0,1}^2) \sqcup (M_2^0, ds_{0,2}^2)$, i.e., $ds_{0,k}^2 = ds_0^2\ |_{M_k^0}$, $k = 1, 2$. Also, for $k = 1, 2$, we let $\omega_{0,k}$, $\hat{\omega}_{0,k}$ be the Kähler forms on M_k^0 associated to $ds_{0,k}^2$ and defined similarly as above. Since $ds_{0,k}^2$ is of hyperbolic growth near the punctures on M_k^0 (cf. Definitions 2.2.1, 2.2.2 and 2.2.3), it follows from [9, theorem 1] that there exists a unique Green's function $g_{0,k}(\cdot, \cdot)$ on $M_k^0 \times M_k^0\backslash\{\text{diagonal}\}$ satisfying conditions (a) to (e) above (with $t = 0$ and M_t replaced by M_k^0) and also the following growth condition:

(f) Near each puncture of M_k^0, there exists a punctured coordinate neighborhood Δ^* centered at the puncture such that for fixed $x \notin \Delta^*$, there exists a constant $C > 0$ such that

$$|g_{0,k}(x, z)| \leq C \max\{1, \log(-\log|z|)\} \quad \text{on } \Delta^*. \tag{2.3.6}$$

We remark that similar descriptions also hold for other noded fibers M_t for t along the coordinate hyperplanes of Δ^{m+1}.

(2.4) Notation as in (2.1), (2.2) and (2.3). We are ready to state our main result in this article as follows:

THEOREM 2.4.1. *Let $\{M_t\}$ be a family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a stable Riemann surface M with one separating node p_1 and m non-separating nodes $p_2, p_3, \ldots, p_{m+1}$ as described in (2.1). Suppose $\{ds_t^2\}$*

is a continuous family of quasi-hyperbolic metrics on $\{M_t\}$ *(cf. Definition 2.2.3). Then for continuous sections* x_t, y_t *of* $\{M_t\}$ *such that* $x_t \neq y_t$ *for all* $t \in \Delta^{m+1}$ *and* $x_0, y_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$, *we have*

$$\lim_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} \left(g_t(x_t, y_t) - \frac{1}{\lambda_{1,t}} \phi_{1,t}(x_t)\phi_{1,t}(y_t) \right)$$

$$= \begin{cases} g_{0,1}(x_0, y_0), & \text{if } x_0, y_0 \in M_1^0, \\ 0, & \text{if } (x_0, y_0) \in M_1^0 \times M_2^0 \text{ or } M_2^0 \times M_1^0, \\ g_{0,2}(x_0, y_0), & \text{if } x_0, y_0 \in M_2^0, \end{cases} \tag{2.4.1}$$

where $g_{0,k}(\cdot, \cdot)$ *is Green's function on* M_k^0 *with respect to* $ds_{0,k}^2$, $k = 1, 2$ *(cf. (2.3)), and* $\phi_{1,t}$ *is any eigenfunction on* M_t *of* L^2*-norm 1 corresponding to the first non-zero eigenvalue* $\lambda_{1,t}$ *of the Laplacian* Δ_t *associated to* ds_t^2.

Theorem 2.4.1 gives rise to the following

COROLLARY 2.4.2. *Let* $\{M_t\}$, $\{ds_t^2\}$, x_t, y_t *be as in Theorem 2.4.1.*

(a) *We have*

$$\lim_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} \lambda_{1,t} g_t(x_t, y_t)$$

$$= \begin{cases} \frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})}, & \text{if } x_0, y_0 \in M_1^0, \\ -\frac{1}{V_{0,1}+V_{0,2}}, & \text{if } (x_0, y_0) \in M_1^0 \times M_2^0 \text{ or } M_2^0 \times M_1^0, \\ \frac{V_{0,1}}{V_{0,2}(V_{0,1}+V_{0,2})}, & \text{if } x_0, y_0 \in M_2^0, \end{cases} \tag{2.4.2}$$

where $V_{0,k} := \mathrm{Vol}(M_k^0, ds_{0,k}^2) < \infty$, $k = 1, 2$ *(cf. Remark 2.2.4(ii)).*

(b) *In particular, there exist constants* $C_{1,k}, C_{2,k}$ $(k = 1, 2)$, $C_3, C_4 > 0$ *such that*

(i) *if* $x_0, y_0 \in M_k^0$, *then for* $k = 1, 2$,

$$0 < C_{1,k} \leq \liminf_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} l_t \cdot g_t(x_t, y_t) \leq \limsup_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} l_t \cdot g_t(x_t, y_t) \leq C_{2,k}; \tag{2.4.3}$$

and

(ii) *if* $(x_0, y_0) \in M_1^0 \times M_2^0$ *or* $M_2^0 \times M_1^0$, *then*

$$-C_3 \leq \liminf_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} l_t \cdot g_t(x_t, y_t) \leq \limsup_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} l_t \cdot g_t(x_t, y_t) \leq -C_4 < 0. \tag{2.4.4}$$

Here, for $t \in (\Delta^*)^{m+1}$, l_t denotes the infinum of the lengths of all simple closed geodesics on M_t (with respect to ds_t^2) which separate M_t into two components.

We also have the following

COROLLARY 2.4.3. *Let $\{M_t\}$ be as in Theorem 2.4.1, and let $\{ds_t^2\}$ be a continuous family of complete Hermitian metrics on $\{M_t\}$. Suppose there exist constants $C_1, C_2 > 0$ such that the sectional curvatures of $\{ds_t^2\}$ are pinched between $-C_1$ and $-C_2$ for all $t \in \Delta^{m+1}$, and ds_0^2 is of hyperbolic growth near the punctures on M^0. Then the conclusions of Theorem 2.4.1 and Corollary 2.4.2 remain valid.*

We remark that in the special case of the family of hyperbolic metrics $\{ds^2_{\mathrm{hyp},t}\}$ on $\{M_t\}$, one can easily check (using Burger's result [1, theorem 1.1]) that Theorem 2.4.1 and Corollary 2.4.2 agree with the corresponding results of Ji [5, theorems 1.1 and 1.2].

3. Approximation of Eigenfunctions on M_t

Let $\mathcal{M} = \{M_t\}$, $\{ds_t^2\}$, $p_1, p_2, \ldots, p_{m+1}, \lambda_{1,t}$ be as in Theorem 2.4.1. In this section, we are going to construct good approximations of the eigenfunctions on M_t corresponding to the first non-zero eigenvalue $\lambda_{1,t}$ of the Laplacian Δ_t with respect to ds_t^2 (cf. (3.2) below).

(3.1) To facilitate subsequent discussion, we first set up some notations. Let $\tilde{M} = M_1 \sqcup M_2$ be as in (2.1). For $1 \leq i \leq m+1$, recall from (2.1) the coordinate functions $z_{i,k} : U_{i,k} \to \Delta$, $k = 1, 2$, on $\tilde{M}$, and the coordinate neighborhoods Δ^{m+1} of p_i in $\mathcal{M}$ such that for $t = (t_1, t_2, \ldots, t_{m+1}) \in \Delta^{m+1}$, $M_t \cap \Delta^{m+2} = \{(t_1, \ldots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \ldots, t_{m+1}) \in \Delta^{m+2} \mid z_{i,1} z_{i,2} = t_i\}$. Fix a small number $\delta > 0$, and define, for $1 \leq i \leq m+1$ and $t = (t_1, t_2, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$,

$$\begin{aligned} \mathrm{I}_{i,t} &:= \Big\{(t_1, \ldots, t_{i-1}, z_{i,1}, t_i/z_{i,1}, t_{i+1}, \ldots, t_{m+1}) \\ &\qquad \in \Delta^{m+2} \mid |t_i|^{(1/2)+2\delta} < |z_{i,1}| < |t_i|^{(1/2)-2\delta}\Big\} \\ &\subset M_t. \end{aligned} \tag{3.1.1}$$

For $1 \leq i \leq m+1$, we denote by

$$\mathrm{pr}_{i,k,t} : \mathrm{I}_{i,t} \to U_{i,k}, \quad k = 1, 2, \ t \in (\Delta^*)^{m+1}, \tag{3.1.2}$$

the holomorphic maps induced by the i-th and $(i+1)$-st coordinate projection maps on Δ^{m+2} respectively. Since p_1 is a separating node, $M_t \backslash \mathrm{I}_{1,t}$ consists of two separated components, which we denote by $\mathrm{II}'_{1,t}$ and $\mathrm{II}'_{2,t}$ (so that $M_t = \mathrm{I}_{1,t} \sqcup \mathrm{II}'_{1,t} \sqcup \mathrm{II}'_{2,t}$). Also, we denote $\mathrm{II}_{1,t} := \mathrm{II}'_{1,t} \backslash \sqcup_{2 \leq i \leq m_1+1} \mathrm{I}_{i,t}$ and $\mathrm{II}_{2,t} := \mathrm{II}'_{2,t} \backslash \sqcup_{m_1+2 \leq i \leq m+1} \mathrm{I}_{i,t}$ (so that we also have $M_t = (\sqcup_{1 \leq i \leq m+1} \mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t} \sqcup \mathrm{II}_{2,t}$). For $k = 1, 2$ and $t =$

$(t_1, t_2, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$, we let $W_{k,t} := M_k^0 \setminus \sqcup_{(i,k')\in \mathcal{I}_k} \{z_{i,k'} \in \Delta \mid |z_{i,k'}| < |t_i|^{(1/2)-2\delta}\}$, where $\mathcal{I}_1 = \{(1,1)\} \cup \{(i,k') \mid i = 2, \ldots, m_1+1, k' = 1, 2\}$ and $\mathcal{I}_2 = \{(1,2)\} \cup \{(i,k') \mid i = m_1+2, \ldots, m+1, k' = 1, 2\}$, and we denote by

$$i_{k,t} : \mathrm{II}_{k,t} \to W_{k,t} \tag{3.1.3}$$

the biholomorphisms induced by the plumbing construction in (2.1). The inverse of $i_{k,t}$ will be denoted by $j_{k,t} : W_{k,t} \to \mathrm{II}_{k,t}$, $k = 1, 2$, $t \in (\Delta^*)^{m+1}$.

