Extension Classes in Adelic Language

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In this paper, we give an adelic interpretation of extension classes of locally free sheaves over curves. This may be viewed as an effective version of the classical cohomology approach for the extension classes of Grothendieck. While the discussion works over any base field, we limit our discussion over the finite field $\mathbb{F}_q$.

1 Locally Free Sheaves in Adelic Language

We start with a quick review of the standard correspondence between the moduli stacks of locally free sheaves on curves and adelic quotients associated to general linear groups. For details, please refer to (the Appendix A of) [2].

Let X be an integral, regular, projective curve of genus g over a finite field $\mathbb{F}_q$. Denote by F the field of rational functions of X, by $\mathbb{A}$ the adelic ring of F, and by $\mathcal{O}$ the integer ring of $\mathbb{A}$.

Let $\mathcal{M}_{X,r}$ be the moduli stack of locally free sheaves of rank r on X. There is a natural identification among elements of $\mathcal{M}_{X,r}$ and the adelic quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O})$ which we denote by

$$\phi:\,\mathcal{M}_{X,r}\,\simeq\,\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O}).$$

Indeed, if $\mathcal{E}$ is a rank r locally free sheaf on X, the fiber E_η of $\mathcal{E}$ over the generic point η of X is an F-linear space of dimension r. Among its $\mathrm{GL}_r(F)$-equivalence class, fix an F-linear isomorphism

$$\xi: E_\eta \longrightarrow F^r. \tag{1}$$

Then there exists a dense open subset U of X such that ξ induces a trivialization $\xi_U:\mathcal{E}\simeq\mathcal{O}_U^{\oplus r}$. of $\mathcal{E}$ on U

To work locally, let x be a closed point of X, and denote its formal neighborhood by $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$, where $\widehat{\mathcal{O}}_x$ is the x-adic completion of the local ring $\mathcal{O}_{X,x}$. Denote by $\widehat{F}_x$ the fraction field of $\widehat{\mathcal{O}}_x$. Since X is separable, the natural inclusion $F\hookrightarrow\widehat{F}_x$ induces a natural morphism $\widehat{\iota}_x:\mathrm{Spec}(\widehat{\mathcal{O}}_x)\longrightarrow X$ such that the pull-back $\widehat{\mathcal{E}}_x:=\widehat{\iota}_x^*\mathcal{E}$ of $\mathcal{E}$ on $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ is locally free of rank r. Hence, the fiber $\widehat{E}_{x,\eta}$ of $\widehat{\mathcal{E}}_x$ over the generic point η of $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ becomes a natural $\widehat{F}_x$-linear space of dimension r. Furthermore, since $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ is affine, for the rank r locally free sheaf $\widehat{\mathcal{E}}_x$, there exists a free $\widehat{\mathcal{O}}_x$- module $\widehat{E}_x^\sim$ of rank r such that $\widehat{\mathcal{E}}_x\simeq\widetilde{\widehat{E}_x^\sim}$ and $\widehat{E}_x^\sim\otimes_{\widehat{\mathcal{O}}_x}\widehat{F}_x=\widehat{E}_{x,\eta}$, where $\widetilde{\widehat{E}_x^\sim}$ denotes the locally free sheaf associated to $\widehat{E}_x^\sim$. Therefore, induced from the free $\widehat{\mathcal{O}}_x$-module structure on $\widehat{E}_x^\sim$, there is a natural isomorphism

$$\widetilde{\xi}_x:\,\widehat{E}_{x,\eta}=\widehat{E}_x^\sim\otimes_{\widehat{\mathcal{O}}_x}\widehat{F}_x\simeq\widehat{F}_x^r. \tag{2}$$

On the other hand, induced from $F \hookrightarrow \widehat{F}_x$, there is a natural identification $E_\eta \otimes_F \widehat{F}_x = \widehat{E}_{x,\eta}$. Hence, using ξ in (1) and the natural inclusion $F \hookrightarrow \widehat{F}_x$, we obtain yet another isomorphism

$$\widehat{\xi}_x : \widehat{E}_{x,\eta} = E_\eta \otimes_F \widehat{F}_x \simeq \widehat{F}_x^r. \tag{3}$$

