Arithmetic Characteristic Curves

Lin WENG

Abstract

For a split reductive group defined over a number field, we first introduce the notations of arithmetic torsors and arithmetic Higgs torsors. Then we construct arithmetic characteristic curves associated to arithmetic Higgs torsors, based on the Chevalley characteristic morphism and the existence of Chevalley basis for the associated Lie algebra. As to be expected, this work is motivated by the works of Beauville-Narasimhan on spectral curves and Donagi-Gaistgory on cameral curves in algebraic geometry. In the forthcoming papers, we will use arithmetic characteristic curves to construct arithmetic Hitchin fibrations and study the intersection homologies and perverse sheaves for the associated structures, following Ngo's approach to the fundamental lemma.

1 Chevelley's Characteristic Morphism

1.1 Over Number Fields

Let F be a number field with $\mathcal{O}_F$ its ring of integers. Denote by $X = \mathrm{Spec}\,\mathcal{O}_F$ the associated uncompleted arithmetic curve.

Let G be a split reductive group over F. Fix a split maximal subtorus T and a maximal split quotient torus T' of G. Denote the Lie algebra of G by $\mathfrak{g} := \mathrm{Lie}\,G$, and set $\mathfrak{t} := \mathrm{Lie}\,T$ be the associated commutative subalgebra of $\mathfrak{g}$.

Recall that, with respect to the adjoint action

$$\begin{array}{rccc} \mathrm{ad}: & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & (g,x) & \mapsto & (\mathrm{ad}(g))(x) := [g,x] \end{array}$$

$\mathfrak{g}$ admits a natural decomposition

$$\mathfrak{g} := \mathfrak{t} \bigoplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha := \big\{x\in\mathfrak{g} : (\mathrm{ad}(h))(x) = \alpha(h)\,x\big\}$$

for α running through a finite subset Φ of the space

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m),$$

of rational characteristics of T.

For a fixed minimal split parabolic subgroup B of G containing T, set $\mathfrak{b} := \mathrm{Lie}\,B$. Then G/B is proper, and there exists a finite subset Φ_+ of Φ, the so-called set of positive roots of (G,B,T), such that

(1) $\Phi = \Phi_+ \bigsqcup (-\Phi_+)$,

(2) $\mathfrak{t} \oplus \oplus_{\alpha\in\Phi_+}\mathfrak{g}_\alpha \subset \mathfrak{b}$, and

(3) Φ_+ admits a subset Δ of simple roots associated to (G, B, T), such that

  (i) $\Phi_+ \subseteq \sum_{\alpha\in\Delta}\mathbb{Z}_{\geq 0}\alpha$, where $\mathbb{Z}_{\geq 0} := \{n \in \mathbb{Z} : n \geq 0\}$, and

  (ii) Δ forms a basis of the $\mathbb{Q}$-linear space

$$X^*(T)_{\mathbb{Q}} := X^*(T) \otimes \mathbb{Q}.$$

Let W be the Weyl group of G, defined as the finite quotient group

$$W := N_G(T)/Z_G(T)$$

where $N_G(T)$, resp. $Z_G(T)$, denotes the normalizer subgroup, resp. the centralizer subgroup, of T in G. It is well known that W is canonically isomorphic to the subgroup of the automorphism group of $X^*(T)_{\mathbb{Q}}$ generated by the reflections

$$\begin{array}{rccc} \sigma_\alpha : & X^*(T)_{\mathbb{Q}} & \longrightarrow & X^*(T)_{\mathbb{Q}} \\ & v & \mapsto & v - \frac{2}{(v,a)}\,\alpha \end{array}$$

It is a canonical result due to Chevalley that, over the base field F, the space of G-invariant polynomials on G coincides with the space of the W-invariant polynomials of T. That is to say,

$$F[G]^G \simeq F[T]^W \tag{1}$$

where the actions of both sides are defined by

$$\begin{array}{ccc} G \times F[G] & \longrightarrow & F[G] \\ (g, \sum_i a_i g_i) & \mapsto & \sum_i a_i(g g_i g^{-1}) \end{array} \quad \text{and} \quad \begin{array}{ccc} W \times F[T] & \longrightarrow & F[T] \\ (\sigma, \sum_i a_i t_i) & \mapsto & \sum_i a_i \sigma(t_i) \end{array}$$

Similarly, in terms of the Lie algebras, we have

$$F[\mathfrak{g}]^G \simeq F[\mathfrak{t}]^W$$

where G acts on $\mathfrak{g}$ in terms of the adjoint action Ad, namely, $\mathrm{Ad}(g)$ is defied as the differential of the conjugation morphism $x \mapsto gxg^{-1}$ of G.

In terms of geometry, the isomorphism (1) naturally induces the scheme theoretic morphism

$$\begin{array}{ccc} & & \operatorname{Spec} F[T] \\ & & \downarrow \\ G & \xrightarrow{\chi} & T/\!/W := \operatorname{Spec} F[T]^W \end{array}$$

or equivalently, for the associated Lie structures,

$$\begin{array}{ccc} \mathfrak{g}_F \simeq \operatorname{Spec} F[\mathfrak{g}] & & \mathfrak{t}_F \simeq \operatorname{Spec} F[\mathfrak{t}] \\ \downarrow & \chi \searrow & \downarrow \pi_F \\ \mathfrak{g}_F/\!/G := \operatorname{Spec} F[\mathfrak{g}]^G & \simeq & \mathfrak{t}_F/\!/W := \operatorname{Spec} F[\mathfrak{t}]^W \end{array} \tag{2}$$

where $\mathfrak{g}_F := \mathfrak{g} \otimes F$ and similarly $\mathfrak{t}_F := \mathfrak{t} \otimes F$.

Example 1. For $G = \mathrm{GL}_n/F$, we have $W \simeq \mathfrak{S}_n$, the symmetric group on n symbols and $\mathfrak{g}_F = \mathfrak{gl}_n(F) = \mathrm{End}(F^n)$. Then χ coincides with the morphism

$$\begin{array}{cccc} \chi: & \mathrm{End}(F^n) & \longrightarrow & \bigoplus_{k=1}^n L_k \\ & A & \mapsto & \det(\lambda I_n - A) \end{array}$$

where L_k denotes the one dimensional vector space generated by the k-th elementary symmetric polynomials, and I_n denotes the unity matrix of size n. That is to say, χ assigns a matrix A to the associated eigen polynomial. In particular, by restricting χ to $\mathfrak{t}_F$ which consists of diagonal matrices $D = \mathrm{diag}(a_1, \ldots, a_n)$, we conclude that

$$\chi(D) = t^n - \sum_{i=1}^n a_i t^{n-1} + \sum_{i<j} a_i a_j t^2 + \ldots + (-1)^n \prod_{i=1}^n a_i$$

For this reason, the morphism χ for general G is called *the Chevelley characteristic morphism.* From above, after restricting χ to $\mathfrak{t}$, the Chevelley characteristic morphism is simply equivalent to the assignments of the unordered eigenvalues.

1.2 Over Integral Bases

The diagram in (2) associated to a split reductive group G/F only works over the pointed base $\mathrm{Spec}\, F$. In this section, we construct a natural extension to the integral base $\mathrm{Spec}\, \mathcal{O}_F$, whose generic fiber (over the generic point $\eta_F := \mathrm{Spec}\, F$) coincides with that of (2).

To start with, we recall the so-called Chevelley basis for $\mathfrak{g}_F$. For simplicity, we assume that $F = \mathbb{Q}$ for the time being.

Definition 1 (See e.g. Ch. VII, §25 of [6])**.** A *Chevelley basis* for $[\mathfrak{g}, \mathfrak{g}]$ is a basis for the $\mathbb{Q}$-linear space $[\mathfrak{g}, \mathfrak{g}]$, consisting of $\{x_\alpha : \alpha \in \Phi\} \bigsqcup \{h_\alpha : \alpha \in \Delta\}$ which satisfy the following properties:

(a) for all $\alpha \in \Phi$, $x_\alpha \in \mathfrak{g}_\alpha$.

(b) for all $\alpha \in \Phi$, $[x_\alpha, x_{-\alpha}] = h_\alpha$ so that $x_\alpha, x_{-\alpha}$ and h_α span a three dimensional simple subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_2(F)$ via

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) for α, $\beta \in \Phi$, if $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$, then

 (i) $c_{\alpha,\beta} = c_{-\alpha,-\beta}$.

 (ii) $c_{\alpha,\beta}^2 = \kappa\,(\ell+1)\dfrac{(\alpha+\beta, \alpha+\beta)}{(\beta,\beta)}$ where the constants ℓ, κ are defined by the α-string $\beta - \ell\alpha, \cdots, \beta + \kappa\alpha$ through β.

We have the following well-known

Theorem 2 (Chevelley)**.** *Over* $[\mathfrak{g}, \mathfrak{g}]$*, we have*

(1) There always exists a Chevelley basis $\{x_\alpha : \alpha \in \Phi\} \bigsqcup \{h_\alpha : \alpha \in \Delta\}$ *on* $[\mathfrak{g}, \mathfrak{g}]$,

(2) All the structural constants lie in $\mathbb{Z}$. *That is to say,*

(i) for all α, $\beta \in \Delta$, $[h_\alpha, h_\beta] = 0$.

(ii) for all $\alpha \in \Phi,\ \beta \in \Delta$, $[h_\beta, x_\alpha] = \langle \alpha, \beta \rangle x_\alpha$, *where* $\langle \alpha, \beta \rangle := 2\dfrac{(\alpha, \beta)}{(\beta, \beta)}$.

(iii) for $\alpha \in \Phi$, $[x_\alpha, x_{-\alpha}]$ *is a* $\mathbb{Z}$*-linear combination of* h_α*'s* ($\alpha \in \Delta$).