(3.2) Next we consider the following function on $M^0 = M_1^0 \sqcup M_2^0$ given by

$$\phi_{1,0}(z) := \begin{cases} \sqrt{\frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})}} & \text{if } z \in M_1^0, \\ -\sqrt{\frac{V_{0,1}}{V_{0,2}(V_{0,1}+V_{0,2})}} & \text{if } z \in M_2^0, \end{cases} \tag{3.2.1}$$

where $V_{0,1}$, $V_{0,2}$ are as in Equation (2.4.2). Throughout the rest of this paper, L^2-norm and inner products on M_t will be with respect to ds_t^2, and they are simply denoted by $\|\|_2$ and $\langle, \rangle$ respectively. From Equation (3.2.1), it is easy to see that $\|\phi_{1,0}\|_2 = 1$ and $\phi_{1,0}$ is orthogonal to the constant functions on M^0.

Notation is as in (3.1). We fix a smooth function $\eta = \eta(a)$ on $\mathbf{R}$ such that $0 < \eta < 1$ for all $a \in \mathbf{R}$, $\eta = 1$ for $a < (1/2) - \delta$, and $\eta = 0$ for $a > (1/2) + \delta$. Then we define the following family of cut-off functions $\{\eta_t\}$ on $\{M_t\}$ as follows: for $t = (t_1, t_2, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$,

$$\eta_t(z) := \begin{cases} 1, & \text{for } z \in \mathrm{II}'_{1,t}, \\ \eta\left(\frac{\log|z_{1,1}|}{\log|t_1|}\right), & \text{for } z = (z_{1,1}, t_1/z_{1,1}, t_2, \ldots, t_{m+1}) \in \mathrm{I}_{1,t}, \\ 0, & \text{for } z \in \mathrm{II}'_{2,t}. \end{cases} \tag{3.2.2}$$

It is easy to see that each η_t is smooth on M_t. Next, for $t \in (\Delta^*)^{m+1}$, we define the following smooth function on M_t given by

$$\tilde{\phi}_{1,t}(z) := \eta_t(z)\sqrt{\frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})}} - (1-\eta_t(z))\sqrt{\frac{V_{0,1}}{V_{0,2}(V_{0,1}+V_{0,2})}} \tag{3.2.3}$$

for $z \in M_t$. Finally, for $t \in (\Delta^*)^{m+1}$, we define the smooth function

$$\kappa_t := \Delta_t \tilde{\phi}_{1,t} \quad \text{on } M_t. \tag{3.2.4}$$

Remark 3.2.1. For any continuous section z_t of $\{M_t\}$ such that $z_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$, it is easy to see from Equations (3.2.1) and (3.2.2) that $\tilde{\phi}_{1,t}(z_t) \to \phi_{1,0}(z_0)$ as $t \to 0$.

(3.3) We are going to derive some estimates on $\mathrm{I}_{i,t}$, $i = 1, \ldots, m+1$, which will be needed later. Recall from (2.1) that for $1 \le i \le m+1$, there is a coordinate neighborhood Δ^{m+2} of p_i in $\mathcal{M}$ centered at p_i and such that for $t = (t_1, t_2, \ldots, t_{m+1}) \in$

Δ^{m+1}, $M_t \cap \Delta^{m+2} = \{(t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \in \Delta^{m+2} \mid z_{i,1}z_{i,2} = t_i\}$ with $z_{i,1}, z_{i,2}$ providing two different coordinate functions. As in [9, proposition 4.2.1], we have

PROPOSITION 3.3.1. *There exist constants* $C_1, C_2 > 0$ *such that for all* $1 \le i \le m+1$, $t = (t_1, t_2, \dots, t_{m+1}) \in (\Delta^*)^{m+1}$ *and* $k = 1, 2$, *we have, on* $M_t \cap \Delta^{m+2}$,

$$C_1 \left(\frac{\pi}{\log |t_k|} \csc \frac{\pi \log |z_{i,k}|}{\log |t_k|} \frac{|dz_{i,k}|}{|z_{i,k}|} \right)^2 \le ds_t^2 \le C_2 \left(\frac{\pi}{\log |t_k|} \csc \frac{\pi \log |z_{i,k}|}{\log |t_k|} \frac{|dz_{i,k}|}{|z_{i,k}|} \right)^2 . \tag{3.3.1}$$

Proof. It follows from a result of Wolpert [12, expansion 4.2] that Equation (3.3.1) holds for the hyperbolic metrics $\{ds^2_{\text{hyp},t}\}$ on $\{M_t\}$. This, together with Equation (2.2.4), implies Proposition 3.3.1 immediately. □

Next we have the following

PROPOSITION 3.3.2. *Let* $\mathrm{I}_{i,t}$, $i = 1, \dots, m+1$, *be as in Equation (3.1.1), and let* κ_t *be as in Equation (3.2.4). We have*

(i) *for* $1 \le i \le m+1$, $\int_{\mathrm{I}_{i,t}} \omega_t \to 0$, $\int_{\mathrm{I}_{i,t}} \hat{\omega}_t \to 0$, *and* (3.3.2)

(ii) $\|\kappa_t\|_2 \to 0$ *as* $t \to 0$. (3.3.3)

Proof. First we recall from Equation (3.1.1) that, for $1 \le i \le m+1$, $t = (t_1, t_2, \dots, t_{m+1}) \in (\Delta^*)^{m+1}$ and $k = 1, 2$, one has

$$\frac{1}{2} - 2\delta < \frac{\log |z_{i,k}|}{\log |t_i|} < \frac{1}{2} + 2\delta \tag{3.3.4}$$

for $z = (t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \in \mathrm{I}_{i,t}$ (so that $z_{i,1}z_{i,2} = t_i$). For fixed i with $1 \le i \le m+1$, we write $\zeta := \log z_{i,1} / \log |t_i|$ on $\mathrm{I}_{i,t}$, so that

$$\zeta = a + ib, \quad \frac{1}{2} - 2\delta < a < \frac{1}{2} + 2\delta, \quad 0 \le b < \frac{2\pi}{|\log |t_i||}, \tag{3.3.5}$$

gives a parametrization for each $\mathrm{I}_{i,t}$, $t \in (\Delta^*)^{m+1}$. From Equation (2.2.4) and the well-known fact that $\mathrm{Vol}(M_t, ds^2_{\text{hyp},t}) = 2\pi(2q-2)$ for $t \in (\Delta^*)^{m+1}$, it follows that there exist constants $C_3, C_4 > 0$ such that $C_3 \le \mathrm{Vol}(M_t, ds_t^2) \le C_4$ for all $t \in (\Delta^*)^{m+1}$. Then, together with Proposition 3.3.1 and Equation (3.3.4), it follows that there exist constants $C_5, C_6 > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$C_5 \frac{i}{2} d\zeta \wedge d\bar{\zeta} \le \omega_t, \quad \hat{\omega}_t \le C_6 \frac{i}{2} d\zeta \wedge d\bar{\zeta} \quad \text{on } \mathrm{I}_{i,t}. \tag{3.3.6}$$

Therefore,

$$\begin{aligned}\int_{\mathrm{I}_{i,t}} \hat{\omega}_t &\leq C_6 \left| \int_{\mathrm{I}_{i,t}} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} \right| \\ &= C_6 \int_0^{(2\pi)/(|\log|t_i||)} \int_{(1/2)-2\delta}^{(1/2)+2\delta} da db \\ &= \frac{8\pi\delta C_6}{|\log|t_i||} \to 0 \quad \text{as } t \to 0. \end{aligned} \tag{3.3.7}$$

Similarly, one has $\int_{\mathrm{I}_{i,t}} \omega_t \to 0$ as $t \to 0$, and this finishes the proof of (i). To verify (ii), we first see from construction of η_t that there exists a constant $C_7 > 0$ such that for all $t \in (\Delta^*)^{m+1}$, $|\partial_\zeta \partial_{\bar{\zeta}} \eta_t| \leq C_7$ on $\mathrm{I}_{1,t}$, where ζ is as in Equation (3.3.5). Write $C_8 := \sqrt{V_{0,2}}/\sqrt{V_{0,1}(V_{0,1} + V_{0,2})} - \sqrt{V_{0,1}}/\sqrt{V_{0,2}(V_{0,1} + V_{0,2})}$. Then it follows from Equations (3.2.2) and (3.2.4) that for all $t \in (\Delta^*)^{m+1}$ and $z \in \mathrm{I}_{1,t}$,

$$|\kappa_t(z)| = \left| C_8 \cdot \frac{1}{\omega_t(\partial/\partial\zeta, \partial/\partial\zeta)} \cdot \partial_\zeta \partial_{\bar{\zeta}} \eta_t(z) \right| \leq \frac{C_8 C_7}{C_5},$$

where C_5 is as in Equation (3.3.6). From Equation (3.2.2), one sees that $\mathrm{supp}(d\kappa_t) \subset \mathrm{I}_{1,t}$, and thus

$$\begin{aligned}\|\kappa_t\|_2^2 &= \int_{\mathrm{I}_{1,t}} |\kappa_t|^2 \omega_t \\ &\leq \left(\frac{C_8 C_7}{C_5} \right)^2 \int_{\mathrm{I}_{1,t}} \omega_t \\ &\to 0 \quad \text{as } t \to 0 \quad \text{(by (3.3.2))}.\end{aligned}$$

This finishes the proof of Proposition 3.3.2. □

(3.4) We recall from (3.1) the biholomorphisms $j_{k,t} : W_{k,t} \to \mathrm{II}_{k,t}$, $k = 1, 2$, $t \in (\Delta^*)^{m+1}$. For $t = (t_1, t_2, \ldots, t_{m+1}) \in \mathbf{C}^{m+1}$, we denote $|t| := \max_{1 \leq i \leq m+1} |t_i|$. As in [9, lemma 4.3.1], we have