The two isomorphisms $\widetilde{\xi}_x$ and $\widehat{\xi}_x$ in (2) and (3), respectively, can be connected by a unique element $\widehat{g}_x \in \mathrm{GL}_r(\widehat{F}_x)/\mathrm{GL}_r(\widehat{\mathcal{O}}_x)$ characterized by the following commutative diagram

$$\begin{array}{ccccc} E_\eta \otimes_F \widehat{F}_x & = \widehat{E}_{x,\eta} = & \widehat{E}_x^{\sim} \otimes_{\widehat{\mathcal{O}}_x} \widehat{F}_x \\ \widehat{\xi}_x \Big\downarrow \simeq & & \simeq \Big\downarrow \widetilde{\xi}_x \\ \widehat{F}_x^r & \overset{g_x}{\simeq} & \widehat{F}_x^r. \end{array}$$

In this way, we obtain an element $(g_x) \in \prod_{x\in X} \mathrm{GL}_x(\widehat{F}_x)$. Furthermore, this element (g_x) belongs to $\mathrm{GL}_r(\mathbb{A})$. Indeed, by (1), for $x \in U$, $\widetilde{\widehat{E}_x^{\sim}} \simeq \widehat{\mathcal{O}}_x^r$. This implies that, for such x, g_x may be choosen to be the identical matrix $I_r := \mathrm{diag}(1, 1, \ldots, 1)$. We denote the adelic class of (g_x) constructed above by $g_{\mathcal{E}}$ for later use.

Conversely, for a class in $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O})$, there exists a dense open subset U of X such that g_x are identity matrix for all $x \in U$. This yields an trivial locally free sheaf $\mathcal{O}_U^{\oplus r}$ on U. On the other hand, for a closed point x of $X \smallsetminus U$, the image $g_x^{-1}(\widehat{\mathcal{O}}_x^{\oplus r})$ is a full rank $\widehat{\mathcal{O}}_x$-module contained in $\widehat{F}_x^r$ since $g_x \in \mathrm{GL}_x(\widehat{F}_x)$, and hence induces a locally free sheaf $\widetilde{g_x^{-1}(\widehat{\mathcal{O}}_x^{\oplus r})}$ of rank r on $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$. All these can be used to construct the locally free sheaf $\mathcal{E}_{(g_x)}$ of rank r on X by the so-called fpqc-gluing using the components of (g_x). It is not difficult to see that the adelic class $g_{\mathcal{E}_{(g_x)}}$ coincides with (that of) (g_x).

2 Extension Classes: Classical Approach

Now, we recall the classical approach to extension classes ([1]).

Let

$$\mathbb{E} : \qquad 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \to 0 \tag{4}$$

be a short exact sequence of locally free sheaves on the curve X. Applying the operator $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$ to $\mathbb{E}$ leads to a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1). \tag{5}$$

Following Grothendieck, the extension $\mathbb{E}$, up to isomorphism, is uniquely determined by the δ-image in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1)$ of the identity map $\mathrm{Id}_{\mathcal{E}_3}$ of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3)$. In addition,

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) \simeq H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1), \tag{6}$$

where, for a locally free sheaf $\mathcal{E}$, $\mathcal{E}^\vee$ denotes its its dual sheaf. Note that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3) \simeq H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3),$$

(5) is equivalent to the following long exact sequence of cohomology groups

$$0\to H^0(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1)\to H^0(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_2)\to H^0(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_3)\xrightarrow{\delta} H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1). \quad (7)$$

Furthermore, we obtain naturally the following isomorphism and the decomposition

$$\mathcal{E}_3^\vee\otimes\mathcal{E}_3=\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3)\simeq\mathcal{O}_X\oplus\mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3).$$