(iv) If α, β *are independent roots and* $\beta - \ell\alpha, \cdots, \beta + \kappa\alpha$ *is the* α*-string through* β, *then* $[x_\alpha, x_\beta] = \begin{cases} 0 & \kappa = 0, \\ \pm(\ell + 1)\, x_{\alpha+\beta} & \alpha + \beta \in \Phi. \end{cases}$

Obviously, once Δ is fixed, h_α are uniquely determined if $\alpha \in \Delta$. In addition, for a general $\alpha \in \Phi$, if x_α is replaced by $c_\alpha x_a$, deduced from the conditions in (c) of Definition 1, $\{c_\alpha\}_{\alpha\in\Phi}$ are bounded by the constrains:

(i) for all $\alpha \in \Phi$, $c_\alpha c_{-\alpha} = 1$,

(ii) for all α, $\beta \in \Phi$, if $\alpha + \beta \in \Phi$, then $c_\alpha c_\beta = \pm c_{\alpha+\beta}$.

Conversely, it is clear that if $\{c_\alpha\}_{\alpha\in\Phi}$ satisfies (these two conditions)i) and (ii) just mentioned, then $\{x_\alpha : \alpha \in \Phi\} \bigsqcup \{h_\alpha : \alpha \in \Delta\}$ forms a Chevelley basis of $[\mathfrak{g}, \mathfrak{g}]$ as well.

To treat the Lie algebra $\mathfrak{g}$ associated to the split reductive group $G/\mathbb{Q}$, it suffices to use the decomposition

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$. Obviously, the integral bases for $\mathfrak{z}$ are parametrized by $\mathrm{GL}_{\dim_{\mathbb{Q}} \mathfrak{z}}(\mathbb{Z})$. Hence, it is natural to define *a Chevelley basis of* $\mathfrak{g}$ to be the union of a Chevelley basis of $[\mathfrak{g}, \mathfrak{g}]$ as above and an integral basis of $\mathfrak{z}$. Denote by $\mathfrak{g}_{\mathbb{Z}}$, resp. $\mathfrak{t}_{\mathbb{Z}}$, resp. $\mathfrak{z}_{\mathbb{Z}}$, the associated lattice of $\mathfrak{g}$, resp. of $\mathfrak{t}$, resp. of $\mathfrak{z}$. Obviously, $\mathfrak{g}_{\mathbb{Z}}$ admits a natural Lie structure and does not depend on the chosen integral Chevalley basis.

Moreover, working with $\mathfrak{g}_{\mathbb{Z}}$, we obtain the following structural diagram over $\mathbb{Z}$:

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{Z}} \simeq \operatorname{Spec} \mathbb{Z}[\mathfrak{g}_{\mathbb{Z}}] & & \mathfrak{t}_{\mathbb{Z}} \simeq \operatorname{Spec} \mathbb{Z}[\mathfrak{t}_{\mathbb{Z}}] \\ \big\downarrow & \chi_{\mathbb{Z}} \searrow & \big\downarrow \pi_{\mathbb{Z}} \\ \mathfrak{g}_{\mathbb{Z}} /\!/ G(\mathbb{Z}) := \operatorname{Spec} \mathbb{Z}[\mathfrak{g}_{\mathbb{Z}}]^{G(\mathbb{Z})} & \simeq & \mathfrak{t}_{\mathbb{Z}} /\!/ W := \operatorname{Spec} \mathbb{Z}[\mathfrak{t}_{\mathbb{Z}}]^W. \end{array}$$

Obviously, associated to the base change $\operatorname{Spec} F \hookrightarrow \operatorname{Spec} \mathbb{Z}$, we recover the diagram in (2).

The same construction works for a general number field F instead of $\mathbb{Q}$. That is to say, we may use the base change $\operatorname{Spec} \mathcal{O}_F \longrightarrow \operatorname{Spec} Z$ to obtain the following diagram over the integral base $\operatorname{Spec} \mathcal{O}_F$:

$$\begin{array}{ccc} \mathfrak{g}_{\mathcal{O}_F} \simeq \operatorname{Spec} \mathcal{O}_F[\mathfrak{g}_{\mathcal{O}_F}] & & \mathfrak{t}_{\mathcal{O}_F} \simeq \operatorname{Spec} \mathbb{Z}[\mathfrak{t}_{\mathbb{Z}}] \\ \big\downarrow & \chi_{\mathcal{O}_F} \searrow & \big\downarrow \pi_{\mathcal{O}_F} \\ \mathfrak{g}_{\mathcal{O}_F} /\!/ G(\mathcal{O}_F) := \operatorname{Spec} \mathcal{O}_F[\mathfrak{g}_{\mathcal{O}_F}]^{G(\mathcal{O}_F)} & \simeq & \mathfrak{t}_{\mathcal{O}_F} /\!/ W := \operatorname{Spec} \mathcal{O}_F[\mathfrak{t}_{\mathcal{O}_F}]^W. \end{array} \tag{3}$$

whose general fiber is the similar diagram over $\operatorname{Spec} F$, which can be obtained from (7) by replacing the integer ring $\mathcal{O}_F$ with its associated number field F.

To end this discussion, via the Minkowski embedding $\mathcal{O}_F \hookrightarrow F \hookrightarrow F_\infty := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, we obtain the Lie algebra $\mathfrak{g}_\infty := \mathfrak{g} \otimes \mathbb{F}_\infty$ and similarly $\mathfrak{t}_\infty$ and $\mathfrak{z}_\infty$. Furthermore, induced from the base change $\operatorname{Spec} F_\infty \to \operatorname{Spec} \mathcal{O}_F$, we obtain the following diagram on F_∞:

$$\begin{array}{ccc} \mathfrak{g}_\infty \simeq \operatorname{Spec} F_\infty[\mathfrak{g}_\infty] & & \mathfrak{t}_\infty \simeq \operatorname{Spec} F_\infty[\mathfrak{t}_\infty] \\ \downarrow & \chi_\infty \searrow & \downarrow \pi_\infty \\ \mathfrak{g}_\infty//G(F_\infty) := \operatorname{Spec} F_\infty[\mathfrak{g}_\infty]^{G(F_\infty)} & \simeq & \mathfrak{t}_\infty//W := \operatorname{Spec} F_\infty[\mathfrak{t}_\infty]^W. \end{array} \tag{4}$$

In particular, similar to the morphisms π_F and $\pi_{\mathcal{O}_F}$, π_∞ is a finite $(|W| : 1)$-morphism, even supposed to be highly ramified in general. For later use, we denote $\mathfrak{t}_{\mathcal{O}_F}//W$, $\mathfrak{t}_F//W$, and $\mathfrak{t}_\infty//W$ by $\mathfrak{c}_{\mathcal{O}_F}$, $\mathfrak{c}_F$ and $\mathfrak{c}_\infty$, respectively.

2 G-Torsors on $\overline{\operatorname{Spec}\mathcal{O}_F}$

Let F be a number field with $\mathcal{O}_F$ its ring of integers and $\mathbb{A}_F$ its adelic ring. Denote by S the set of inequivalent normalized valuations of F, and by S_{fin}, resp. S_∞, the subsets of S consisting of non-archimedean, resp. archimedean, valuations. For each $v \in S$, denote by F_v the v-completion of F. When $v \in S_\infty$, F_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$; and if $v \in S_{\text{fin}}$, F_v is a discrete valuation fields. Accordingly, the valuation is called real, or complex, or v-adic. For v-adic valuations, denote by $\mathcal{O}_v$ the associated valuation ring of F_v, by $\mathfrak{P}_v$ its maximal ideal, and by $k_v := \mathcal{O}_v/\mathfrak{P}_v$ its residue field. It is well-known that $\mathcal{O}_v$, being a discrete valuation ring, admits only two prime ideals, namely, $\mathfrak{P}_v$ and $\{0\}$, and k_v is a finite extension of $\mathbb{F}_p$, for a certain prime number $p \in \mathbb{Z}$. We call $[k_v : \mathbb{F}_p]$ the residue extension degree of v. Based on all these, we have $\mathbb{A}_F = \prod'_{v\in S} F_v$ where $\prod'_{v\in S}$ denotes the restricted product of the F_v's with respect to the $\mathcal{O}_c$'s. That is to say, an element $a = (a_v) \in \prod_{v\in S} F_v$ belongs to $\mathbb{A}_F$ if and only if $a_v \in \mathcal{O}_v$ for all but finitely many $v \in S$. It is well known that, induced from the locally compact topologies on the F_v's, $\mathbb{A}_F$ is locally compact.

Let G be a split reductive group over F with a pinning $(T, B, \{x_\alpha\}_{\alpha\in\Delta})$. Here T is a maximal split subtorus of G, B is a minimal split parabolic subgroup of G containing T, and Δ denotes the set of simple roots of the root system Φ associated to (G, T, B) and x_α denotes a non-zero vector of the proper subspace $\operatorname{Lie}(U)_\alpha$ of the Lie algebra $\operatorname{Lie}(U)$ corresponding to the eigenvalue α, where U denotes the unipotent radical of B. We assume that $\{x_\alpha\}_{\alpha\in\Delta}$ can be extended to a Chevelley basis of $\mathfrak{g}_F$. Set then

$$x_+ := \sum_{\alpha\in\Delta} x_\alpha. \tag{5}$$

2.1 Torsors over Local and Global Fields

Let $\overline{F}$ be the algebraic closure on F contained in $\mathbb{C}$, and set $G_F = \operatorname{Gal}(\overline{F}/F)$ be the absolute Galois group of F. For each $v \in S$, fix an algebraic closure $\overline{F}_v$ of F_v, and denote by $G_{F_v} = \operatorname{Gal}(\overline{F}_v/F_v)$ the local absolute Galois group of F at v.