LEMMA 3.4.1. *For* $0 < t_0 < 1$ *and* $k = 1, 2$, *there exist constants* $C_k, C'_k > 0$ *such that for all* $t \in (\Delta^*)^{m+1}$ *with* $|t| \leq t_0$,

$$C_k ds_{0,k}^2 \leq j_{k,t}^* ds_t^2 \leq C'_k ds_{0,k}^2 \quad \text{on } W_{k,t}. \tag{3.4.1}$$

Proof. For $k = 1, 2$, we let $z_{i,k} : U_{i,k} \to \Delta$ be as in (2.1). For $t = (t_1, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$ with $|t| \leq t_0$, we write $W_{k,t} = N_k \sqcup_{(i,k') \in \mathcal{I}_k} U_{i,k',t}$, where the index set $\mathcal{I}_k$ is as in (3.1) and $U_{i,k',t} := \{z_{i,k'} \in U_{i,k'} : |t_i|^{\frac{1}{2}-2\delta} \leq |z_{i,k'}| < 1\}$ and $N_k := W_{k,t} \setminus \sqcup_{(i,k') \in \mathcal{I}_k} U_{i,k',t}$. Note that N_k does not vary with t. First, from the compactness of $N_k \times \{t \in \Delta^{m+1} : |t| \leq t_0\}$ and the continuity of $\{ds_t^2\}$, it follows that Equation (3.4.1) holds on N_k. For $k = 1, 2$, it follows from Remark 2.2.4 that there exist constants $C_{k,1}, C_{k,2} > 0$ such that for all $1 \leq i \leq m+1$,

$$C_{k,1}\frac{|dz_{i,k}|^2}{|z_{i,k}|^2(\log|z_{i,k}|)^2} \leq ds_{0,k}^2 \leq C_{k,2}\frac{|dz_{i,k}|^2}{|z_{i,k}|^2(\log|z_{i,k}|)^2} \quad \text{on } U_{i,k}. \tag{3.4.2}$$

Observe that $0 < \pi \log|z_{i,k'}|/\log|t_i| < \pi((1/2) - 2\delta)$ on each $U_{i,k',t}$ (cf. Equation (3.1.1)), and that there exist constants $C_3, C_4 > 0$ such that $C_3 < \theta \csc\theta < C_4$ for all $0 < \theta < \pi((1/2) - 2\delta)$. Together with Equation (3.4.2) and Proposition 3.3.1, it is easy to verify that Equation (3.4.1) also holds on each $U_{i,k',t}$ for $(i,k') \in \mathcal{I}_k$, $t \in (\Delta^*)^{m+1}$ with $|t| \leq t_0$, and this finishes the proof of Lemma 3.4.1. □

Remark 3.4.2. Since $\pi((1/2) - 2\delta) < \pi \log|z_{i,k}|/\log|t_i| < \pi((1/2) + 2\delta)$ on $\mathrm{I}_{i,t}$, $k = 1, 2$, $i = 1, \ldots, m+1$, and there exist constants $C_1, C_2 > 0$ such that $C_1 < \theta\csc\theta < C_2$ for $\pi((1/2) - 2\delta) < \theta < \pi((1/2) + 2\delta)$, one can easily verify as in Lemma 3.4.1 that for $k = 1, 2$, shrinking Δ^{m+1} if necessary, there exist constants $C_{k,1}, C_{k,2} > 0$ such that for all $i = 1, \ldots, m+1$ and $t \in (\Delta^*)^{m+1}$, one has $C_{k,1} ds_t^2 \leq \mathrm{pr}_{i,k,t}^* ds_{0,k}^2 \leq C_{k,2} ds_t^2$ on $\mathrm{I}_{i,t}$.

PROPOSITION 3.4.3. *We have*

(i) $\int_{M_t} \tilde{\phi}_{1,t}\omega_t \to 0$, *and* (3.4.3)

(ii) $\|\tilde{\phi}_{1,t}\|_2 \to 1$ *as* $t \to 0$. (3.4.4)

Proof. First we recall from (3.1) that $M_t = (\sqcup_{1\leq i\leq m+1}\mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t} \sqcup \mathrm{II}_{2,t}$ for $t \in (\Delta^*)^{m+1}$, so that

$$\int_{M_t} \tilde{\phi}_{1,t}\omega_t = \sum_{1\leq i\leq m+1}\int_{\mathrm{I}_{i,t}} \tilde{\phi}_{1,t}\omega_t + \int_{\mathrm{II}_{1,t}} \tilde{\phi}_{1,t}\omega_t + \int_{\mathrm{II}_{2,t}} \tilde{\phi}_{1,t}\omega_t. \tag{3.4.5}$$

From Equation (3.2.3), one sees that there exists a constant $C > 0$ such that for all $t \in (\Delta^*)^{m+1}$, $|\tilde{\phi}_{1,t}(z)| \leq C$ for all $z \in \mathrm{I}_{i,t}$. Together with Proposition 3.3.2(i), one easily deduces that for all $1 \leq i \leq m+1$,

$$\int_{\mathrm{I}_{i,t}} \tilde{\phi}_{1,t}\omega_t \to 0 \quad \text{and} \quad \int_{\mathrm{I}_{i,t}} (\tilde{\phi}_{1,t})^2\omega_t \to 0 \quad \text{as } t \to 0. \tag{3.4.6}$$

From Equation (3.2.2), one also has

$$\int_{\mathrm{II}_{1,t}} \tilde{\phi}_{1,t}\omega_t = \sqrt{\frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})}}\int_{W_{1,t}} \frac{j_{1,t}^*\omega_t}{\omega_{0,1}}\,\omega_{0,1}. \tag{3.4.7}$$

Observe that $\{W_{1,t}\}$ form an increasing sequence of compact subsets exhausting M_1^0 as $t \to 0$ and the right-hand side of Equation (3.4.7) can be regarded as an integral over M_1^0 by extending the integrand to be zero on $M_1^0 \backslash W_{1,t}$. By Lemma 3.4.1 and the continuity of $\{ds_t^2\}$, $j_{1,t}^*\omega_t/\omega_{0,1}$ is uniformly bounded from above by a constant and converges pointwise to the constant function 1 as $t \to 0$. Since $V_{0,1} < \infty$, it follows from the dominated convergence theorem that

$$\int_{\mathrm{II}_{1,t}} \tilde{\phi}_{1,t}\omega_t \to \sqrt{\frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})}} \cdot V_{0,1}$$
$$= \sqrt{\frac{V_{0,1}V_{0,2}}{V_{0,1}+V_{0,2}}} \quad \text{as } t \to 0. \tag{3.4.8}$$

Similarly, using Equation (3.2.3), one can easily see that

$$\int_{\mathrm{II}_{2,t}} \tilde{\phi}_{1,t}\omega_t \to -\sqrt{\frac{V_{0,1}V_{0,2}}{V_{0,1}+V_{0,2}}}, \tag{3.4.9}$$

$$\int_{\mathrm{II}_{1,t}} (\tilde{\phi}_{1,t})^2\omega_t \to \frac{V_{0,2}}{V_{0,1}+V_{0,2}}, \tag{3.4.10}$$

and

$$\int_{\mathrm{II}_{2,t}} (\tilde{\phi}_{1,t})^2\omega_t \to \frac{V_{0,1}}{V_{0,1}+V_{0,2}} \quad \text{as } t \to 0. \tag{3.4.11}$$

By combining Equations (3.4.5), (3.4.6), (3.4.8) and (3.4.9), one obtains Proposition 3.4.3(i) immediately. Similarly, Proposition 3.4.3(ii) can be obtained by combining Equations (3.4.6), (3.4.10) and (3.4.11). Thus we have finished the proof of Proposition 3.4.3. □

Remark 3.4.4. We remark that one can easily modify the proof of Proposition 3.4.3 to show that $\mathrm{Vol}(M_t, \omega_t) \to V_{0,1} + V_{0,2}$ as $t \to 0$.

(3.5) Let $\{M_t\}, \{ds_t^2\}$ be as in Theorem 2.4.1. For $t \in (\Delta^*)^{m+1}$, we let $0 = \lambda_{0,t} < \lambda_{1,t} \leq \lambda_{2,t} \leq \ldots$ be the set of eigenvalues of the Laplacian Δ_t on M_t, counting multiplicity. Also, for each $t \in (\Delta^*)^{m+1}$, we fix an orthonormal set $\{\phi_{l,t}\}_{0\leq l<\infty}$ of eigenvectors of Δ_t on M_t with $\phi_{l,t}$ corresponding to $\lambda_{l,t}$. Moreover, we will let $\phi_{0,t} := 1/\sqrt{\mathrm{Vol}(M_t, \omega_t)}$. It follows from standard elliptic theory that $\{\phi_{l,t}\}_{0\leq l<\infty}$ forms a complete orthonormal basis of the Hilbert space of L^2 functions on M_t endowed with the inner product $\langle , \rangle$ induced by ds_t^2. First we have

LEMMA 3.5.1. *There exists a constant $\alpha > 0$ such that $\lambda_{2,t} \geq \alpha$ for all $t \in (\Delta^*)^{m+1}$.*

Proof. It is well known and follows from results in [4, 8] that in our case of degenerating Riemann surfaces with one separating node (and m non-separating nodes), there exists a constant $\beta > 0$ such that $\lambda_{2,t}^{\text{hyp}} \geq \beta$ for all $t \in (\Delta^*)^{m+1}$, where $\lambda_{2,t}^{\text{hyp}}$ denotes the second non-zero eigenvalue of the hyperbolic Laplacian on M_t (see, e.g., [5, corollary 2]). This, together with Equation (2.2.4) and the minimax principle, implies Lemma 3.5.1 immediately. □

For $t \in (\Delta^*)^{m+1}$, we define the following smooth function on M_t given by

$$\begin{aligned}\mu_t &:= \tilde{\phi}_{1,t} - \langle \tilde{\phi}_{1,t}, \phi_{0,t}\rangle \phi_{0,t} - \langle \tilde{\phi}_{1,t}, \phi_{1,t}\rangle \phi_{1,t} \\ &= \tilde{\phi}_{1,t} - \frac{1}{\mathrm{Vol}(M_t, \omega_t)} \int_{M_t} \tilde{\phi}_{1,t}\omega_t - \langle \tilde{\phi}_{1,t}, \phi_{1,t}\rangle \phi_{1,t},\end{aligned} \tag{3.5.1}$$

where $\tilde{\phi}_{1,t}$ is as in Equation (3.2.3).