Here $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3)$ denotes the sheaf of endmorphisms of $\mathcal{E}_3$ and $\mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3)$ denotes the sub-sheaf of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3)$ resulting from the so-called trace zero endmorphisms. Since $H^0(X,\mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3))=0$, the morphism δ in (5) is equivalent to the induced morphism

$$\delta: H^0(X,\mathcal{O}_X)\longrightarrow H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1), \quad (8)$$

and the extension $\mathbb{E}$, up to equivalence, is uniquely determined by $\delta(1)$, the δ-image of the unit element 1 of $H^0(X,\mathcal{O}_X)$. Here, as usual, if

$$\mathbb{E}': \qquad 0\to\mathcal{E}_1\longrightarrow\mathcal{E}_2'\longrightarrow\mathcal{E}_3\to 0$$

is another extension of $\mathcal{E}_3$ by $\mathcal{E}_1$, and there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_3 & \to & 0\\
 & & \| & & \psi\big\downarrow\simeq & & \| & & \\
0 & \to & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{E}_3 & \to & 0
\end{array}$$

then ϕ is called an equivalence between two extensions $\mathbb{E}$ and $\mathbb{E}'$ of of $\mathcal{E}_3$ by $\mathcal{E}_1$. Normally, we denote an equivalence by $\psi:\mathbb{E}\simeq\mathbb{E}'$.

3 Extension Classes: Adelic Descriptions

We first give a more concrete local description of the boundary map

$$\delta: H^0(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_3)\to H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1). \quad (9)$$

For this purpose, we review the adelic interpretation of $H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1)$. Denote by $g_{\mathcal{E}}\in\mathrm{GL}_r(\mathbb{A})$ an adelic representor associated to a rank r locally free sheaf $\mathcal{E}$ introduced in §1. Let r_i the rank of $\mathcal{E}_i$ $(i=1,2,3)$. Then

$$H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1)=\mathbb{A}^{r_1r_3}\big/(\mathbb{A}^{r_1r_3}(g_{\mathcal{E}_3^\vee\otimes\mathcal{E}_1})+F^{r_1r_3}), \quad (10)$$

where

$$\mathbb{A}^{r_1r_3}(g_{\mathcal{E}_3^\vee\otimes\mathcal{E}_1});=\Big\{a\in\mathbb{A}^{r_1r_3}: g_{\mathcal{E}_3^\vee\otimes\mathcal{E}_1}(a)\in\mathcal{O}^{r_1r_3}\Big\}. \quad (11)$$

By (5) and (7), it is not difficult to see that this space is isomorphic to

$$\prod\nolimits'_{x\in X}\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})\Big/\Big(\prod_{x\in X}\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim},\widehat{E}_{1,x}^{\sim})+\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta})\Big). \quad (12)$$

Here $\prod'$ denotes the restrict product of $\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})$ with respect to $\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim},\widehat{E}_{1,x}^{\sim})$. Hence, we should find a natural morphism from $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}_3,\mathcal{E}_3)$ to (12) which gives the boundary map (5) for the extension classes.

By applying $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$, or the same $\mathcal{E}_3^\vee \otimes$, to the extension $\mathbb{E}$, we obtain a short exact sequence of locally free sheaves

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \to 0, \tag{13}$$

since the the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$ and $\mathcal{E}_3^\vee \otimes$ are left and right exactness, respectively. Furthermore, by applying the derived functor of $\Gamma(X, \cdot)$ to (13), we arrive at the long exact sequence (5), namely,

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1). \tag{14}$$

This boundary map can be simply constructed using the following well-known

Lemma 1 (Snake Lemma)**.** *Let R be a commutative ring. Assume that*

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \to & 0 \\ & & \phi_1 \big\downarrow & & \big\downarrow \phi_2 & & \big\downarrow \phi_3 & & \\ 0 & \to & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \to & 0 \end{array} \tag{15}$$

is a commutative diagram of R-modules with exact rows. Then, for the kernel and cockerel of ϕ_i, there is a long exact sequence

$$0 \to \mathrm{Ker}(\phi_1) \to \mathrm{Ker}(\phi_2) \to \mathrm{Ker}(\phi_3) \xrightarrow{\delta} \mathrm{Coker}(\phi_1) \to \mathrm{Coker}(\phi_2) \to \mathrm{Coker}(\phi_3) \to 0.$$

In particular, the boundary mapping δ is defined by

$$\begin{array}{cccc} \delta: & \mathrm{Ker}(\phi_3) & \longrightarrow & \mathrm{Coker}(\phi_1) \\ & a_3 & \mapsto & \phi_2(a_2) \in B_1 \mod \phi_1(A_1), \end{array} \tag{16}$$

where $a_2 \in A_2$ is a left of the element $a_3 \in A_3$.[1]