Definition 3. For $K = F$ or F_v, a K-scheme $\mathcal{G}$ is called a *G-torsor* if $\mathcal{G}$ is equipped with a faithful, transitive and G_K-compatible action of $G(\overline{K})$ on $\mathcal{G}(\overline{K})$.

Example 2. *For $K = F$ or F_v, set $\mathcal{G} = G$. Then*

$$\begin{array}{ccc} G(\overline{K}) \times \mathcal{G}(\overline{K}) \times \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \mathcal{G}(\overline{K}) \\ (g, d, \sigma) & \mapsto & (g\, d)^\sigma = g^\sigma\, d^\sigma \end{array}$$

gives a G-torsor structure on $\mathcal{G}$ over K.

Since G is defined over F, there is a continuous action of G_F on $G(\overline{F})$. As usual, set $H^0(F,G) := G(\overline{F})^{\mathrm{Gal}(\overline{F}/F)}$ be the collection of G_F-invariant points of $G(\overline{F})$ and $H^1(F,G) := Z^1(F,G)/B^1(F,G)$ be the set of equivalence classes of 1-cocycles, where $Z^1(F,G)$, resp. $B^1(F,G)$, denotes the set of 1-cocycles, resp. 1-coboundaries, of G_F on $G(\overline{F})$, i.e. a continuous map

$$\begin{array}{cccc} \phi: & G_F & \longrightarrow & G(\overline{F}) \\ & \sigma & \mapsto & a_\sigma \end{array}$$

satisfying

$$a_{\alpha\tau} = a_\sigma \cdot a_\tau^\sigma,$$

and two 1-cocycles a_σ and a'_s are said to be equivalence if there exists an element g of $G(\overline{F})$ such that

$$a'_\sigma = g^{-1} \cdot a_\sigma \cdot g^\sigma$$

In other words, a_σ is an 1-coboundary if $\alpha_\sigma = g^{-1} g^\sigma$.

Theorem 4 (See e.g. §2 of [7])**.** *For $K = F$ or F_v, there exists a natural bijection between the set of isomorphism classes of G-torsors on K and the set $H^1(K,G)$, which sends the trivial G-torsor on K to the trivial class in $H^1(K,G)$.*

Here, naturality means that the bijection is compatible with the changes of the field K and the group G. For reader's convenience, we sketch a proof.

Proof. Let $\mathcal{G}$ be a G-torsor on K. Fix a point $d_0 \in \mathcal{G}(\overline{K})$. Then, for any $\sigma \in G_K$, there exists a unique $a_\sigma \in G(\overline{K})$ such that $d_0^\sigma = d_0\, a_\sigma$ since G_K acts on $G(\overline{K})$ which itself is a group. It is not difficult to check that $\sigma \mapsto a_\sigma$ is an 1-cocycle and hence induces an element in $H^1(K,G)$.

Conversely, if $\sigma \mapsto a_\sigma$ is an 1-cocycle, then on $G \times_K \overline{K}$, we obtain a new action of G_K through $(\sigma, g) \mapsto \alpha_\sigma\, g^\sigma$. Since G is quasi-projective, by Weil's theorem on descent, there exists a K-scheme $\mathcal{G}$, or better, a G-torsor on K, such that $\mathcal{G} \times_K \overline{K} = G \times_K \overline{K}$ is G_K-equivalent (with respect to the twisted action of G_K on $G \times_K \overline{K}$). □

There is a canonical morphism $H^1(F,G) \to H^1(F_v,G)$ induced by the inclusion $F \hookrightarrow F_v$ for each $v \in S$, which themselves then induce a natural morphism

$$H^1(F,G) \to \prod_{v\in S} H^1(F_v,G) \tag{6}$$

Denote by $\mathrm{Ker}^1(F,G)$ the kernel of this morphism. It is a result of Borel-Serre that $\mathrm{Ker}^1(F,G)$ is finite.

Corollary 5. *There are only finite many G-torsors on F, up to isomorphisms, such that, for all $v \in S$, the induced G-torsors on F_v are trivial.*

2.2 G-Torsors on $\overline{\operatorname{Spec}\mathcal{O}_F}$

We begin with the following

Definition 6. A *G-torsor on* $\overline{\operatorname{Spec}\mathcal{O}_F}$ is a scheme $\mathcal{G}$, equipped with a flat surjective morphism $\pi : \mathcal{G} \to \overline{\operatorname{Spec}\mathcal{O}_F}$ and a family of flat surjective morphism $\pi_v : \mathcal{G}_{\mathcal{O}_v} \to \overline{\operatorname{Spec}\mathcal{O}_v}$, together with actions of $G(F_v)$ on $\mathcal{G}_v$ for all $v \in S_{\text{fin}}$, such that the induced morphism

$$\begin{array}{ccc} \mathcal{G}_x \times G_x & \longrightarrow & \mathcal{G}_x \times_{\overline{\operatorname{Spec}\mathcal{O}_x}} \mathcal{G}_x \\ (d,g) & \mapsto & (d, d^g) \end{array}$$

are isomorphisms for all points $x \in \operatorname{Spec}\mathcal{O}_F$, closed or generic.

In particular, if $\mathcal{G}$ is a G-torsor on $\overline{\operatorname{Spec}\mathcal{O}_F}$, the fiberwise $\{G(F_x)\}_{x\in X}$ acts on $\{\mathcal{G}(F_x)\}_{x\in X}$ with respect to π, and the action on each generic fiber is faithful and transitive. Moreover, for each $v \in S_{\text{fin}}$, over the local integral base, induced by the natural inclusion $\mathcal{O}_F \hookrightarrow \mathcal{O}_v$, we obtain a composition of morphisms $\operatorname{Spec}\mathcal{O}_v \to \operatorname{Spec}\mathcal{O}_F \to \overline{\operatorname{Spec}\mathcal{O}_F}$. Thus, for the G-torsors $\mathcal{G}_{\mathcal{O}_v}$ on $\operatorname{Spec}\mathcal{O}_v$, it is not too difficulty to deduce the following result, whose proof we left to the reader.

Lemma 7. *(1) For each finite place $v \in S_{\text{fin}}$, induced from the natural morphisms $\mathcal{O}_v \hookrightarrow F_v$ and $\mathcal{O}_v \to k_v$, we have*

$$(\mathcal{G}_{\mathcal{O}_v})_{\eta_v} \simeq \mathcal{G}_{\eta_v} \qquad \text{and} \qquad (\mathcal{G}_{\mathcal{O}_v})_{k_v} \simeq \mathcal{G}_{k_v}.$$

(2) For each infinite place $\sigma \in S_\infty$, induced by the natural embedding $F \hookrightarrow F_\sigma$, we have

$$(\mathcal{G}_\eta)_\sigma \simeq \mathcal{G}_\sigma.$$

In particular, $(\mathcal{G}_\eta)_\infty \simeq \mathcal{G}_\infty$.

2.3 Inner Form

When working over integral base $\operatorname{Spec}\mathcal{O}_F$, our choice of a Chevalley basis $\{x_\alpha : \alpha \in \Phi\} \bigsqcup \{h_\alpha : \alpha \in \Delta\}$ determines a pinning $(T, B, \{x_\alpha\}_{\alpha\in\Delta})$ of G. To deal with the associated compatibility problem, in the automorphism group $\operatorname{Aut}(G)$ of G, we consider the so-called *outer automorphism group* $\operatorname{Out}(G)$ defined as the collection of the automorphisms of G which preserves the pinning $(T, B, \{x_\alpha\}_{\alpha\in\Delta})$. There is a natural split short exact sequence

$$1 \to G^{\text{ad}} \to \operatorname{Aut}(G) \xrightarrow{\pi} \operatorname{Out}(G) \to 1. \tag{7}$$

Indeed, G^{ad} is identified with the image of G under the adjoint representation of G,[1] and hence also fits into the short exact sequence

$$1 \to G^{\text{ad}} \to \operatorname{Aut}(G) \xrightarrow{\pi} \operatorname{Aut}(\Phi, \Delta) \to 1. \tag{8}$$

[1] The adjoint group G^{ad} of G is the Zariski connected component of $\operatorname{Aut}(G)$, and may also be identified with the connected component of the automorphism group of $\mathfrak{g} := \operatorname{Lie}(G)$. In particular, $G^{\text{ad}}(F)$ coincides with the group of inner automorphisms of G(F) defined over F, and acts simply transitively on the triples $(T, B, \{x_\alpha\})$ of all pinnings of $G(F)$.

where $\mathrm{Aut}(\Phi, \Delta)$ denotes the automorphic of the root system (Φ, Δ). In addition, the pinning $\{x_\alpha\}_{\alpha\in\Delta}$ identifies $\mathrm{Aut}(\Phi, \Delta)$ with $\mathrm{Aut}(G, T, B, \{x_\alpha\})$ and hence introduces a section $s : \mathrm{Aut}(\Phi, \Delta) \to \mathrm{Aut}(G)$ of the morphism $\pi : \mathrm{Aut}(G) \to \mathrm{Aut}(\Phi, \Delta)$ in (8).

By taking Galois cohomology, we obtain the morphisms

$$H^1(F, \mathrm{Aut}(G)) \xrightarrow{H^1_\pi} H^1(F, \mathrm{Out}(G)) \quad \text{and} \quad H^1(F, \mathrm{Out}(G)) \xrightarrow{H^1_s} H^1(F, \mathrm{Aut}(G)).$$

Hence, naturally associated to an element $\xi \in H^1(F, \mathrm{Aut}(G))$ are the elements $H^1_\pi(\xi) \in H^1(F, \mathrm{Out}(G))$ and $(H^1_s \circ H^1_\pi)(\xi) \in H^1(F, \mathrm{Aut}(G))$. It is not too difficult to see that this element belongs to $H^1(F, G^{\mathrm{ad}})$. Denote the induced G^{ad}-torsor on F by G^ξ, which, for later use, we call *the inner form of* G associated to ξ.