PROPOSITION 3.5.2. *Let μ_t, κ_t, α be as in Equations (3.5.1), (3.2.4) and Lemma 3.5.1 respectively. Then we have*

(i) $\|\Delta_t \mu_t\|_2 \leq \|\kappa_t\|_2$, *and* (3.5.2)
(ii) $\|\mu_t\|_2 \leq \frac{1}{\alpha}\|\kappa_t\|_2$ *for all* $t \in (\Delta^*)^{m+1}$. (3.5.3)
(iii) *In particular,* $\|\mu_t\|_2 \to 0$ *as* $t \to 0$.

Proof. From Equation (3.2.4), we have

$$\langle \kappa_t, \phi_{1,t}\rangle = \langle \Delta_t \tilde{\phi}_{1,t}, \phi_{1,t}\rangle = \langle \tilde{\phi}_{1,t}, \Delta_t \phi_{1,t}\rangle = \lambda_{1,t}\langle \tilde{\phi}_{1,t}, \phi_{1,t}\rangle. \tag{3.5.4}$$

Together with Equations (3.2.4) and (3.5.1), one easily checks that

$$\Delta_t \mu_t = \kappa_t - \langle \kappa_t, \phi_{1,t}\rangle \phi_{1,t} \quad \text{on } M_t. \tag{3.5.5}$$

This implies, in particular, that

$$\begin{aligned}\|\Delta_t \mu_t\|_2^2 &= \langle \kappa_t - \langle \kappa_t, \phi_{1,t}\rangle \phi_{1,t}, \kappa_t - \langle \kappa_t, \phi_{1,t}\rangle \phi_{1,t}\rangle \\ &= \langle \kappa_t, \kappa_t\rangle - 2|\langle \kappa_t, \phi_{1,t}\rangle|^2 + |\langle \kappa_t, \phi_{1,t}\rangle|^2 \langle \phi_{1,t}, \phi_{1,t}\rangle \\ &= \|\kappa_t\|_2^2 - |\langle \kappa_t, \phi_{1,t}\rangle|^2 \quad (\text{since } \|\phi_{1,t}\|_2 = 1),\end{aligned}$$

which implies Proposition 3.5.2(i). Next, by construction in Equation (3.5.1), one sees that μ_t is orthogonal to both $\phi_{0,t}$ and $\phi_{1,t}$. Together with Parseval's identity, we have

$$\begin{aligned}\|\mu_t\|_2^2 &= \sum_{l \geq 2} |\langle \mu_t, \phi_{l,t}\rangle|^2 \\ &= \sum_{l \geq 2} \frac{1}{(\lambda_{l,t})^2} |\langle \Delta_t \mu_t, \phi_{l,t}\rangle|^2 \quad (\text{as in (3.5.4)}) \\ &\leq \frac{1}{\alpha^2} \|\Delta_t \mu_t\|_2^2 \quad (\text{by Lemma 3.5.1 and Parseval's identity}) \\ &\leq \frac{1}{\alpha^2} \|\kappa_t\|_2^2 \quad (\text{by Proposition 3.5.2(i)}),\end{aligned} \tag{3.5.6}$$

which leads to Proposition 3.5.2(ii). Finally, Proposition 3.5.2(iii) follows readily from Proposition 3.3.2(ii) and Proposition 3.5.2(ii) by letting $t \to 0$. □

PROPOSITION 3.5.3. *Let x_t be a continuous section of $\{M_t\}$ such that $x_0 \notin \{p_1, \ldots, p_{m+1}\}$. Then $\mu_t(x_t) \to 0$ as $t \to 0$.*

Proof. Since $x_0 \notin \{p_1, \ldots, p_{m+1}\}$, it follows easily from the construction of $\{M_t\}$ in (2.1) that one can find a continuous family of coordinate discs $\Delta(x_t, r) \subset M_t$ centered at x_t and of fixed radius $r > 0$ for $0 \leq |t| < t_0$ and such that $\{p_1, \ldots, p_{m+1}\} \cap \Delta(x_0, r) = \emptyset$, shrinking t_0 and r if necessary. Then by the relative compactness of $\bigcup_{0\leq|t|<t_0} \Delta(x_t, r)$ and the continuity of $\{ds_t^2\}$, there exist constants $C_1, C_2 > 0$ such that for all $t \in (\Delta^*)^{m+1}$ with $|t| < t_0$,

$$C_1 dz \otimes d\bar{z} \leq ds_t^2 \leq C_2 dz \otimes d\bar{z} \quad \text{on } \Delta(x_t, r). \tag{3.5.7}$$

Using standard Nash–Moser iteration technique (cf. [3, theorem 8.2.4]), one can deduce from Equation (3.5.7) that there exists a constant $C = C(C_1, C_2) > 0$ such that for all $t \in (\Delta^*)^{m+1}$ with $|t| < t_0$,

$$\begin{aligned} |\mu_t(x_t)| &\leq C\left(\sqrt{\int_{\Delta(x_t,r)} \mu_t^2 \omega_t} + \sqrt{\int_{\Delta(x_t,r)} (\Delta_t \mu_t)^2 \omega_t}\right) \\ &\leq C(\|\mu_t\|_2 + \|\Delta_t \mu_t\|_2) \\ &\leq C(\|\mu_t\|_2 + \|\kappa_t\|_2) \quad \text{(by Proposition 3.5.2(i))} \\ &\to 0 \quad \text{as } t \to 0 \quad \text{(by Propositions 3.3.2(ii) and 3.5.2(iii)).} \end{aligned} \tag{3.5.8}$$

Thus we have finished the proof of Proposition 3.5.3. □

(3.6) Notations are as before. To facilitate ensuing discussion, we summarize our discussion in Section 3 as follows. Let $\{\phi_{l,t}\}_{0\leq l<\infty}$ be as in (3.5), and let $\tilde{\phi}_{1,t}, \phi_{1,t}$ be as in (3.2).

PROPOSITION 3.6.1. *We have*

(i) $\langle \tilde{\phi}_{1,t}, \phi_{0,t} \rangle \to 0$, *and*
(ii) $|\langle \tilde{\phi}_{1,t}, \phi_{1,t} \rangle| \to 1$ *as* $t \to 0$.

Proof. Proposition 3.6.1(i) follows easily from Proposition 3.4.2(i) and Remark 3.4.3 (cf. also Equation (3.5.1)). From Equation (3.5.1), it is easy to check that

$$\|\mu_t\|_2^2 = \|\tilde{\phi}_{1,t}\|_2^2 - |\langle \tilde{\phi}_{1,t}, \phi_{0,t} \rangle|^2 - |\langle \tilde{\phi}_{1,t}, \phi_{1,t} \rangle|^2. \tag{3.6.1}$$

Then Proposition 3.6.1(ii) can be obtained from Equation (3.6.1) by letting $t \to 0$ and using Propositions 3.4.2(ii), 3.5.2(iii) and 3.6.1(i). □

We remark that in Proposition 3.6.2(ii), $\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle$ itself does not tend to a limit since $\phi_{1,t}$ is determined only up to sign. Also, from Proposition 3.6.1(ii), we may assume that $\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle \neq 0$ for $t \in (\Delta^*)^{m+1}$, shrinking Δ^{m+1} if necessary. Then for $t \in (\Delta^*)^{m+1}$, we define the following smooth function

$$\psi_t := \phi_{1,t} - \frac{1}{\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle}\tilde{\phi}_{1,t} \quad \text{on } M_t. \tag{3.6.2}$$

We have

PROPOSITION 3.6.2.

(i) $\|\psi_t\|_2 \to 0$ *as* $t \to 0$.
(ii) *Let* x_t *be a continuous section of* $\{M_t\}$ *such that* $x_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$. *Then*

$$\psi_t(x_t) \to 0 \quad \text{as } t \to 0.$$

Proof. Rewriting Equation (3.5.1) using Equation (3.6.2), one has

$$\psi_t = \frac{-\langle\tilde{\phi}_{1,t},\phi_{0,t}\rangle\phi_{0,t} - \mu_t}{\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle}, \tag{3.6.3}$$

which implies that

$$0 \le \|\psi_t\|_2^2 \le \frac{2(|\langle\tilde{\phi}_{1,t},\phi_{0,t}\rangle|^2 + \|\mu_t\|_2^2)}{|\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle|^2}. \tag{3.6.4}$$

Then Proposition 3.6.2(i) can be obtained from Equation (3.6.4) by letting $t \to 0$ and using Propositions 3.5.2(iii), 3.6.1(i) and (ii). Similarly, Proposition 3.6.2(ii) can be obtained easily from Equation (3.6.3) by letting $t \to 0$ and using Remark 3.2.1, Propositions 3.5.3, 3.6.1(i) and (ii). □