More precisely, to apply this lemma, we introduce the following commutative diagram with exact rows

$$\begin{array}{cccccccc} & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \to & \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x}, \widehat{E}^{\sim}_{1,x}) & \to & \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x}, \widehat{E}^{\sim}_{2,x}) & \to & \mathrm{End}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x}) & \xrightarrow{\delta} \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & \\ 0 & \to & \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1;x,\eta}) & \to & \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{2;x,\eta}) & \to & \mathrm{End}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}) & \to 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & \\ \xrightarrow{\delta} & & \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x}) & \to & \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{2,x}) & \to & \mathrm{End}(\overline{E}_{3,x}) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & & 0 & & 0. & \end{array} \tag{17}$$

Here, for $i = 1, 2, 3$,

$$\mathrm{Hom}(\overline{E}_{i,x}, \overline{E}_{j,x}) := \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{i;x,\eta}, \widehat{E}_{j;x,\eta}) / \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{i,x}, \widehat{E}^{\sim}_{j,x}).$$

[1]Certainly, δ is well-defined. Indeed, since $a_3 \in \mathrm{Ker}(\phi_3)$ implies that a_3 has the zero image in $\mathrm{Coker}(\phi_3)$. This implies that the element $\phi_2(a_2)$ of $\mathrm{Coker}(\phi_2)$ belongs to the sub-module $\mathrm{Coker}(\phi_1)$.

Obviously, by (12), we have

$$\prod'_{x\in X}\mathrm{Hom}(\overline{E}_{3,x},\overline{E}_{1,x})/\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta})\simeq$$
$$\prod'_{x\in X}\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})\Big/\Big(\prod'_{x\in X}\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x},\widehat{E}^{\sim}_{1,x})+\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta})\Big). \tag{18}$$

Accordingly, we set a_2 (in the footnote of the previous page) to be the identity morphism $\mathrm{Id}_{\widehat{E}^{\sim}_{3,x}}$ of $\mathrm{End}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x})$ in the Snake Lemma. Map this $\mathrm{Id}_{\widehat{E}^{\sim}_{3,x}}$ to the identify map $\mathrm{Id}_{\widehat{E}_{3;x,\eta}}$ of $\mathrm{End}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta})$, which has the zero image in $\mathrm{End}(\overline{E}_{3,x})$. Now using the exact sequence in the middle row, we can lift $\mathrm{Id}_{\widehat{E}_{3;x,\eta}}$ to an element $\widehat{s}_{3,2;x,\eta}$ of $\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1;x,\eta})$. This is nothing but $\phi_2(a_2)$. Denote its image in $\mathrm{Hom}(\overline{E}_{3,x},\overline{E}_{2,x})$ by $\overline{s}_{3,2;x,\eta}$, which certainly admits zero as its image in $\mathrm{End}(\overline{E}_{3,x})$, since so is the element $\mathrm{Id}_{\widehat{E}_{3;x,\eta}}$ of $\mathrm{End}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta})$ above. Thus by the exactness of the last row, we obtain an element $\kappa_x=\delta(\mathrm{Id}_{\widehat{E}^{\sim}_{3,x}})$ in $\mathrm{Hom}(\overline{E}_{3,x},\overline{E}_{1,x})$. Therefore, applying the natural quotient morphism

$$\prod'_{x\in X}\mathrm{Hom}(\overline{E}_{3,x},\overline{E}_{1,x})\longrightarrow\prod'_{x\in X}\mathrm{Hom}(\overline{E}_{3,x},\overline{E}_{1,x})/\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta}) \tag{19}$$

we obtain an element $([\kappa_x])\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3,\mathcal{E}_1)$, which is nothing but the extension class for the extension $\mathbb{E}$ of $\mathcal{E}_3$ by $\mathcal{E}_1$, by (18), (12), (10) and (6). This then completes our proof of the following main result of this paper.

