# Codes and Stability

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**Abstract**

We introduce new yet easily accessible codes for elements of $\mathrm{GL}_r(\mathbb{A})$ with $\mathbb{A}$ the adelic ring of a (dimension one) function field over a finite field. They are linear codes, and coincide with classical algebraic geometry codes when $r = 1$. Basic properties of these codes are presented. In particular, when offering better bounds for the associated dimensions, naturally introduced is the well-known stability condition. This condition is further used to determine the minimal distances of these codes. To end this paper, for reader's convenience, we add two appendices on some details of the adelic theory of curves and classical AG codes, respectively.

## 1 Introduction

Let $F/\mathbb{F}_q$ be a function field of an integral regular projective curve $X$ over $\mathbb{F}_q$. Denote by $\mathbb{A}$ its adelic ring with $\mathcal{O}$ the ring of integers, and, for a positive integer $r$, denote by $\mathrm{GL}_r(\mathbb{A})$ the general linear group with coefficients in $\mathbb{A}$. Fix a degree $n$ divisor $D = \sum_{i=1}^n p_i$.

For an element $g = (g_p)_p \in \mathrm{GL}_r(\mathbb{A})$, we introduce a subspace $\mathbb{A}^r(g)$ of $\mathbb{A}^r$ by

$$\mathbb{A}^r(g) := \big\{a \in \mathbb{A}^r \mid ga \in \mathcal{O}^r\big\}, \tag{1}$$

and define the comology groups of $g$ on $F$ by

$$H^0(F,g) := \mathbb{A}^r(g) \cap F^r \qquad \text{and} \qquad H^1(F,g) := \mathbb{A}^r\big/(\mathbb{A}^r(g) + F^r). \tag{2}$$

In addition, we introduce the space of $r$-multiple differentials $\Omega_F^r(g)$ by

$$\Omega_F^r(g) := \big\{(\omega_j) \in \Omega_F^r : \mid (\phi(\omega_j)) \in g\mathcal{O}^r\big\} \tag{3}$$

where $\Omega_F$ denote the space of rational differentials on $X$ and $\phi : \Omega_F \to F, \omega \mapsto \frac{\omega}{\omega_0}$ is an $F$-isomorphism with $\omega_0$ a $D$-special rational differential. It is known, see e.g. §2 that $H^i(F,g)$ satisfy the standard topology duality and the Riemann-Roch relation. For example, we have

$$\Omega_F^r(g) \simeq H^1(F, \iota_{(\omega_0)} g^{-1}) \tag{4}$$

where, for a divisor $E$ on $X$, we set $\iota_E = \big(\pi_p^{\mathrm{ord}_p(E)}\big) \in \mathbb{I} := \mathrm{GL}_1(\mathbb{A})$ to be its characterizing idele.

If $g$ is $D$-balanced, we introduce the following codes

$$\begin{aligned} C_{F,r}(D,g) :=& \Big\{ \big(f_j(p_1)\big), \dots, \big(f_j(p_n)\big) \mid f = (f_j) \in H^0(F,g) \Big\}, \\ C_{\Omega,r}(D,g) :=& \Big\{ \big(\omega_{j,p_1}(1)\big), \dots, \big(\omega_{j,p_n}(1)\big) \mid (\omega_j) \in \Omega_F^r(g(-D)) \Big\}. \end{aligned} \tag{5}$$

Here $g(-D) := \iota_D^{-1} g$. In this paper, we show the following results on basic properties of the above codes.

**Theorem.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-balanced.*

*(0) If $E$ is a divisor with support away from the $p_i$'s, then*

$$C_{F,1}(D, \iota_E) = C_L(D, E) \qquad \text{and} \qquad C_{\Omega,1}(D, \iota_E) = C_\Omega(D, E).$$

*(1) $C_{F,r}(D, g)$ and $C_{\Omega,r}(D, g)$ are linear codes of length $rn$ and mutually dual to each other. That is,*

$$C_{\Omega,r}(D, g)^\perp = C_{F,r}(D, g) \quad \text{and} \quad C_{\Omega,r}(D, g) = C_{F,r}(D, \iota_{(\omega_0)+D} g^{-1}).$$

*(2) The dimensions $k_{D,g}$ and $k_{D,g}^\perp$ of $C_{F,r}(D, g)$ and $C_{\Omega,r}(D, g)$ are given by*

$$h^0(F, g) - h^0(F, g(-D)) \qquad \text{and} \qquad h^1(F, g(-D)) - h^1(F, g),$$

*respectively. In particular,*

$$\dim_{\mathbb{F}_q} C_{F,r}(D, g) + \dim_{\mathbb{F}_q} C_{\Omega,r}(D, g) \; = \; n\, r.$$