4. Proof of Theorem 2.4.1

(4.1) Notation is as in Sections 2 and 3. Let $\{M_t\}, \{x_t\}, \{y_t\}, \lambda_{1,t}, \phi_{1,t}$ be as in Theorem 2.4.1. As it is clear that the proofs of Theorem 2.4.1 in the two cases when $y_0 \in M_1^0$ and when $y_0 \in M_2^0$ are the same, we will consider only the first case and assume $y_0 \in M_1^0$ in ensuing discussion. First we recall from (3.1) that $M_t = (\sqcup_{1\le i\le m_1+1}\mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t} \sqcup \mathrm{II}'_{2,t}$ for $t \in (\Delta^*)^{m+1}$. Since $y_0 \in M_1^0$, it follows that $y_t \in \mathrm{II}_{1,t}$ for $|t|$ sufficiently small. Shrinking Δ^{m+1} if necessary, we may thus assume that $y_t \in \mathrm{II}_{1,t}$ for all $t \in (\Delta^*)^{m+1}$, and we let

$$y'_t := i_{1,t}(y_t) \tag{4.1.1}$$

denote the associated continuous curve on M_1^0. Let $\eta : \mathbf{R} \to \mathbf{R}$ be the smooth function in (3.2). For each $1 \le i \le m+1$ and $t = (t_1, t_2, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$, we define, similar to Equation (3.2.3), the smooth function $\eta_{i,t}$ on $\mathrm{I}_{i,t}$ given by

$$\eta_{i,t}(z) := \eta\left(\frac{\log|z_{i,1}|}{\log|t_i|}\right) \quad \text{for} \quad z = (t_1, \ldots, t_{i-1}, z_{i,1}, t_i/z_{i,1}, t_{i+1}, \ldots, t_{m+1}) \in \mathrm{I}_{i,t}. \tag{4.1.2}$$

For $i = 1, \ldots, m+1$, $k = 1, 2$ and $t \in (\Delta^*)^{m+1}$, let $\mathrm{pr}_{i,k,t}$ and $i_{k,t}$ be as in Equations (3.1.2) and (3.1.3), respectively. For $t \in (\Delta^*)^{m+1}$, we define the following function on $M_t \backslash \{y_t\}$ given by

$$\tilde{g}_{t,y_t}(z) := \begin{cases} \eta_{1,t}(z) g_{0,1}(\mathrm{pr}_{1,1,t}(z), y_t'), & \text{if } z \in \mathrm{I}_{1,t}; \\ \eta_{i,t}(z) g_{0,1}(\mathrm{pr}_{i,1,t}(z), y_t') & \text{if } z \in \mathrm{I}_{i,t}, 2 \le i \le m_1 + 1; \\ \quad + (1 - \eta_{i,t}(z)) g_{0,1}(\mathrm{pr}_{i,2,t}(z), y_t'), & \\ g_{0,1}(i_{1,t}(z), y_t'), & \text{if } z \in \mathrm{II}_{1,t}; \\ 0, & \text{if } z \in \mathrm{II}_{2,t}'. \end{cases} \tag{4.1.3}$$

It is easy to see that $\tilde{g}_{t,y_t}$ is smooth on $M_t \backslash \{y_t\}$. Then for $t \in (\Delta^*)^{m+1}$, we define the following function on $M_t \backslash \{y_t\}$ given by

$$u_t(z) := g_t(z, y_t) - \frac{1}{\lambda_{1,t}} \phi_{1,t}(z) \phi_{1,t}(y_t) - \tilde{g}_{t,y_t}(z) \quad \text{for} \quad z \in M_t. \tag{4.1.4}$$

From the growth condition (a) in (2.3) for $g_t(\cdot, y_t)$ near y_t and that for $g_{0,1}(\cdot, y_t')$ near y_t', it follows easily that u_t extends smoothly across y_t, and we denote its smooth extension on M_t by the same symbol u_t.

Remark 4.1.1. It follows easily from Equation (4.1.3) and the continuity of $g_{0,1}$ that for a continuous section x_t of $\{M_t\}$ such that $x_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$,

$$\lim_{\substack{t \to 0 \\ t \in (\Delta^*)^{m+1}}} \tilde{g}_{t,y_t}(x_t) = \begin{cases} g_{0,1}(x_0, y_0), & \text{if } x_0 \in M_1^0, \\ 0, & \text{if } x_0 \in M_2^0. \end{cases}$$

(4.2) For $t \in (\Delta^*)^{m+1}$, let u_t be as in Equation (4.1.4). Then it follows from condition (b) in (2.3) for $g_t(\cdot, y_t)$ and $g_{0,1}(\cdot, y_t')$ that for $z \in M_t \backslash \{y_t\}$,

$$\Delta_t u_t(z) = -\frac{1}{\mathrm{Vol}(M_t, \omega_t)} - \phi_{1,t}(z)\phi_{1,t}(y_t) + \frac{\sqrt{-1}\partial\bar{\partial}\tilde{g}_{t,y_t}(z)}{\omega_t(z)}. \tag{4.2.1}$$

Here ratios of (1, 1)-forms make sense since M_t is one-dimensional.

LEMMA 4.2.1. *Let u_t be as in Equation (4.1.4). Then*

(i) $\int_{\mathrm{II}_{1,t}} |\Delta_t u_t|^2 \omega_t \to 0$, *and*

(ii) $\int_{\mathrm{II}_{2,t}} |\Delta_t u_t|^2 \omega_t \to 0$, *as* $t \to 0$.

Proof. By Equation (4.1.3) and condition (b) in (2.3) for $g_{0,1}(\cdot, y_t')$, we have

$$\sqrt{-1}\,\partial\bar{\partial}\tilde{g}_{t,y_t} = \begin{cases} \frac{i_{1,t}^*\omega_{0,1}}{V_{0,1}} & \text{on } \mathrm{II}_{1,t}\backslash\{y_t\}, \\ 0 & \text{on } \mathrm{II}_{2,t}. \end{cases} \tag{4.2.2}$$

To prove (i), we substitute Equations (3.6.2) and (4.2.2) into Equation (4.2.1), which gives

$$\begin{aligned} \Delta_t u_t(z) &= -\frac{1}{\mathrm{Vol}(M_t,\omega_t)} - \left(\frac{\tilde{\phi}_{1,t}(z)}{\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle} + \psi_t(z)\right)\left(\frac{\tilde{\phi}_{1,t}(y_t)}{\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle} + \psi_t(y_t)\right) \\ &\quad + \frac{i_{1,t}^*\omega_{0,1}}{V_{0,1}\cdot\omega_t(z)} \quad \text{on } \mathrm{II}_{1,t}\backslash\{y_t\}, \end{aligned} \tag{4.2.3}$$

where ψ_t is as in Equation (3.6.2). As in Equation (4.1.4), since both sides of Equation (4.2.3) are smooth on $\mathrm{II}_{1,t}$, Equation (4.2.3) actually holds on $\mathrm{II}_{1,t}$. From Equations (3.2.1), (3.2.3), Remark 3.2.1, Lemma 3.4.1, Remark 3.4.4, Propositions 3.6.1(ii) and 3.6.2(ii), it follows that shrinking Δ^{m+1} if necessary, there exists a constant $C_1 > 0$ such that

$$|\Delta_t u_t(z)| \le C_1 \quad \text{for all} \quad z \in \mathrm{II}_{1,t},\ t \in (\Delta^*)^{m+1}. \tag{4.2.4}$$

Let $j_{k,t} : W_{k,t} \to \mathrm{II}_{k,t}, k = 1, 2$, be as in (3.1). Then from Equation (4.2.3), we have, for $z \in W_{1,t} \subset M_1^0$,

$$\begin{aligned} (j_{1,t}^*\Delta_t u_t)(z) &\to -\frac{1}{V_{0,1}+V_{0,2}} - \frac{V_{0,2}}{V_{0,1}(V_{0,1}+V_{0,2})} + \frac{1}{V_{0,1}} \\ &= 0 \quad \text{as } t \to 0. \end{aligned} \tag{4.2.5}$$

Here the first term on the right-hand side of Equation (4.2.5) follows from Remark 3.4.4, the second term follows from Propositions 3.6.1(ii), 3.6.2(ii), Equation (3.2.1), Remark 3.2.1, and the last term follows from the continuity of $\{ds_t^2\}$. Obviously, one has

$$\int_{\mathrm{II}_{1,t}} |\Delta_t u_t|^2 \omega_t = \int_{W_{1,t}} \left(|j_{1,t}^*\Delta_t u_t|^2 \frac{j_{1,t}^*\omega_t}{\omega_{0,1}}\right)\omega_{0,1}. \tag{4.2.6}$$

The right-hand side of Equation (4.2.6) can be regarded as an integral over M_1^0 by letting the integrand to be zero on $M_1^0\backslash W_{1,t}$. From Lemma 3.4.1, Equations (4.2.4) and (4.2.5), it follows that the integrand on the right-hand side of Equation (4.2.6) is uniformly bounded from above and converges pointwise to zero on M_1^0 as $t \to 0$. Since $V_{0,1} < \infty$, it follows from the dominated convergence theorem that

$$\int_{\mathrm{II}_{1,t}} |\Delta_t u_t|^2 \omega_t \to 0 \quad \text{as } t \to 0, \tag{4.2.7}$$

and this finishes the proof of (i). Next we proceed to prove (ii). As in Equation (4.2.4), one easily sees from Equations (4.2.4) and (4.2.3) that there exists a constant $C_2 > 0$ such that

$$|\Delta_t u_t(z)| \leq C_2 \quad \text{for all} \quad z \in \mathrm{II}_{2,t},\ t \in (\Delta^*)^{m+1}. \tag{4.2.8}$$

Also, as in Equation (4.2.5), one can verify from Equations (4.2.2) and (4.2.3) that for $z \in W_{2,t} \subset M_2^0$,

$$\begin{aligned}(j_{2,t}^*\Delta_t u_t)(z) &\to -\frac{1}{V_{0,1}+V_{0,2}} + \frac{1}{V_{0,1}+V_{0,2}} + 0 \\ &= 0 \quad \text{as } t \to 0.\end{aligned} \tag{4.2.9}$$

Then one can use the dominated convergence theorem as in (i) to deduce from Equations (4.2.8) and (4.2.9) that

$$\begin{aligned}\int_{\mathrm{II}_{2,t}} |\Delta_t u_t|^2 \omega_t &= \int_{W_{2,t}} \left(|j_{2,t}^*\Delta_t u_t|^2 \frac{j_{2,t}^*\omega_t}{\omega_{0,2}} \right) \omega_{0,2} \\ &\to 0 \quad \text{as } t \to 0,\end{aligned}$$

which gives (ii). Thus we have finished the proof of Lemma 4.2.1. □

Recall from (2.1) the coordinate mappings $z_{i,k} : U_{i,k} \to \Delta$ near p_i, $i = 1, \ldots, m+1$, $k = 1, 2$, and let $\{z_t'\} \subset\subset M_k^0$ be a continuous curve. We shall need the following lemma:

LEMMA 4.2.2. *There exist constants $C_1, C_2 > 0$ such that for all $t \in (\Delta^*)^{m+1}$, $i = 1, \ldots, m+1$ and $k = 1, 2$,*

(i) $$|g_{0,k}(z_{i,k}, z_t')| \leq C_1 \max\{1, \log(-\log|z_{i,k}|)\}, \textit{ and} \tag{4.2.10}$$

(ii) $$\left|\frac{\partial g_{0,k}(z_{i,k}, z_t')}{\partial z_{i,k}}\right|, \left|\frac{\partial g_{0,k}(z_{i,k}, z_t')}{\partial \bar{z}_{i,k}}\right| \leq \frac{C_2}{|z_{i,k}||\log|z_{i,k}||} \quad \textit{on } U_{i,k}\backslash\{p_i\}. \tag{4.2.11}$$

Proof. Same as [9, lemma 4.2.2], and it follows from conditions (2.2.2), (2.2.3) for $\hat{\omega}_{0,k}$ and conditions (b), (e) in (2.3) for $g_{0,k}(\cdot,\cdot)$. □

Now we have

LEMMA 4.2.3. *Let u_t be as in Equation (4.1.4). Then for $i = 1, 2, \ldots, m+1$,*

$$\int_{\mathrm{I}_{i,t}} |\Delta_t u_t|^2 \omega_t \to 0 \quad \textit{as } t \to 0.$$

Proof. By Equation (4.1.3), we have, for $z \in \mathrm{I}_{i,t}$,

$$\sqrt{-1}\partial\bar{\partial}\tilde{g}_{t,y_t}(z) = \begin{cases} \sqrt{-1}\partial\bar{\partial}\big(\eta_{1,t}(z)g_{0,1}(\mathrm{pr}_{1,1,t}(z), y_t')\big), & i = 1, \\ \sqrt{-1}\partial\bar{\partial}\big(\eta_{i,t}(z)g_{0,1}(\mathrm{pr}_{i,1,t}(z), y_t') \\ \quad + (1-\eta_{i,t}(z))g_{0,1}(\mathrm{pr}_{i,2,t}(z), y_t')\big), & 2 \le i \le m_1+1, \\ 0, & m_1+2 \le i \le m+1. \end{cases} \tag{4.2.12}$$

As in Equation (4.2.3), by substituting Equation (3.6.2) into Equation (4.2.1), we have

$$\begin{aligned} \Delta_t u_t(z) &= -\frac{1}{\mathrm{Vol}(M_t, \omega_t)} - \left(\frac{\tilde{\phi}_{1,t}(z)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(z)\right)\left(\frac{\tilde{\phi}_{1,t}(y_t)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(y_t)\right) \\ &\quad + \frac{\sqrt{-1}\partial\bar{\partial}\tilde{g}_{t,y_t}(z)}{V_{0,1}\cdot\omega_t(z)} \\ &=: \tau_{1,t}(z) + \tau_{2,t}(z) \quad \text{on } \mathrm{I}_{i,t}, \end{aligned} \tag{4.2.13}$$

where

$$\tau_{1,t}(z) := -\frac{1}{\mathrm{Vol}(M_t, \omega_t)} - \left(\frac{\tilde{\phi}_{1,t}(z)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(z)\right)\left(\frac{\tilde{\phi}_{1,t}(y_t)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(y_t)\right),$$

and

$$\tau_{2,t}(z) := \frac{\sqrt{-1}\partial\bar{\partial}\tilde{g}_{t,y_t}(z)}{V_{0,1}\cdot\omega_t(z)} \quad \text{on } \mathrm{I}_{i,t}. \tag{4.2.14}$$

Here, ψ_t is as in Equation (3.6.2). For simplicity, we will only prove Lemma 4.2.3 for the case when $i = 2, \ldots, m_1 + 1$, since the calculations in the other cases are similar. First, using Remark 3.4.4, Equation (3.2.3), Propositions 3.6.1(ii), 3.6.2(ii), Equation (4.2.14) and the fact that $y_t \in \mathrm{II}_{1,t}$, it is easy to see that there exist constants $C_1, C_2 > 0$ such that

$$|\tau_{1,t}(z)| \le C_1 + C_2|\psi_t(z)| \quad \text{for} \quad z \in \mathrm{I}_{i,t}. \tag{4.2.15}$$

Then it follows that

$$\begin{aligned} \int_{\mathrm{I}_{i,t}} |\tau_{1,t}(z)|^2\omega_t &\le \int_{\mathrm{I}_{i,t}} |\, C_1 + C_2|\psi_t(z)|\,|^2\,\omega_t \quad \text{(by (4.2.15))} \\ &\le 2C_1^2\int_{\mathrm{I}_{i,t}}\omega_t + 2C_2^2\|\psi_t\|_2^2 \\ &\to 0 \quad \text{as } t \to 0 \\ &\quad \text{(by Propositions 3.3.2(i) and 3.6.2(i))}. \end{aligned} \tag{4.2.16}$$

For fixed i with $2 \le i \le m_1+1$ and $t = (t_1, \dots, t_{m+1}) \in (\Delta^*)^{m+1}$, we consider the parametrization of $\mathrm{I}_{i,t}$ in (3.3.5) given by $\zeta(= a+ib) = \log z_{i,1}/\log|t_i|$, $(1/2)-2\delta < a < (1/2)+2\delta$, $0 \le b < 2\pi/|\log|t_i||$. Recall also from Equation (3.1.1) that one has $z_{i,2} = t_i/z_{i,1}$ for $z = (t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \in \mathrm{I}_{i,t}$. By Equation (4.2.12) and condition (b) in (2.3) for $g_{0,1}(\cdot, y_t')$, one easily sees that for $2 \le i \le m_1+1$, $t = (t_1, \dots, t_{m+1}) \in (\Delta^*)^{m+1}$ and $z = (t_1, \dots, t_{i-1}, z_{i,1}, z_{i,2}, t_{i+1}, \dots, t_{m+1}) \in \mathrm{I}_{i,t}$,

$$\begin{aligned}\sqrt{-1}\partial\bar\partial \tilde g_{t,y_t}(z) &= \sqrt{-1}\partial\bar\partial\eta_{i,t}(z)\big(g_{0,1}(z_{i,1}, y_t') - g_{0,1}(z_{i,2}, y_t')\big)\\ &\quad + \sqrt{-1}\partial\eta_{i,t}(z)\wedge\big(\bar\partial g_{0,1}(z_{i,1}, y_t') - \bar\partial g_{0,1}(z_{i,2}, y_t')\big)\\ &\quad - \sqrt{-1}\bar\partial\eta_{i,t}(z)\wedge\big(\partial g_{0,1}(z_{i,1}, y_t') - \partial g_{0,1}(z_{i,2}, y_t')\big)\\ &\quad + \frac{\eta_{i,t}(z)\omega_{0,1}(z_{i,1})}{V_{0,1}} + \frac{(1-\eta_{i,t}(z))\omega_{0,1}(z_{i,2})}{V_{0,1}}. \qquad (4.2.17)\end{aligned}$$

It is easy to see from Equation (3.2.2) that there exists a constant $C_3 > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$|\partial_\zeta\eta_{i,t}|,\ |\partial_{\bar\zeta}\eta_{i,t}|,\ |\partial_\zeta\partial_{\bar\zeta}\eta_{i,t}| \le C_3 \quad \text{on } \mathrm{I}_{i,t}. \qquad (4.2.18)$$

By Lemma 4.2.2(ii), we have, for $t \in (\Delta^*)^{m+1}$,

$$\begin{aligned}\partial_\zeta g_{0,1}(z_{i,1}, y_t')| &= |\partial_{z_{i,1}} g_{0,1}(z_{i,1}, y_t')\cdot(\partial z_{i,1}/\partial\zeta)|\\ &\le \frac{C_4}{|z_{i,1}||\log|z_{i,1}||}\cdot|z_{i,1}||\log|t_i||\\ &\le C_5 \quad \text{on } \mathrm{I}_{i,t} \quad \text{(cf. (3.3.4))}. \qquad (4.2.19)\end{aligned}$$

Here the constants $C_4, C_5 > 0$ are independent of t. One can easily see that similar inequality also holds for $\partial_{\bar\zeta} g_{0,1}(z_{i,1}, y_t')$, $\partial_\zeta g_{0,1}(z_{i,2}, y_t')$, $\partial_{\bar\zeta} g_{0,1}(z_{i,2}, y_t')$ on $\mathrm{I}_{i,t}$. Then by Equations (3.3.6), (4.2.3), (4.2.14), (4.2.18), (4.2.19), one easily sees that there exists a constant $C_6 > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$|\tau_{2,t}(z)| \le C_6 \quad \text{for all} \quad z \in \mathrm{I}_{i,t}. \qquad (4.2.20)$$

Then, using Equation (4.2.20), one can proceed as Equation (3.3.7) to show that there exists a constant $C_7 > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$\begin{aligned}\int_{\mathrm{I}_{i,t}} |\tau_{2,t}|^2\omega_t &\le C_7\frac{|\log(-\log|t_i|)|^2}{|\log|t_i||}\\ &\to 0 \quad \text{as } t \to 0. \qquad (4.2.21)\end{aligned}$$

Thus for $i = 2, \dots, m_1+1$, we have

$$\int_{\mathrm{I}_{i,t}} |\Delta_t u_t|^2\omega_t = \int_{\mathrm{I}_{i,t}} |\tau_{1,t} + \tau_{2,t}|^2\omega_t \quad \text{(by (4.2.13))}$$

$$\leq \; 2\int_{\mathrm{I}_{i,t}} |\tau_{1,t}|^2\omega_t + 2\int_{\mathrm{I}_{i,t}} |\tau_{2,t}|^2\omega_t$$
$$\to \; 0 \quad \text{as } t \to 0 \quad \text{(by (4.2.16) and (4.2.21))}. \tag{4.2.22}$$