For example, an element $H^1(F, \mathrm{Aut}(G))$ induces naturally a G- torsor on F. Hence, if we take ξ as the trivial G-torsor, namely, G itself to start with, then the element corresponding to $(H^1_s \circ H^1_\pi)(\xi)$ defines a split group G^{ad} on F.

3 Compatible Metrics

In this subsection, we assume that our base field is the field of real numbers, unless otherwise stated explicitly.

3.1 Maximal Compact Subgroup

We first recall some basic facts on maximal compact subgroups of a real reductive group, mainly following [2].

Let G be a real Lie group with finitely many connected components. Then any compact subgroup of G is contained in a maximal compact subgroup. Moreover, if K is a maximal subgroup of G, then G is diffeomorphic to the direct product of K with a euclidean space, any maximal compact subgroup is conjugate to K in G and $G/G^0 \simeq K/K^0$, where $\bullet^0$ denotes the connected component of $\bullet$ containing the unit element.

In addition, if G_1 is a closed normal subgroup of G admitting only finitely many connected components, then the maximal compact subgroups of G_1 are the intersections of G_1 with maximal compact subgroups of G. Similarly, if G_1 is a closed subgroup of G with finitely many connected components such that all maximal compact subgroups of G are conjugate by elements of G_1, the maximal compact subgroups of G_1 are the intersections of G_1 with the maximal compact subgroups of G. Consequently, in both cases, by taking a maximal compact subgroup K of G containing a maximal compact subgroup of G_1, we conclude that $G_1 \cap K$ is a maximal compact subgroup in G_1 for at one and hence for all maximal compact subgroups of G by conjugacy.

More generally, if $G \to G'$ is a surjective morphism (of Lie groups) whose kernel admits only finitely many connected components, then the maximal compact subgroups of G' are the images of the maximal compact subgroups of G.

3.2 The Cartan Involution

We here recall some basic facts on *the* Cartan involution associated to an algebraic group, mainly following [5].

Let G be an algebraic group defined over a base field $F \subseteq \mathbb{R}$. Denote by RG the *radical of* G and R_uG the *unipotent radical of* G and R_dG the so-called *split radical of* G, namely, the greatest connected k-split subgroup of RG. By definition, a *Levi subgroup* of G is a maximal reductive k-subgroup of G. Let

$$G^1 = \bigcap_{\chi \in X(G)_k} \mathrm{Ker}(\chi^2), \tag{9}$$

with $X(G)_F := \mathrm{Hom}_F(G, \mathbb{G}_m)$, the group of F-morphism of G into $\mathbb{G}_m$. Then G^1 is a normal subgroup of G, and is defined over F. Note that for a character χ in $X(G)_F$, its restriction to G^1 is of order ≤ 2, hence is trivial on $(G^1)^0$. Consequently,

$$(G^1)^0 = \Big(\bigcap_{\chi \in X(G)_F} \mathrm{Ker}(\chi)\Big)^0. \tag{10}$$

Since any character in $X(G)_F$ is trivial on R_uG, we have $G^1 = L^1 \ltimes R_uG$ for any Levi subgroup L of G. Hence, if A is a maximal F-split torus of RG, then

(i) $G(\mathbb{R}) = A(\mathbb{R})^0 \ltimes G(\mathbb{R})^1$ and

(ii) $G(\mathbb{R})^1$ contains all compact subgroups of $G(\mathbb{R})$. More generally,

(iii) if A_1 and A_2 are two F-tori in RG such that A_1 is F-split, A_1A_2 is a torus and $A_1 \cap A_2$ is finite, then here exists a normal F-subgroup G_1 of G containing A_2 and G^1 such that $G(\mathbb{R}) = A_1(\mathbb{R})^0 \ltimes G_1(\mathbb{R})$.

Therefore, if P is a parabolic F-subgroup of G and A is a maximal F-split torus of the split radical R_dG of G, then, for a maximal compact subgroup K of $G(\mathbb{R})$, we have

(a) $K \cap P$ is a maximal compact subgroup of $P(\mathbb{R})$, and

(b) $G(\mathbb{R}) = KP(\mathbb{R}) = KA(\mathbb{R})^0P(\mathbb{R})$. Furthermore,

(c) if $KaP(\mathbb{R})^0 = Ka'P(\mathbb{R})^0$ for some a, $a' \in A(\mathbb{R})^0$, then $a = a'$ and the map $G(\mathbb{R}) \to A(\mathbb{R})^0$ sending g to $a = a(g)$ characterized by $g \in KaP(\mathbb{R})^0$ is real analytic.

As a direct consequence, when G is a reductive group, there exists one and only one involutive automorphism θ_K of $G(\mathbb{R})$ associated to K satisfying the following properties.

(1) θ_K is "algebraic," i.e. the restriction to $G(\mathbb{R})$ of an involutive automorphism of algebraic groups of the Zariski-closure of $G(\mathbb{R})$ in G.

(2) The fixed point set of θ_K is K.

(3) If G_1 is a normal $\mathbb{R}$-subgroup of G, then $\theta_K(G_1(\mathbb{R})) = G_1(\mathbb{R})$.

(4) θ_K leaves $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}$ stable. Here, we use the same θ_K to denote the induced involution on $\mathfrak{g} := \mathrm{Lie}(G)$, and set $\mathfrak{z}$ denotes the center of $\mathfrak{g}$.

(5) If $\mathfrak{o}$ is the (-1)-eigenspace of θ_K in $\mathfrak{z}$, then $V = \exp \mathfrak{o}$ is a split component of G.

(6) If $\mathfrak{p}$ is the (-1)-eigenspace of θ_K in $\mathrm{Lie}(G(\mathbb{R}))$, then, for $\mathfrak{k} := \mathrm{Lie}(K)$, there is a decomposition $\mathrm{Lie}(G(\mathbb{R})) = \mathfrak{k} \oplus \mathfrak{p}$ and

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad \mathfrak{p}^k \subset \mathfrak{p} \quad (\forall k \in K)$$

(7) The map $(k, X) \mapsto k \cdot \exp(X)$ is an isomorphism of analytic manifolds of $K \times \mathfrak{p}$ onto $G(\mathbb{R})$.

Note that in the case when G is semi-simple, θ_K is the usual Cartan involution. Motivated by this, we call θ_K *the Cartan involution of* $G(\mathbb{R})$ *with respect to* K. Moreover, the existence of the Cartan involution θ_K implies an existence of a non-degenerate $G_{\mathbb{R}}$-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ satisfying the follows.

(a) $\langle \cdot, \cdot \rangle$ is invariant under G and θ_K, and is real on $\mathfrak{g} \times \mathfrak{g}$.

(b) The quadratic form of $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. In particular, if we set

$$(X, Y) := -\langle X, \theta_K Y \rangle \qquad \text{and} \qquad \|X\|^2 := -\langle X, \theta_K X \rangle \quad \forall X, Y \in \mathfrak{g} \tag{11}$$

then $(\cdot, \cdot)$ is a positive definite K-invariant and θ_K-invariant scalar product for $\mathfrak{g} \times \mathfrak{g}$ with $\| \cdot \|$ its associated norm.

(c) For $\mathfrak{g}^1 := \mathrm{Lie}(G^1)$ and $\langle \cdot, \cdot \rangle^1$ and θ_K^1 the restriction of $\langle \cdot, \cdot \rangle$ and θ_K to $\mathfrak{g}^1 \times \mathfrak{g}^1$ and G^1, respectively, we have that $(G^1, K, \theta_K^1, \langle \cdot, \cdot \rangle^1)$ inherit all the properties of $(G, K, \theta_K, \langle \cdot, \cdot \rangle)$ above.

In addition, since $\langle \cdot, \cdot \rangle$ is G-invariant, the following infinitesimal invariance holds.

(d) $\langle \cdot, \cdot \rangle$ is characteristic, namely,

$$\begin{aligned} \langle \phi(X), \phi(Y) \rangle =& \langle X, Y \rangle \qquad \forall \phi \in \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}),\ \forall X,\, Y \in \mathfrak{g} \\ \langle [X, Y], Z \rangle =& \langle X, [Y, Z] \rangle \qquad \forall X,\, Y,\, Z \in \mathfrak{g}. \end{aligned} \tag{12}$$

Therefore, $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}$ are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$. In addition, since $\langle \cdot, \cdot \rangle$ is θ_K-invariant, $\mathfrak{k}$ and $\mathfrak{p}$ is mutually orthogonal. Conversely, we may reconstruct the bilinear form $\langle \cdot, \cdot \rangle$ using all the above conditions. To be more precise, starting with the Cartan-Killing form on $[\mathfrak{g}, \mathfrak{g}]$, we may extend it to obtain $\langle \cdot, \cdot \rangle$ as the direct sum of the Cartan-Killing form with a symmetric non-degenerate bilinear form on $\mathfrak{z}$, which is negative definite on $\mathfrak{z} \cap \mathfrak{k}$ and positive definite on $\mathfrak{z} \cap \mathfrak{p}$. Finally, we may extend this latest $\langle \cdot, \cdot \rangle$ to the total space $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$. For later use, we call $\langle \cdot, \cdot \rangle =: \langle \cdot, \cdot \rangle_K$ the *canonical form on* $\mathfrak{g}$ *associated to* K. For later use, when $\langle \cdot, \cdot \rangle_K$ is viewed as a linear form from $\mathfrak{g}$ to $\mathfrak{g}^*$, we write it as H_K.