Theorem 2. *The natural bijections*

$$\begin{array}{ccc}\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3,\mathcal{E}_1) & \simeq & H^1(X,\mathcal{E}_3^\vee\otimes\mathcal{E}_1)\\ \Phi\big\downarrow\simeq & & \simeq\big\downarrow \\ \multicolumn{3}{c}{\prod'_{x\in X}\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})\Big/\Big(\prod'_{x\in X}\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}^{\sim}_{3,x},\widehat{E}^{\sim}_{1,x})+\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta})\Big)}\end{array}$$

such that
$$\Phi(\delta(\mathrm{Id}_{\mathcal{E}_3}))=\big([\kappa_x]\big)_{x\in X}. \tag{20}$$

Proof. Indeed, the commutative diagram of bijections are direct consequence of (12), (10) and (6). And the relation (20) comes direct from the construction of κ_x. □

We end this subsection by a useful construction of the inverse map of Φ. Let $s_\eta=(s_{x,\eta})$ be be an element of $\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})$.

To clarifying the structures involved, first we *assume that s_η is regular for all but one closed point $x_0\in X$.* That is to say, there exists one and only one closed point $x_0\in X$ such that $s_{x,\eta}$ are regular when $x\neq x_0$. Choose an open neighborhood U_0 of x_0 in X such that $s_{x_0,\eta}$ is regular over $U_0\smallsetminus\{x_0\}$. Shrinking U_0 if necessary, we may and hence will assume that U_0 is affine. Denote its affine ring by A_{U_0}. Within the $\widehat{F}_x$-linear space $\widehat{E}_{1;x_0,\eta}\oplus\widehat{E}_{3;x_0,\eta}$ of dimension (r_1+r_3), construct an A_{U_0}-module generated by the sub-modules $E^{\sim}_{1,U_0}\oplus\{0\}$ and $\big\{(s_{x_0,\eta}(b),b):b\in E^{\sim}_{3,U_0}\big\}$, where, for $i=1$, 3, $E^{\sim}_{i,U_0}=\Gamma(U_0,\mathcal{E}_i|_{U_0})$ so that

$\mathcal{E}_i|_{U_0} \simeq \widetilde{E_{i,U_0}^{\sim}}$. In other words, this new A_{U_0}-module contained in $E_{1;x,\eta} \oplus E_{3;x,\eta}$ is given by

$$E_{1,U_0}^{\sim} \rtimes_{x_0,s_{x_0},\eta} E_{3,U_0}^{\sim} := \{(a + s_\eta(b), b) : a \in E_{1,U_0}^{\sim},\ b \in E_{3,U_0}^{\sim}\}. \tag{21}$$

Obviously, such an A_{U_0}-module is free and hence induces a locally free sheaf which we denote by $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0}$. That is,

$$\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0} := E_{1,U_0}^{\sim} \widetilde{\rtimes_{x_0,s_{x_0},\eta}} E_{3,U_0}^{\sim}. \tag{22}$$

By our construction, obviously,

(i) $\mathcal{E}_1|_{U_0}$ is a locally free $\mathcal{O}_{U_0}$ sub-sheaf of $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0}$ such that

$$\big(\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0}\big) \,/\, (\mathcal{E}_1|_{U_0}) \simeq \mathcal{E}_3|_{U_0}. \tag{23}$$

(ii) Since s_η is regular over $U \smallsetminus \{x_0\}$,

$$\big(\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0}\big)\,\big|_{U_0 \smallsetminus \{x_0\}} = \mathcal{E}_1|_{U_0} \oplus \mathcal{E}_3|_{U_0}. \tag{24}$$

In particular, it is possible to glue the locally free sheaves $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0},\eta} \mathcal{E}_3|_{U_0}$ on U_0 and $(\mathcal{E}_1 \oplus \mathcal{E}_3)|_{X \smallsetminus \{x_0\}}$ on $X \smallsetminus \{x_0\}$ over $U_0 \smallsetminus \{x_0\}$. Denote the resulting locally sheaf by $\mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3$. Then we obtain a natural short exact sequence

$$\mathbb{E}_{s_\eta} : 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3 \longrightarrow \mathcal{E}_3 \to 0. \tag{25}$$

Furthermore, it is not too difficult to see that s_η is equivalent to $([\kappa_x])_{x\in X}$ associated to $\mathbb{E}_{s_\eta}$ constructed before Theorem 2.