Similarly, one can show that for $i = 1$ and $i = m_1+2, \ldots, m+1$, $\int_{\mathrm{I}_{i,t}} |\Delta_t u_t|^2\omega_t \to 0$ as $t \to 0$, and this finishes the proof of Lemma 4.2.3. □

Now we summarize our discussion in (4.2) in the following

PROPOSITION 4.2.4. *Let u_t be as in Equation (4.1.4). Then $\|\Delta_t u_t\|_2 \to 0$ as $t \to 0$.*

Proof. We have

$$\|\Delta_t u_t\|_2^2 \;=\; \sum_{1\leq i\leq m+1} \int_{\mathrm{I}_{i,t}} |\Delta_t u_t|^2\omega_t + \int_{\mathrm{II}_{1,t}} |\Delta_t u_t|^2\omega_t + \int_{\mathrm{II}_{2,t}} |\Delta_t u_t|^2\omega_t$$
$$\to \; 0 \quad \text{as } t \to 0 \quad \text{(by Lemmas 4.2.1 and 4.2.3)}.$$

(4.3) Notation is as before. Let u_t be as in Equation (4.1.4), $\phi_{0,t} (= 1/\sqrt{\mathrm{Vol}(M_t, \omega_t)})$ be as in (3.5), and let $\phi_{1,t}$ be as in Theorem 2.4.1. We have

PROPOSITION 4.3.1.

$$\langle u_t, \phi_{0,t}\rangle \to 0 \quad \text{as } t \to 0. \tag{4.3.1}$$

Proof. Using condition (c) in (3.2) for $g_t(\cdot, y_t)$ and the fact that $\phi_{0,t}$ is orthogonal to $\phi_{1,t}$, we have, for all $t \in (\Delta^*)^{m+1}$,

$$\int_{M_t} \left(g_t(z, y_t) - \frac{1}{\lambda_{1,t}}\phi_{1,t}(z)\phi_{1,t}(y_t) \right) \phi_{0,t}(z)\omega_t = 0. \tag{4.3.2}$$

Let $\tilde{g}_{t,y_t}$ be as in Equation (4.1.3). Using Lemma 4.2.2(i) and Equation (3.3.4), one can then proceed as in [9, proposition 4.2.3(ii)] to show that for $1 \leq i \leq m+1$,

$$\int_{\mathrm{I}_{i,t}} \tilde{g}_{t,y_t}(z)\omega_t \to 0 \quad \text{as } t \to 0. \tag{4.3.3}$$

Also, using the dominated convergence theorem, the continuity of $\{ds_t^2\}$, condition (c) in (3.2) for $g_{0,1}(z, y_t')$, Proposition 3.3.1, Lemmas 3.4.1 and 4.2.2(i), one can proceed as in [9, proposition 4.3.2(iii)] to show that

$$\int_{\mathrm{II}_{1,t}} \tilde{g}_{t,y_t}(z)\omega_t \to 0 \quad \text{as } t \to 0. \tag{4.3.4}$$

Since $\mathrm{supp}(\tilde{g}_{t,y_t}) \subset (\sqcup_{1\leq i\leq m_1+1}\mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t}$ (cf. Equation (4.1.3)), one can combine Equations (4.3.3), (4.3.4) and Remark 3.4.4 to get

$$\int_{M_t} \tilde{g}_{t,y_t}(z)\phi_{0,1}(z)\omega_t$$

$$= \frac{1}{\sqrt{\mathrm{Vol}(M_t,\omega_t)}}\left(\sum_{1\le i\le m_1+1}\int_{\mathrm{I}_{i,t}} + \int_{\mathrm{II}_{1,t}}\right)\tilde{g}_{t,y_t}(z)\omega_t$$

$$\to \frac{1}{\sqrt{V_{0,1}+V_{0,2}}}\cdot 0 = 0 \quad \text{as } t\to 0. \tag{4.3.5}$$

Then Equation (4.3.1) follows from Equations (4.1.4), (4.3.2) and (4.3.5) by letting $t \to 0$, and we have finished the proof of Proposition 4.3.1. □

Also we have:

PROPOSITION 4.3.2.

$$\langle u_t, \phi_{1,t}\rangle \to 0 \; as \; t \to 0. \tag{4.3.6}$$

Proof. From the self-adjointedness of Δ_t and the identity $\Delta_t\phi_{1,t} = \lambda_{1,t}\phi_{1,t}$, we have, for all $t \in (\Delta^*)^{m+1}$,

$$\int_{M_t}\left(g_t(z,y_t) - \frac{1}{\lambda_{1,t}}\phi_{1,t}(z)\phi_{1,t}(y_t)\right)\phi_{1,t}(z)\omega_t$$

$$= \frac{1}{\lambda_{1,t}}\int_{M_t} g_t(z,y_t)\Delta_t\phi_{1,t}(z)\omega_t - \frac{\phi_{1,t}(y_t)}{\lambda_{1,t}}\int_{M_t}(\phi_{1,t}(z))^2\omega_t$$

$$= \frac{1}{\lambda_{1,t}}\left(\phi_{1,t}(y_t) - \int_{M_t}\phi_{1,t}(z)\hat{\omega}_t\right) - \frac{\phi_{1,t}(y_t)}{\lambda_{1,t}}$$

(by condition (b) in (3.2) for $g_t(\cdot, y_t)$ and $\|\phi_{1,t}\|_2 = 1$)

$$= 0 \quad (\text{since } \langle\phi_{1,t},\phi_{0,t}\rangle = 0). \tag{4.3.7}$$

Let $\tilde{\phi}_{1,t}$ be as in (3.2.3). It is easy to check that there exists a constant $C > 0$ such that $|\tilde{\phi}_{1,t}(z)| \le C$ for all $z \in M_t$ and $t \in (\Delta^*)^{m+1}$. Then as in Equation (4.3.3), one can use Lemma 4.2.2(i), Equation (3.3.4) and proceed as in [9, proposition 4.2.3(ii)] to show that for $i = 1,\ldots,m_1+1$,

$$\int_{\mathrm{I}_{i,t}} \tilde{g}_{t,y_t}(z)\tilde{\phi}_{1,t}(z)\omega_t \to 0 \quad \text{as } t \to 0. \tag{4.3.8}$$

From Equations (3.2.2) and (3.2.3), one easily checks that $\tilde{\phi}_{1,t}(z) = \sqrt{V_{0,2}/V_{0,1}(V_{0,1}+V_{0,2})}$ on $\mathrm{II}_{1,t}$ for $|t|$ sufficiently small. Thus by Equation (4.3.4), we also have

$$\int_{\mathrm{II}_{1,t}} \tilde{g}_{t,y_t}(z)\tilde{\phi}_{1,t}(z)\omega_t \to 0 \quad \text{as } t \to 0. \tag{4.3.9}$$

Since $\mathrm{supp}(\tilde{g}_{t,y_t}) \subset (\sqcup_{1\le i\le m_1+1}\mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t}$ (cf. Equation (4.1.3)) and $|\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle| \ge C > 0$ for some constant C independent of t (cf. Proposition 3.6.1(ii)), it follows from Equations (4.3.8) and (4.3.9) that

$$\int_{M_t} \tilde{g}_{t,y_t}(z)\frac{\tilde{\phi}_{1,t}(z)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle}\omega_t \to 0 \quad \text{as } t \to 0. \tag{4.3.10}$$

Using Lemma 4.2.2(i), one can easily adapt the argument in [9, proposition 4.2.3(i)] to show that for $1 \le i \le m_1 + 1$ and $t = (t_1, \ldots, t_{m+1}) \in (\Delta^*)^{m+1}$,

$$\begin{aligned}\int_{\mathrm{I}_{i,t}} (\tilde{g}_{t,y_t}(z))^2\omega_t \;&\le\; C_1\frac{|\log(-\log|t_i|)|^2}{|\log|t_i||}\\ &\to\; 0 \quad \text{as } t \to 0.\end{aligned} \tag{4.3.11}$$

Also, using the continuity of $\{y_t'\}$, the growth condition (a) in (2.3) for $g_{0,1}(\cdot, y_t')$ near y_t' and Lemma 4.2.2(i), one can easily adapt the argument in [9, proposition 4.3.2(iii)] to show that for all $t \in (\Delta^*)^{m+1}$,

$$\begin{aligned}&\int_{\mathrm{II}_{1,t}} (\tilde{g}_{t,y_t}(z))^2\omega_t\\ &\quad\le C_2 + C_3\int_0^{2\pi}\int_0^{1/2}(\log r^2)^2 r\,dr\,d\theta\\ &\qquad + C_4\int_0^{2\pi}\int_0^{1/2}(\log(-\log r))^2\cdot\frac{r\,dr\,d\theta}{r^2(\log r)^2}\\ &\quad\le C_5.\end{aligned} \tag{4.3.12}$$

Since $\mathrm{supp}(\tilde{g}_{t,y_t}) \subset (\sqcup_{1\le i\le m_1+1}\mathrm{I}_{i,t}) \sqcup \mathrm{II}_{1,t}$, it follows from Equation (4.3.11) and (4.3.12) that

$$\|\tilde{g}_{t,y_t}(z)\|_2 \le C_6 \quad \text{for all } t \in (\Delta^*)^{m+1}. \tag{4.3.13}$$

Here the constants $C_1, C_2, C_3, C_4, C_5, C_6 > 0$ are all independent of t. Let ψ_t be as in Equation (3.6.2). Then

$$\begin{aligned}\left|\int_{M_t} \tilde{g}_{t,y_t}(z)\psi_t(z)\omega_t\right| \;&\le\; \|\tilde{g}_{t,y_t}(z)\|_2\cdot\|\psi_t\|_2\\ &\le\; C_6\|\psi_t\|_2 \quad \text{(by (4.3.13))}\\ &\to\; 0 \quad \text{as } t \to 0 \quad \text{(by Proposition 3.6.2(i))}.\end{aligned} \tag{4.3.14}$$