Finally, let us point out that the Cartan involution can be applied in many ways. For example, if G_1 is a $\mathbb{R}$-subgroup of G containing $R_u G$ such that all maximal compact subgroups K of $G(\mathbb{R})$ are conjugate under $G_1(\mathbb{R})$, then, for a Levi subgroup M of $G(\mathbb{R})$, it makes sense to take about the Cartan involution θ_K of M with respect to K. Moreover, in this case, the subgroup $(G_1 \cap M) \cap \theta_K(G_1 \cap K)$ is the unique θ_K-stable Levi subgroup of $G_1(\mathbb{R})$ contained in M. Consequently, if P is a parabolic $\mathbb{R}$-subgroup of G, and K is a maximal compact subgroup of $G(\mathbb{R})$ and M is a Levi subgroup of $G(\mathbb{R})$ containing K, then $M \cap P$ contains one and only one tLevi subgroup of $P(\mathbb{R})$ stable under θ_K. For this reasons, we will fix a maximal compact subgroup K of G in the sequel.

3.3 Fine Involutions for Maximal Compact Subgroups

We are now ready to introduce new structures called fine involutions and their associated compatible metrics for general reductive groups, which may be viewed as natural generalizations of the known structures for semi-simple groups (see e.g. [4]).

Let G be a reductive group defined over a subfield $F \subset \mathbb{R}$. Fix a maximal compact subgroup K of G. Motivated by the Cartan involution associated to the maximal compact group K of G, we give the following:

Lemma 8. *Let H be a positive definite real symmetric bilinear form on $\mathfrak{g}$. Assume H is K-compatible with respect to the Lie structure of $\mathfrak{g}$. Then, for $\theta_H := -H_K^{-1}H$,*

(1) $\theta_H^2 = 1$.

(2) $H_K H^{-1} H_K = H$. That is to say, $H_K\theta_H = -H$.

(3) Let $\mathfrak{k}_H$, resp. $\mathfrak{p}_H$, be the $(+1)$- eigenspace, resp. the (-1)-eigenspace, of θ_H on $\mathfrak{g}$. Then

(a) $\mathfrak{g} = \mathfrak{k}_H \oplus \mathfrak{p}_H$.

(b) On $\mathfrak{k}_H$, resp. $\mathfrak{p}_H$, $H_K = -H$, resp. $H_K = H$, is negative definite, resp. positive definite.

(c) $\mathfrak{g} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ is an orthogonal decomposition with respect to H_K.

(4) H is compatible with H_K, i.e. $H_K : \mathfrak{g} \to \mathfrak{g}^*$ is an isometry with respect to the metric H on $\mathfrak{g}$ and the metric H^{-1} on $\mathfrak{g}^*$.

Proof. (1) By the infinitesimal invariance of H_K, namely, two relations in (12), we have ${}^t\theta_H H_K \theta_H = H_K$ since θ_H is compatible with the Lie structure on $\mathfrak{g}$. On the other hand, ${}^t\theta_H H_K = -H_K H^{-1} H_K = H_K\theta_H$. Therefore $\theta_H^2 = 1$.

(2) This is a direct consequence of (1). Indeed, since $\theta_H^2 = 1$, we have $(H^{-1}H_K)(H^{-1}H_K) = \mathrm{Id}_{\mathfrak{g}}$. Therefore, $H_K H^{-1} H_K = H$ and hence $H^t_K heta_H = -H$.

(3) (a) This is a standand result in linear algebra.

(b) This is a direct consequence of (2). Indeed, since $H_K\theta_H = -H_K H^{-1} H_K = -H$, we have that $H_K = -H$ on $\mathfrak{k}$, resp. $H_K = H$ on $\mathfrak{p}$, by the fact that, on $\mathfrak{k}$, resp. on $\mathfrak{p}$, $\theta_H = 1$. Consequently, H_K is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

(c) This is a direct consequence of (b) and (c).

(4) This is a reinterpretation of (2) and (3). Indeed, by (3), $H_K : \mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism. Moreover, by (2), $H_K H^{-1} H_K = H$, we obtain the following commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{H_K} & \mathfrak{g}^* \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H^{-1}} \\ \mathfrak{g}^* & \xleftarrow{H_K} & \mathfrak{g}. \end{array}$$

This implies that $H_K : \mathfrak{g} \to \mathfrak{g}^*$ is an isometry with respect to the metric H on $\mathfrak{g}$ and the metric H^{-1} on $\mathfrak{g}^*$. $\square$

Definition 9. An element θ in the automorphisms group $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is called a *fine involutions* of K (with respect to the Lie structure on $\mathfrak{g}$) if there exists a positive definite real symmetric bilinear form H on $\mathfrak{g}$ such that $\theta = -H^{-1}H_K$ and satisfies all the properties (1), (2), (3) and (4) in Lemma 8. Here, $H_K : \mathfrak{g} \to \mathfrak{g}^*$ denotes the linear isomorphism associated to the bilinear form $\langle\cdot,\cdot\rangle_K$ on $\mathfrak{g}$ induced from the Cartan involution θ_K associated with K. Moreover, if this is the case, H is called a *K-compatible metric on $\mathfrak{g}$ (with respect to its Lie structure)*, [2] and we denote θ by θ_H.

From the discussion above, it is not difficult to see that fine involutions and admissible metrics on $\mathfrak{g}$ associated to K works exactly in the same way for G^1 as well, since G^1 is reductive and all maximal compact subgroups of G are contained in G^1. Indeed, the corresponding constructions on G^1 may be viewed as the restrictions of the structures from $\mathfrak{g}$ to $\mathfrak{g}^1 := \mathrm{Lie}(G^1)$. For later use, set

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{v}. \tag{13}$$

3.4 Compatible Metrics for Maximal Compact Subgroups

Denote by $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$, resp. $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K}$ the moduli space of K-compatible euclidean metrics on $\mathfrak{g}$, resp. on $\mathfrak{g}^1$. Since they contains (the isometric class of) $(\cdot,\cdot)_K$ in (11), both $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$ and $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K}$ are not empty. Moreover, $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$, resp. $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K}$, admits a natural interpretation as a subspace of the space of (isomety classes of) euclidean metrics on $\mathfrak{g}$, resp. on $\mathfrak{g}^1$. Our main result of this section is the following:

Proposition 10. *Let G be a reductive group defined over a subfield $F \subset \mathbb{R}$ and let K be a maximal compact subgroup of G. Set $\mathfrak{g} = \mathrm{Lie}(G)$. We have*

(1) There are natural actions of G on $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$ and $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K}$.

(2) The action of G on $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$ in (1) induces a natural diffeomorphism

$$G^1(\mathbb{R})/K \simeq \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K} \tag{14}$$

Proof. (1) Recall that, for $\phi \in \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$, we have ${}^t\phi H_K \phi = H_K$ since H_K is characteristic by (12). Hence, for any $H \in \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$, we have

$$(-\,{}^t\phi H\phi)H_K = \phi^{-1}\theta\phi.$$

This implies that ${}^t\phi H\phi$ belongs to $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$ as well. Consequently, the assignment

$$(H,g) \longmapsto {}^t(\mathrm{Ad}\,g)H(\mathrm{Ad}\,g) \qquad \forall H \in \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K},\ \forall g \in G \tag{15}$$

defines a natural action of G on $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K}$. Here, as usual, for $g \in G$, $\mathrm{Ad}\,g$ denotes its adjoint. This proves (1).

(2) In terms of the fine involutions $\theta = \theta_H$, the action above is equivalent to

$$(\theta,g) \longmapsto {}^t(\mathrm{Ad}\,g)\theta(\mathrm{Ad}\,g) \qquad \forall g \in G \tag{16}$$

[2] Here as usual, we view the bilinear H as a linear map from $\mathfrak{g}$ to $\mathfrak{g}^*$. Hence $H^{-1}H_K$ is indeed a linear endomorphism of $\mathfrak{g}$.

Recall that, for a general G, if we set

$$\mathfrak{v}_\theta := \{\, X \in \mathfrak{z} : \theta X = -X \,\}\,.$$

Then $V_\theta := \exp(\mathfrak{v}_\theta)$ is a split component of G. [3] Moreover, by Proposition 2.1.10 of [5], the assignment $\theta \mapsto \mathfrak{v}_\theta$ gives a bijection from the set of fine involutions associated to K to the set of split components of G. In particular, when $G = G^1$, this map gives a bijection from the set of fine involutions to the set of maximal compact subgroups of G. Hence, in our case, since G is connected and reductive, all maximal compact subgroups of G are conjugate to each other. This implies that G and hence G^1 acts transitively on $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K}$. Thus, to complete our proof, it suffices to show that the stabilizer group of θ_K in G^1 is exactly K itself. For this we choose $\Theta_K : G^1(\mathbb{R}) \to G^1(\mathbb{R})$ to be a Cartan involution satisfying $d\Theta_K = \theta_K$. By definition,

$$K = \left\{\, g \in G^1(\mathbb{R}) : \Theta(g) = g \,\right\}.$$

On the other hand, for $g \in G^1$, $\theta_K = (\mathrm{Ad}g)^{-1}\theta_K(\mathrm{Ad}g)$ if and only if $g^{-1}\Theta_K g$ is in the center of $G(\mathbb{R})^0$, the connected component of $G(\mathbb{R})$ containing the unit element. But this center is trivial by our assumption, hence g belongs to the stabilizer group of θ_K if and only if $g \in K$. $\square$

From the proof, we conclude that $\mathfrak{v}$ in (13) is identified with $\mathfrak{v}_H$ for a certain compatible H of K. As a direct consequence, we obtain the following:

Corollary 11. *Denote by $\mathcal{M}_\mathfrak{v}$ be the moduli space of euclidean metric on $\mathfrak{v}$. Then*

(1) $\mathcal{M}_\mathfrak{v} \simeq \mathrm{GL}_{\dim_\mathbb{R}\mathfrak{v}}(\mathbb{R})/O_{\dim_\mathbb{R}\mathfrak{v}}(\mathbb{R})$, where $O_n(\mathbb{R})$ denotes the orthogonal group of degree n.