Now we are ready to treat the general case. *Assume that, as we may, there exist closed points $x_1, \ldots, x_n$ such that*

(a) for $x \notin \{x_1, \ldots, x_n\}$, $s_{x,\eta}$ is regular on X,

(b) for each $i = 1, \ldots, n$, $s_{x_i,\eta}$ is regular for all but one closed point $x_i \in X$

Similar to the single closed point case above, for each i, choose an affine open neighborhood U_i of x_i in X such that $U_i \cap U_j = \emptyset$ if $i \neq j$, and on each U_i, a locally free sheaf $\mathcal{E}_1|_{U_i} \rtimes_{x_i,s_{x_i},\eta} \mathcal{E}_3|_{U_i}$ can be constrcted using $s_{x_i,\eta}$. Thus, by gluing these locally free sheaves on the U_i's with the free sheaf $(\mathcal{E}_1 \oplus \mathcal{E}_3)|_{X \smallsetminus \{x_1,\ldots,x_n\}}$ on $X \smallsetminus \{x_1, \ldots, x_n\}$ on the overlaps $U_i \smallsetminus \{x_i\}$, we obtain a locally free sheaf $\mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3$ of rank $(r_1 + r_3)$ on X, which is an extension of $\mathcal{E}_3$ by $\mathcal{E}_1$. In other words, there is the following short exact sequence of locally free sheaves on X constructed from s_η:

$$\mathbb{E}_{s_\eta} : \qquad 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \rtimes_{s\eta} \mathcal{E}_3 \longrightarrow \mathcal{E}_3 \to 0. \tag{26}$$

In addition, by our construction, it is not difficult to see that $(s_{i,\eta})$ is equivalent to the element $([\kappa_x])$ of $\mathbb{E}_{s_\eta}$constructed before Theorem 2. In this we have constructed the inverse of Φ. For example, if all the $s_{x,\eta}$'s are regular, then $s_{x,\eta}$ is equivalent to zero and the associated extension is trivial.

4 Locally Free Sheaves Induced from Extensions: Adelic Descriptions

As in (4), let

$$\mathbb{E}: \qquad 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \to 0, \tag{27}$$

be a short exact sequence of locally free sheaves on X. For each $i = 1, 2, 3$, let $g_i = (g_{i,x}) \in \mathrm{GL}_{r_i}(F)\backslash\mathrm{GL}_{r_i}(\mathbb{A})/\mathrm{GL}_{r_i}(\mathcal{O})$ be the adelic elements associated to $\mathcal{E}_i$ introduced in § 1.

Theorem 3. *Let* $([\kappa_x])$ *be the extension class in the space*

$$\prod\nolimits'_{x\in X}\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta})\Big/\Big(\prod\nolimits'_{x\in X}\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{1,x}^{\sim})+\mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta})\Big) \tag{28}$$

associated to the extension $\mathbb{E}$. *Then, in* $\mathrm{GL}_{r_i}(F)\backslash\mathrm{GL}_{r_i}(\mathbb{A})/\mathrm{GL}_{r_i}(\mathcal{O})$*, the adelic element* $g_2 = (g_{2,x})$ *of the locally free sheaf* $\mathcal{E}_2$ *is represented by*

$$g_{2,x} = \begin{pmatrix} g_{1,x} & \kappa_x \\ 0 & g_{3,x} \end{pmatrix} \qquad \forall x \in X. \tag{29}$$

Here κ_x *are viewed as elements of the spaces* $M_{r_1\times r_3}(\widehat{F}_x)$ *of* $r_1 \times r_3$*-matrices with entries in* $\widehat{F}_x$*.*

Proof. This is a direct consequence of the proof of Theorem 2, particularly, the construction of Φ^{-1} at the end of the previous subsection. Indeed, by (21), we have $g_{2,x}$ is a upper triangular matrices with diagonal blocks $g_{1,x}$ and $g_{3,x}$ and with κ_x as the right-upper block as stated in the theorem. Finally, the reason that $([\kappa_x]) \in M_{r_1\times r_3}(\mathbb{A})$ comes from the fact that the restrict product $\prod'$ is used in the quotient space $\frac{\prod'_{x\in X}\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta},\widehat{E}_{1,;x,\eta})}{\prod'_{x\in X}\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim},\widehat{E}_{1,x}^{\sim})+\mathrm{Hom}_F(E_{3,\eta},E_{1,\eta})}$. □

References

[1] R. Hartshorn, *Algebraic Geometry*, GTM 52, Springer 1977

[2] L. Weng, Codes and Stability, arXiv:1806.04319

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