Then one has

$$\begin{aligned}
&\int_{M_t} \tilde{g}_{t,y_t}(z)\phi_{1,t}(z)\omega_t \\
&\quad = \int_{M_t} \tilde{g}_{t,y_t}(z)\left(\frac{\tilde{\phi}_{1,t}(z)}{\langle\tilde{\phi}_{1,t},\phi_{1,t}\rangle} + \psi_t(z)\right)\omega_t \quad \text{(by (3.6.2))} \\
&\quad \to 0 \quad \text{as } t \to 0 \quad \text{(by (4.3.10) and (4.3.14)).} \qquad (4.3.15)
\end{aligned}$$

Then Equation (4.3.6) follows easily from Equations (4.1.4), (4.3.7) and (4.3.15), and thus we have finished the proof of Proposition 4.3.2. □

(4.4) Finally we are ready to give the proof of Theorem 2.4.1 as follows:

Proof of Theorem 2.4.1. Let $\{M_t\}$, $p_1, \ldots, p_{m+1}, x_t, y_t, \lambda_{1,t}, \phi_{1,t}$ be as in Theorem 2.4.1. First we consider the case when $y_0 \in M_1^0$. Let $\phi_{0,t} = 1/\sqrt{\mathrm{Vol}(M_t, \omega_t)}$ on M_t be as before, and let u_t be as in Equation (4.1.4). For $t \in (\Delta^*)^{m+1}$, we define the smooth function given by

$$w_t := u_t - \langle u_t, \phi_{0,t}\rangle\phi_{0,t} - \langle u_t, \phi_{1,t}\rangle\phi_{1,t} \quad \text{on } M_t. \qquad (4.4.1)$$

From Equation (4.4.1) and the self-adjointedness of Δ_t, one easily checks that

$$\begin{aligned}
\Delta_t w_t &= \Delta_t u_t - 0 - \langle u_t, \phi_{1,t}\rangle\lambda_{1,t}\phi_{1,t} \\
&= \Delta_t u_t - \langle \Delta_t u_t, \phi_{1,t}\rangle\phi_{1,t} \quad \text{on } M_t. \qquad (4.4.2)
\end{aligned}$$

From Equation (4.4.2), it is easy to check that

$$\begin{aligned}
\|\Delta_t w_t\|_2^2 &= \|\Delta_t u_t\|_2^2 - |\langle \Delta_t u_t, \phi_{1,t}\rangle|^2 \\
&\leq \|\Delta_t u_t\|_2^2. \qquad (4.4.3)
\end{aligned}$$

From Equation (4.4.1), it is easy to see that w_t is orthogonal to the eigenfunctions $\phi_{0,t}$ and $\phi_{1,t}$. Then one can use Parseval's identity and proceed as in Equation (3.5.6) to show that

$$\|w_t\|_2^2 \leq \frac{1}{\alpha^2}\|\Delta_t w_t\|_2^2 \quad \text{for all } t \in (\Delta^*)^{m+1}, \qquad (4.4.4)$$

where $\alpha > 0$ is the constant in Lemma 3.5.1. Since $x_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$, one can proceed as in Equation (3.5.8) in Proposition 3.5.3 using the standard Nash–Moser iteration technique (cf., e.g., [3, theorem 8.24]) to show that there exists a constant $C > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$\begin{aligned}
|w_t(x_t)| &\leq C(\|w_t\|_2 + \|\Delta_t w_t\|_2) \\
&\leq C\left(\frac{1}{\alpha}\|\Delta_t w_t\|_2 + \|\Delta_t w_t\|_2\right) \quad \text{(by (4.4.4))} \\
&\leq C\left(\frac{1}{\alpha} + 1\right)\|\Delta_t u_t\|_2 \quad \text{(by (4.4.3))} \\
&\to 0 \quad \text{as } t \to 0 \quad \text{(by Proposition 4.2.4).} \qquad (4.4.5)
\end{aligned}$$

Also, since $x_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$, one has

$$\begin{aligned}
&|\langle u_t, \phi_{1,t}\rangle \phi_{1,t}(x_t)| \\
&\quad \le |\langle u_t, \phi_{1,t}\rangle| \left(\left| \frac{\tilde{\phi}_{1,t}(x_t)}{\langle \tilde{\phi}_{1,t}, \phi_{1,t}\rangle} \right| + |\psi_t(x_t)| \right) \quad \text{(by (3.6.2))} \\
&\quad \le C|\langle u_t, \phi_{1,t}\rangle| \quad \text{(by (3.2.3), Propositions 3.6.1(ii) and 3.6.2(ii))} \\
&\quad \to 0 \quad \text{as } t \to 0 \quad \text{(by Proposition 4.3.2).} \qquad (4.4.6)
\end{aligned}$$

Here the constant $C > 0$ does not depend on t. Similarly, it follows from Remark 3.4.4 and Proposition 4.3.1 that

$$\begin{aligned}
\langle u_t, \phi_{0,t}\rangle \phi_{0,t}(x_t) &= \langle u_t, \phi_{0,t}\rangle \frac{1}{\sqrt{\mathrm{Vol}(M_t, \omega_t)}} \\
&\to 0 \frac{1}{\sqrt{V_{0,1} + V_{0,2}}} \\
&= 0 \quad \text{as } t \to 0. \qquad (4.4.7)
\end{aligned}$$

Then one has, from Equation (4.4.1),

$$\begin{aligned}
u_t(x_t) &= w_t(x_t) + \langle u_t, \phi_{0,t}\rangle \phi_{0,t}(x_t) + \langle u_t, \phi_{1,t}\rangle \phi_{1,t}(x_t) \\
&\to 0 + 0 + 0 \quad \text{(by (4.4.5), (4.4.6) and (4.4.7))} \\
&= 0 \quad \text{as } t \to 0. \qquad (4.4.8)
\end{aligned}$$

Finally, Theorem 2.4.1 in the case when $y_0 \in M_1^0$ follows readily from Remark 4.1.1, Equations (4.1.4) and (4.4.8). It is clear that Theorem 2.4.1 in the case when $y_0 \in M_2^0$ can be proved similarly, and thus we have finished the proof of Theorem 2.4.1. □

5. Deduction of Corollaries 2.4.2 and 2.4.3

(5.1) First we deduce Corollary 2.4.2 as follows:

Proof of Corollary 2.4.2. Let $\lambda_{1,t}$ and l_t be as in Theorem 2.4.1 and Corollary 2.4.2 respectively, and denote by $\lambda_{1,t}^{\mathrm{hyp}}$ and l_t^{hyp} the corresponding objects on M_t with respect to $ds^2_{\mathrm{hyp},t}$. It follows from results in [8] that there exist constants $C_1, C_2 > 0$ such that

$$C_1 l_t^{\mathrm{hyp}} \le \lambda_{1,t}^{\mathrm{hyp}} \le C_2 l_t^{\mathrm{hyp}} \quad \text{for all } t \in (\Delta^*)^{m+1}. \qquad (5.1.1)$$

Also, from Equation (2.2.4) and the minimax principle, there exist constants $C_3, C_4, C_5, C_6 > 0$ such that for all $t \in (\Delta^*)^{m+1}$,

$$C_3 l_t^{\mathrm{hyp}} \le l_t \le C_4 l_t^{\mathrm{hyp}}, \quad \text{and} \quad C_5 \lambda_{1,t}^{\mathrm{hyp}} \le \lambda_{1,t} \le C_6 \lambda_{1,t}^{\mathrm{hyp}}. \qquad (5.1.2)$$

In our separating node case, it is well known that $l_t^{\mathrm{hyp}} \to 0$ as $t \to 0$, and thus by Equations (5.1.1) and (5.1.2),

$$\lambda_{1,t} \to 0 \quad \text{as } t \to 0. \tag{5.1.3}$$

Multiplying both sides of Equation (2.4.1) by $\lambda_{1,t}$, one easily sees from Equation (5.1.3) and Theorem 2.4.1 that

$$\lim_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} \big(\lambda_{1,t} g_t(x_t, y_t) - \phi_{1,t}(x_t)\phi_{1,t}(y_t)\big) = 0. \tag{5.1.4}$$

Since $x_0, y_0 \notin \{p_1, p_2, \ldots, p_{m+1}\}$, one has

$$\begin{aligned}
&\lim_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} \phi_{1,t}(x_t)\phi_{1,t}(y_t)\\
&\quad = \lim_{\substack{t\to 0\\ t\in(\Delta^*)^{m+1}}} \left(\frac{\tilde{\phi}_{1,t}(x_t)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(x_t)\right)\left(\frac{\tilde{\phi}_{1,t}(y_t)}{\langle\tilde{\phi}_{1,t}, \phi_{1,t}\rangle} + \psi_t(y_t)\right)\\
&\qquad \text{(by (3.6.2))}\\
&\quad = \phi_{1,0}(x_0)\phi_{1,0}(y_0)\\
&\qquad \text{(by Remark 3.2.1, Propositions 3.6.1(ii) and 3.6.2(ii))}.
\end{aligned} \tag{5.1.5}$$

Then Corollary 2.4.2(a) follows readily from Equations (3.2.1), (5.1.4) and (5.1.5). Also Corollary 2.4.2(b) follows easily from Corollary 2.4.2(a) and Equation (5.1.2). □

Finally, we have

Proof of Corollary 2.4.3. Let $\{ds_t^2\}$ be as in Corollary 2.4.3. By Schwarz Lemma of Yau [13], the curvature condition on $\{ds_t^2\}$ implies that $\{ds_t^2\}$ satisfies Equation (2.2.4). Together with the hypothesis on ds_0^2 on M^0, it follows that $\{ds_t^2\}$ forms a continuous family of quasi-hyperbolic metrics on $\{M_t\}$, and Corollary 2.4.3 follows immediately from Theorem 2.4.1 and Corollary 2.4.2. □

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