(2) The natural map defined by

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g};K} & \longrightarrow & \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1;K} \times \mathcal{M}_\mathfrak{v} \\ H & \mapsto & (H|_{\mathfrak{g}^1}, H|_\mathfrak{v}) \end{array}$$

is bijecive.

4 Arithmetic G-Torsors on $\overline{\operatorname{Spec}\mathcal{O}_F}$

4.1 Integral Structures on Lie Algebras

Let F be an algebraic number field with $\mathcal{O}_F$ the ring of integers, and let G be a connected split reductive group over F with $\mathfrak{g}$ its Lie algebra. Our aim here is to introduce a $G(\mathcal{O}_F)$-invariant projective $\mathcal{O}_F$-module $\mathfrak{g}_{\mathcal{O}_F}$ in $\mathfrak{g} \otimes_F \mathbb{R}$ which is closed under the Lie operation.

For simplicity, assume $F = \mathbb{Q}$. Since G is defined over $\mathbb{Q}$, its Lie algebra $\mathfrak{g}$ admits a natural rational structure $\mathfrak{g}_\mathbb{Q}$ and the adjoint representation

[3] Denote by $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ the adjoint action of G on $\mathfrak{g}$. Then $\mathrm{Ker}(\mathrm{Ad})$ acts trivially on $\mathfrak{g}$. A *split component* of G is defined to be a maximal closed linear subspace of $\mathrm{Ker}(\mathrm{Ad})$. By Proposition 2.1.5 of [5], if V is a split component of G, then $G = G^1 \cdot V$.

$G \to \mathrm{End}_{\mathrm{Lie}}(\mathfrak{g}_{\mathbb{Q}})$ is a morphism defined over $\mathbb{Q}$. Consequently, there always exist $G(\mathbb{Z})$-invariant integral structures in $\mathfrak{g}_{\mathbb{Q}}$, since, for any integral structure in $\mathfrak{g}_{\mathbb{Q}}$, the image under the action of $G(\mathbb{Z})$ is again an integral structure in $\mathfrak{g}_{\mathbb{Q}}$. Obviously, the summation of two $G(\mathbb{Z})$-invariant integral structures in $\mathfrak{g}_Q$ is again a $G(\mathbb{Z})$-invariant integral structure. Moreover, if $\mathfrak{g}_{\mathbb{Z}}$ is a $G(\mathbb{Z})$-invariant integral structure in $\mathfrak{g}_Q$, we have $[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_Z] \subset \mathfrak{g}_{\mathbb{Q}}$. Hence, by clearing up denominators, we can instead assume that $[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_Z] \subseteq \mathfrak{g}_{\mathbb{Z}}$ from the beginning. In this way, we obtain a unique maximal $G(\mathbb{Z})$-invariant integral structure $\mathfrak{g}_{\mathbb{Z}}$ in $\mathfrak{g}_Q$ satisfying the condition that $[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_Z] \subseteq \mathfrak{g}_{\mathbb{Z}}$.

Put this in a more concrete form, since our reductive group G is defined over $\mathbb{Q}$, we may use the structural decomposition

$$\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Q}}^{\mathrm{ss}} \oplus \mathfrak{z}_{\mathbb{Q}}$$

where $\mathfrak{g}_{\mathbb{Q}}^{\mathrm{ss}}$ denotes the rational structure on $\mathfrak{g}^{\mathrm{ss}}$ induced by the semi-simple Lie sub-algebra of $\mathfrak{g}$. Since $\mathfrak{v} \subset \mathfrak{z}$, this decomposition is compatible with $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Q}}^1 \oplus \mathfrak{v}_{\mathbb{Q}}$ induced by (13). Moreover, since $\mathfrak{z}$ is abelian, we obtain a natural decomposition

$$\mathfrak{z}_{\mathbb{Q}} = \mathfrak{v}_{\mathbb{Q}} \oplus \mathfrak{g}_{\mathbb{Q}}^1 / {\mathfrak{g}_{\mathbb{Q}}}^{\mathrm{ss}}. \tag{17}$$

Now by applying the Chevalley (integral) basis for semi-simple Lie algebras, we obtain a canonical integral structure $\mathfrak{g}_{\mathbb{Z}}^{\mathrm{ss}}$ on ${\mathfrak{g}_{\mathbb{Q}}}^{\mathrm{ss}}$. Hence, what is left is to introduce an integral structure on $\mathfrak{z}_{\mathbb{Q}}$ which is compatible with the decomposition (17). But this is trivial since $\mathfrak{z}$ is an abelian sub Lie algebra defined over $\mathbb{Q}$. We thus obtain an induced integral structure on $\mathfrak{z}_{\mathbb{Q}}$, which we denoted by $\mathfrak{z}_{\mathbb{Z}}$. Similar arguments then lead to the integral structures $\mathfrak{g}_{\mathbb{Z}}^1$ and $\mathfrak{v}_{\mathbb{Z}}$ on $\mathfrak{g}_{\mathbb{Q}}^1$ and $\mathfrak{v}_{\mathbb{Q}}$, respectively. Consequently, we have

$$\mathfrak{g}_{\mathbb{Z}} := \mathfrak{g}_{\mathbb{Z}}^{\mathrm{ss}} \oplus \mathfrak{z}_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{Z}}^1 \oplus \mathfrak{v}_{\mathbb{Z}}. \tag{18}$$

The above discussion works well if we replace $G/\mathbb{Q}$ by a split reductive group G/F with F a general number field. To indicate the dependence on F, we rewrite the associated Lie algebra by $\mathfrak{g}_F$. Since it admits a natural F-linear space structure, through the Minkowski embedding $F \hookrightarrow F_\infty := \prod_{\sigma \in S_\infty} F_\sigma$, we obtain a Lie algebra

$$\mathfrak{g}_\infty := \mathfrak{g}_F \otimes_{\mathbb{Q}} \mathbb{R} := \prod_{\sigma \in S_\infty} \mathfrak{g} \otimes F_\sigma. \tag{19}$$

We introduce an $\mathcal{O}_F$-lattice structure on $\mathfrak{g}_F$ by setting

$$\mathfrak{g}_{\mathcal{O}_F} := \mathfrak{g}_{\mathbb{Z}} \otimes \mathcal{O}_F \hookrightarrow \mathfrak{g}_F \hookrightarrow \mathfrak{g}_\infty. \tag{20}$$

Then

$$\mathfrak{g}_{\mathcal{O}_F} = \mathfrak{g}_{\mathcal{O}_F}^{\mathrm{ss}} \oplus \mathfrak{z}_{\mathcal{O}_F} = \mathfrak{g}_{\mathcal{O}_F}^1 \oplus \mathfrak{v}_{\mathcal{O}_F} \tag{21}$$

where $\mathfrak{g}_{\mathcal{O}_F}^{\mathrm{ss}} := \mathfrak{g}_{\mathbb{Z}}^{\mathrm{ss}} \otimes \mathcal{O}_F$, $\mathfrak{z}_{\mathcal{O}_F} = \mathfrak{z}_{\mathbb{Z}} \otimes \mathcal{O}_F$, $\mathfrak{g}_{\mathcal{O}_F}^1 = \mathfrak{g}_{\mathbb{Z}}^1 \otimes \mathcal{O}_F$ and $\mathfrak{v}_{\mathcal{O}_F} = \mathfrak{v}_{\mathbb{Z}} \otimes \mathcal{O}_F$. Using a similar argument as above, we conclude that $\mathfrak{g}_{\mathcal{O}_F}$ is a projective $\mathcal{O}_F$-submodule in $\mathfrak{g}_F$ such that $[\mathfrak{g}_{\mathcal{O}_F}, \mathfrak{g}_{\mathcal{O}_F}] \subseteq \mathfrak{g}_{\mathcal{O}_F}$. In the sequel, $\mathfrak{g}_{\mathcal{O}_F}$ will be called *the canonical infinitesimal $\mathcal{O}_F$-structure* of G/F.

4.2 Arithmetic G-Torsors

Let G be a split reductive group over a number field F. For each $\sigma \in S_\infty$, we fix a maximal compact subgroup K_σ of $G^1(F_\sigma)$, set $r_{G,\sigma}^{G^{\mathrm{ss}}} := \operatorname{rank}(G) - \operatorname{rank}(G^{\mathrm{ss}})$. We denote a real, resp. complex, $\sigma \in S_\infty$ by $\sigma : \mathbb{R}$, resp. $\sigma : \mathbb{C}$. By §3.4, we obtain the moduli spaces $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}_{F_\sigma};K_\sigma}$, resp. $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1_{F_\sigma};K_\sigma}$, of the compatible metrics with respect to K_σ on $\mathfrak{g}_{F_\sigma}$, resp. $\mathfrak{g}^1_{F_\sigma}$, and natural isomorphisms

$$\begin{aligned}
\prod_{\sigma\in S_\infty} \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1_{F_\sigma};K_\sigma} &\xrightarrow{\simeq} \prod_{\sigma\in S_\infty} G^1(F_\sigma)/K_\sigma,\\
\prod_{\sigma\in S_\infty} \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}_{F_\sigma};K_\sigma} &\xrightarrow{\simeq} \prod_{\sigma\in S_\infty} \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^1_{F_\sigma};K_\sigma}\\
&\quad\times\Big(\big(\mathrm{GL}_{r_{G,\sigma}^{G^{\mathrm{ss}}}}(\mathbb{R})/O_{r_{G,\sigma}^{G^{\mathrm{ss}}}}(\mathbb{R})\big)^{r_1}\times\big(\mathrm{GL}_{r_{G,\sigma}^{G^{\mathrm{ss}}}}(\mathbb{C})/U_{r_{G,\sigma}^{G^{\mathrm{ss}}}}(\mathbb{C})\big)^{r_2}\Big).
\end{aligned}\tag{22}$$

Here $O_n(\mathbb{R})$, resp. $U_n(\mathbb{C})$, denotes the orthogonal group, resp. the unitary group, of degree n, and r_1, resp. r_2, denotes the number of real, resp. complex, places of F. For later use, set

$$G^1(F_\infty)/K(F_\infty) := \prod_{\sigma\in S_\infty} G^1(F_\sigma)/K_\sigma \quad \text{and} \quad \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^\bullet_\infty;K_\infty} := \prod_{\sigma\in S_\infty} \mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}^\bullet_{F_\sigma};K_\sigma} \tag{23}$$

where, to simplify our notations, we use $\mathfrak{g}^\bullet$ as a running symbol for $\mathfrak{g}$ and $\mathfrak{g}^1$.

Definition 12. Let G/F be a connected (split) reductive group and let $K_\infty := (K_\sigma)_{\sigma\in S_\infty}$ be a family of maximal compact subgroups of $(G(F_\sigma))_{\sigma\in S_\infty}$. By a *$K_\infty$-compatible arithmetic G-torsor* over $\overline{\operatorname{Spec}\mathcal{O}_F}$, or simply over $\mathcal{O}_F$, we mean a tuple $(\mathcal{G}, (H_\sigma)_{\sigma\in S_\infty})$ consisting of a G-torsor $\mathcal{G}$ on $\overline{\operatorname{Spec}\mathcal{O}_F}$ and an element $(H_\sigma)_{\sigma\in S_\infty}$ of $\mathcal{M}^{\mathrm{tot}}_{\mathfrak{g}_\infty;K_\infty}$.

Even apparently not quite related, $H_\infty := (H_\sigma)_{\sigma\in S_\infty}$ may be viewed as a family of $(K_\sigma)_{\sigma\in S_\infty}$-compatible metrics on the tangent bundles of the G-torsor $\mathcal{G}_\infty := \prod_{\sigma\in S_\infty} \mathcal{G}_\sigma$. To explain this, we first recall that $\mathcal{G}_\eta$ admits a natural $G(F)$-torsor structure. This, via the Minkowski embedding, induces a natural $G(F_\infty)$-torsor structure on $\mathcal{G}_\infty$. Hence, for a fixed base point of $\mathcal{G}_\infty$, the the tangent space of $\mathcal{G}_\infty$ at this point is canonically identified with $\mathfrak{g}_\infty$. Consequently, we obtain a natural metric on this tangent space. Moreover, since $\mathcal{G}_\infty$ is a $G(F_\infty)$-torsor, its tangent bundle is a flat bundle. Thus, with the help of the so-called parallel transforms, we obtain a natural metric on the tangent bundle of $\mathcal{G}_\infty$. This metric is uniquely determined by H_∞.

Furthermore, working over G^1, at infinite places, H_∞ is a K_∞-compatible metric on $\mathfrak{g}^1_\infty$. Thus $(\mathfrak{g}^1_{\mathcal{O}_F}, H_\infty)$ for an $\mathcal{O}_F$-lattices in $\mathfrak{g}^1_\infty$ whose projective $\mathcal{O}_F$-module component $\mathfrak{g}^1_{\mathcal{O}_F}$ also admits a natural Lie structure over $\mathcal{O}_F$ because, by our construction, $[\mathfrak{g}^1_{\mathcal{O}_F}, \mathfrak{g}^1_{\mathcal{O}_F}] \subseteq \mathfrak{g}^1_{\mathcal{O}_F}$.

Definition 13. Let $(\Lambda, (H_\sigma))$ be a pair consisting of a projective $\mathcal{O}_F$-module $\Lambda \subset \mathfrak{g}_F$ and a family of K_∞-compatible metrics on $\mathfrak{g}_\infty$. If Λ is $G(\mathcal{O}_F)$-invariant under the adjoint action, and $[\Lambda, \Lambda] \subset \Lambda$, then $(\Lambda, (H_\sigma))$ is called a *K_∞-compatible principle G^1-lattices over $\mathcal{O}_F$*.

Denote by $\mathcal{M}^{\mathrm{tot}}_{G,F}$, resp. $\mathcal{M}^{\mathrm{tot}}_{G^1,F}$, the moduli stack of K_∞-compatible arithmetic G-torsors, resp. G^1-torsors, on $\overline{\operatorname{Spec}\mathcal{O}_F}$. Then we have the following

Theorem 14. *Let G be a split reductive group on F. Then there exist the following natural identifications:*

(1) $\mathcal{M}^{\mathrm{tot}}_{G^1,F} \simeq \prod_{\xi\in\mathrm{Ker}^1(F,G^1)} G^{1,\xi}(F)\backslash G^1(\mathbb{A})/\prod_{v\in S_{\mathrm{fin}}} G^1(\mathcal{O}_v)\times K_\infty.$

(2) $\mathcal{M}^{\mathrm{tot}}_{G,F} \simeq \displaystyle\prod_{\xi\in\mathrm{Ker}^1(F,G)} G^\xi(F)\backslash G(\mathbb{A})/\big(\prod_{v\in S_{\mathrm{fin}}} G(\mathcal{O}_v)\times K_\infty\times O_{r_{G^{\mathrm{ss}}_{G,\sigma}}}(\mathbb{R})^{r_1}\times U_{r_{G^{\mathrm{ss}}_{G,\sigma}}}(\mathbb{C})^{r_2}\big).$

Proof. (1) For each $v\in S_{\mathrm{fin}}$, let $X_v=\mathrm{Spec}(\mathcal{O}_v)$ and set $X_v^\bullet$ be the complementary open subset of $\{v\}$ in X_v. Then for each element g_v of the affine Grassmannian $G(F_n)/G(\mathcal{O}_v)$, we obtain a $G(F_v)$-torsor $\mathcal{E}_v$ on X_v equipped with a trivialization on $X_v^\bullet$ with the trivial G torsor, so that if an automorphicm of $\mathcal{E}_v$ is trivial on $X_v^\bullet$, then it is necessarily trivial. Recall that there exists a natural morphism $G(F_v)/G(\mathcal{O}_v)\longrightarrow G(F)\backslash\prod' G(F_v)/G(\mathcal{O}_v)$ which maps $g_v\in G(F_v)/G(\mathcal{O}_v)$ on the tuple consisting of g_v at v and the unit elements of $G(F_{v'})/G(\mathcal{O}_{v'})$ for all places $v'\in S_{\mathrm{fin}}\smallsetminus\{v\}$. Then, by the compatibility condition in Lemma 7, and the fact that $\operatorname{Spec}\mathcal{O}_F$ is affine and $\mathcal{O}_F$ is Dedekind domain, we conclude that the local maps induces an identifications of $G(F)\backslash\prod' G(F_v)/G(\mathcal{O}_v)$ with the moduli spaces of G^1-torsors on $\overline{\operatorname{Spec}\mathcal{O}_F}$, by adopting a result of Beauville-Laszlo [?] in the case when $G=\mathrm{GL}_n$ and that of Heinloth in [?] for general cases. Therefore, to conclude our proof, it suffices to apply Proposition 10 to take care of the factor of K_∞-compatible metrics (H_σ), since $G(\mathbb{A}_F)$ is nothing but $\prod'_{v\in S} G(F_v)$.

(2) is a direct consequence of (1) and its proof, if we apply Corollary 11. □

In the sequel, to simplify our presentations, we will use arithmetic G-torsors instead of the full version of K_∞-compatible arithmetic G-torsors over $\overline{\operatorname{Spec}\mathcal{O}_F}$, if no confusion arises.

4.3 Slopes of Arithmetic G-Torsors

Let T_G be the maximal split torus in the center Z_G of G and let T'_G be the maximal split quotient torus of G. Then

$$T_G=\mathrm{Hom}(\mathbb{G}_m,Z_G)\otimes\mathbb{G}_m \qquad\text{and}\qquad T'_G=\mathrm{Hom}(\mathrm{Hom}(G^{\mathrm{ab}},\mathbb{G}_m),\mathbb{G}_m).$$

Here $G^{\mathrm{ab}}:=G/[G,G]$ denotes the maximal abelian quotient of G. It is well known that the composition

$$T_G\hookrightarrow Z_G\hookrightarrow G\twoheadrightarrow G^{\mathrm{ab}}\twoheadrightarrow T'_G$$

is an isogeny, i,.e. a morphism with finite kernel and cokernel. Consequently, if we set

$$\begin{aligned} X_*(T_G)&:=\mathrm{Hom}(\mathbb{G}_m,T_G), \qquad & X_*(T'_G)&:=\mathrm{Hom}(\mathbb{G}_m,T'_G)\\ X^*(T_G)&:=\mathrm{Hom}(T_G,\mathbb{G}_m), \qquad & X_*(T'_G)&:=\mathrm{Hom}(T'_G,\mathbb{G}_m)\end{aligned} \tag{24}$$

then $X_*(T_G)\hookrightarrow X_*(T'_G)$ is a free abelian group of the same rank. Moreover, since $\mathrm{Hom}(\mathbb{G}_m,\mathbb{G}_m)\simeq\mathbb{Z}$, there is a non-degenerate pairing

$$\begin{array}{rccc}\langle\cdot,\cdot\rangle: & X_*(T_G^\bullet)\times X^*(T_G^\bullet) & \longrightarrow & \mathbb{Z}\\ & (\chi,\mu) & \mapsto & \langle\chi,\mu\rangle\end{array} \tag{25}$$

Here $T_G^\bullet = T_G$ or T'_G. Set now

$$X_*(T_G^\bullet)_\infty := \prod_{\sigma\in S_\infty} X_*(T_G^\bullet)_{F_\sigma} \quad \text{and} \quad X^*(T_G^\bullet)_\infty := \prod_{\sigma\in S_\infty} X^*(T_G^\bullet)_{F_\sigma}, \tag{26}$$

then (25) induces a non-degenerating pairing

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle: & X_*(T_G^\bullet)_\infty \times X^*(T_G^\bullet)_\infty & \longrightarrow & \mathbb{C} \\ & (\chi,\mu) & \mapsto & \langle\chi,\mu\rangle. \end{array} \tag{27}$$

Obviously, there is a natural action of $\prod_{\sigma\in S_\infty}\mathrm{Gal}(F_\sigma/\mathbb{R})$ on $X^*(T_G^\bullet)_\infty$, and

$$\Big\langle X_*(T_G^\bullet), X^*(T_G^\bullet)_\infty^{\prod_{\sigma\in S_\infty}\mathrm{Gal}(F_\sigma/\mathbb{R})}\Big\rangle \subseteq \mathbb{R}. \tag{28}$$

For our own convenience, we denote the invariant space $X^*(T_G^\bullet)_\infty^{\prod_{\sigma\in S_\infty}\mathrm{Gal}(F_\sigma/\mathbb{R})}$ by $X^*(T_G^\bullet)_\infty^{\mathrm{ar}}$.

Definition 15. Let $\overline{\mathcal{G}} = (\mathcal{G}, (H_\sigma))$ be a K_∞-compatible arithmetic G- torsor over $\overline{\mathrm{Spec}\,\mathcal{O}_F}$. An element $\mu \in X_*(T'_G)_\infty$ is called the *slope of* $\overline{\mathcal{G}}$, denoted by $\mu(\overline{\mathcal{E}})_\infty$, if, for all $\chi \in X^*(A'_G)$, we have

$$\langle\chi,\mu\rangle = \deg_{\mathrm{ar}}(\overline{\mathcal{G}}_\chi) \tag{29}$$

where $\overline{\mathcal{E}}_\chi$ denotes an arithmetic $\mathbb{G}_m$-torsor on $\overline{\mathrm{Spec}\,\mathcal{O}_F}$ induced by the reduction of structure group $G \twoheadrightarrow T'_G \xrightarrow{\chi} \mathbb{G}_m$, and $\deg_{\mathrm{ar}}$ denotes the arithmetic degree.

This definition makes sense, since an arithmetic $\mathbb{G}_m$-torsor on $\overline{\mathrm{Spec}\,\mathcal{O}_F}$ is simply a metrized line bundle on $\overline{\mathrm{Spec}\,\mathcal{O}_F}$. Hence its arithmetic degree $\deg_{\mathrm{ar}}(\overline{\mathcal{G}}_\chi)$ is well-defined.

Remark 1. (1) Arithmetic G-torsors are first introduced in my book [10]. There is a serious overlap between this section and §16.2 of [10], even the context here is much clearer.

(2) As to be expected, the slope can be use to definite stability of arithmetic G-torsors ([10]). We omit the details, since it will not be used in our current work.

5 Arithmetic Characteristic Curves

Let G/F be a split reductive group with pinning $(T, G, \{x_\alpha\}_{\alpha\in\Delta})$ such that $\{x_\alpha\}_{\alpha\in\Delta}$ can be extended to a Chevalley basis of $\mathfrak{g} = \mathrm{Lie}(G)$.

Let $(\mathcal{G}, H)$ be an arithmetic G-torsor on $\overline{X}$. Denote by $\mathfrak{g}_X$ the induced locally free sheaf on X, or equiva;ently, the associated projective $\mathcal{O}_F$-module in $\mathfrak{g}_\infty$.

Let $(\mathcal{L}, \rho)$ be a metrized invertible sheaf on $\overline{X}$. Denote by L its associated rank one projective $\mathcal{O}_F$-module. Since $\mathcal{O}_F$ is Dedekind, we may identify L with a certain fractional ideal of F. Denote this fractional ideal by the same letter L, by an abuse of notations.

For an element $\varphi \in \mathfrak{g}_X \otimes_{\mathcal{O}_F} L \subset \mathfrak{g}_F \subset \mathfrak{g}_\infty$, we denote its images under the Chevalley characteristic morphisms χ_F and χ_∞ by $\chi(\varphi_F) \in \mathfrak{c}_F$ and $\chi(\varphi_\infty) \in \mathfrak{c}_\infty$, respectively. It is not difficulty to see that $\chi(\varphi) \in \mathfrak{c}(\mathcal{O}_F)$.

The element φ can also be used to identify X with the horizontal section associated to $\varphi\,\mathcal{O}_X$ in $\mathfrak{g}_X \otimes \mathcal{L}$. In this way, we obtain a morphism $a_\varphi : X \to \mathfrak{c}_{\mathcal{O}_F}$.

Definition 16. Let G/F be a split reductive group.

(1) The pair $(\mathcal{G}, \varphi)$ consisting of an arithmetic G-torsor $\mathcal{G}$ and an element $\varphi \in \mathfrak{g}_X \otimes_{\mathcal{O}_F} \mathcal{L}$ with $\mathcal{L}/\mathrm{Spec}\,\mathcal{O}_F)$ an invertible line sheaf is called an arithmetic Higgs G-torsor.

(2) The *characteristic arithmetic curve* associated to $(\mathcal{G}; \mathcal{L}; \varphi)$ is defined to be the scheme X_φ of arithmetic dimension one obtained from the morphism $X \stackrel{a_\varphi}{\to} \mathfrak{c}_{\mathcal{O}_F}$ through the base change $\mathfrak{t}_{\mathcal{O}_F} \to \mathfrak{c}_{\mathcal{O}_F}$. That is to say,

$$X_\varphi = X \times_{\mathfrak{c}_{\mathcal{O}_F}} \mathfrak{t}_{\mathcal{O}_F}$$

induced from the product diagram

$$\begin{array}{ccc} X_\varphi & \stackrel{\chi_\varphi}{\longrightarrow} & X \\ \downarrow & & \downarrow a_\varphi \\ \mathfrak{t}_{\mathcal{O}_F} & \stackrel{\chi}{\longrightarrow} & \mathfrak{c}_{\mathcal{O}_F}. \end{array} \tag{30}$$

Obviously, the morphism $\chi_\varphi : X_\varphi \to X$ is a finite $(|W| : 1)$-covering, even highly ramified in general. Here, as usual, W denotes the Weyl group of G/F.

Remark 2. (1) The above construction is motivated by the construction of spectral curve (for $G = \mathrm{GL}_n$) and cameral curve (for general reductive G) by Beauville-Narasimhan ([1]) and Donagi- Gaistgory ([3]), respectively.

(2) When $G = \mathrm{GL}_n$, with the identification $\mathfrak{t} \simeq \mathrm{Spec}\,F[\mathfrak{t}]$, the Chevalley characteristic morphism may be viewed as the assignment for diagonal matrices to their unorded eigenvalues. For this reason, we sometimes also call X_φ the *arithmetic eigen curve* of X associated to φ.

In the forthcoming papers ([11], [12]), we will use arithmetic characteristic curves to construct arithmetic Hitchin fibrations and study the intersection homologies and perverse sheaves for the associated structures, following (Laumon-)Ngo's approach to the fundamental lemma ([9]) using Hitchin fibrations ([8]).

References

[1] A. Beauville, M.S. Narasimhan, S. Ramanan, Spectral curves and the generalised theta divisor. J. Reine Angew. Math. 398 (1989), 169-179.

[2] A. Borel, J.P. Serre, Corners and arithmetic groups. Comment. Math. Helv. 48 (1973), 436 - 491.

[3] R.Y. Donagi, D. Gaitsgory, The gerbe of Higgs bundles. Transform. Groups 7 (2002), no. 2, 109-153.

[4] D.R. Grayson, Reduction theory using semistability. II. Comment. Math. Helv. 61 (1986), no. 4, 661-676.

[5] R. Gangolli, V.S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 101. Springer-Verlag, Berlin, 1988.

[6] J.E. Humphreys, *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978.

[7] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields.* Grundlehren der Mathematischen Wissenschaften 323. Springer-Verlag, Berlin, 2000.

[8] Fibration de Hitchin et endoscopie. Invent. Math. 164 (2006), no. 2, 399-453.

[9] B.C. Ngô, Le lemme fondamental pour les algebres de Lie. Publ. Math. Inst. Hautes Etudes Sci. No. 111 (2010), 1-169.

[10] L. Weng, *Zeta Functions of Reductive Groups and Their Zeros*, World Scientific 2018.

[11] L. Weng, Arithmetic Hitchin Fibrations I: Basic Properties, in preparations

[12] L. Weng, Arithmetic Hitchin Fibrations II: Perverse Sheaves and Intersection Homology, in preparations

Lin WENG
Institute for Fundamental Research
The L Academy
and
Faculty of Mathematics
Kyushu University
Fukuoka, 819-0395
JAPAN
E-Mail: weng@math.kyushu-u.ac.jp