*(3) Assume, in addition, $g$ is semi-stable.*

*(a) For the dimensions of the codes $C_{F,r}(D, g)$ and $C_{\Omega,r}(D, g)$,*

*(i) If $\deg(g) < rn$, then*

$$k_{D,g} = h^0(F, g) \geq \deg(g) - r(g-1).$$

*(ii) If $\deg(g) < 2r(g-1)$, then*

$$k_{D,g}^\perp = h^1(F, g(-D)) \geq r\big(\deg(D) + (g-1)\big) - \deg(g).$$

*(iii) If $rn > \deg(g) > 2r(g-1)$, then*

$$k_{D,g} = \deg(g) - r(g-1) \text{ and } k_{D,g}^\perp = r\big(\deg(D) + (g-1)\big) - \deg(g).$$

*(b) For the minimal distance $d_{D,g}$ of the code $C_{F,r}(D, g)$, we have*

$$d_{D,g} \geq nr - \deg(g).$$

*In particular, if $\deg(g) < rn$, we have*

$$r(n - (g-1)) \leq k_{D,g} + d_{D,g} \leq rn + 1.$$

Certainly, the above results coincides with these for classical algebraic geometry codes, since, when $r = 1$, the stability condition is automatic.

We end this introduction by the following remarks. After completing this paper, as a preparation for open lists, we make a searcher on the internet and find the paper of V. Savin on 'Algebraic-geometric codes from vector bundles and their decoding' at arXiv:0803.1096. While there are some idea overlaps, our codes and approaches are quite different. Also I would like to thank J. Yamada for his explanations of AG codes in our weekly seminar a few days ago, which for the first time making AG codes known to me. Immediately, I realized that these codes admit natural generalizations using adelic cohomologies we developed several years ago. Current work is the outcome of this line of thoughts.

# 2 Rank $r$ Codes

## 2.1 Adelic Cohomology Theory

Let $F/\mathbb{F}_q$ be the function field of an integral regular projective curve $X$ over $\mathbb{F}_q$. Denote its associated adelic ring by $\mathbb{A}$ and set $\mathcal{O} = \{a \in \mathbb{A} : a_p \in \mathcal{O}_p \ \forall p\}$, where $\mathcal{O}_p$ denotes the integer ring of the local field $F_p$ of $F$ at $p \in X$.

Recall that, for an element $g \in \mathrm{GL}_r(\mathbb{A})$, in [13] (see also [10]), we introduce the subspace $\mathbb{A}^r(g)$ of $\mathbb{A}^r$ by

$$\mathbb{A}^r(g) := \Big\{ a \in \mathbb{A}^r : ga \in \mathcal{O}^n \Big\}, \tag{6}$$

and make the following

**Definition 1.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$.*

*(1) The 0-th cohomology of $g$ over $F$ is defined by*

$$H^0(F,g) := \mathbb{A}^r(g) \cap F^r. \tag{7}$$

*(2) The 1st cohomology of $g$ over $F$ is defined by*

$$H^1(F,g) := \mathbb{A}^r/(\mathbb{A}^r(g) + F^r). \tag{8}$$

*Here $F^r$ is viewed as a subspace of $\mathbb{A}^r$ through the natural diagonal embedding $F^r \hookrightarrow \mathbb{A}^r$, $f \mapsto (f)$.*

Let $\mathcal{E}(g)$ be the rank $r$ locally free sheaf on $X$ associated to $g$.[1] Then using the ind-pro topology on $\mathbb{A}^r$, or better on $\mathbb{A}$ defined by $\mathbb{A} = \lim\limits_{\overrightarrow{D}} \lim\limits_{\overleftarrow{D \geq D'}} \mathbb{A}(D)/\mathbb{A}(D')$, we have the following

**Theorem 2.** *([13] and [10]) Let $g \in \mathrm{GL}_r(\mathbb{A})$, and denote by $\mathcal{E}(g)$ the associated rank $r$ locally free sheaf on $X$.*

*(0) The $\mathbb{F}_q$-linear space $H^0(F,g)$ and $H^1(F,g)$ are finite dimensional. For later use, set $h^i(F,g) := \dim_{\mathbb{F}_q} H^i(X,g)$ for $i = 0,1$.*

*(1) There are natural $\mathbb{F}_q$-linear isomorphisms*

$$H^0(F,g) \simeq H^0(X,\mathcal{E}(g)) \qquad \text{and} \qquad H^1(F,g) \simeq H^1(X,\mathcal{E}(g)); \tag{9}$$

*(2) Induced by the natural non-degenerate pairing (with respect to a non-trivial rational differential $\omega_0$ on $X$)*

$$\begin{array}{rccc} \langle \cdot,\cdot \rangle_{\omega_0} : & \mathbb{A}^r \times \mathbb{A}^r & \longrightarrow & \mathbb{F}_q \\ & ((a_i),(b_i)) & \longmapsto & \sum_{i=1}^r \omega_0(a_i b_i), \end{array} \tag{10}$$

*we have*

*(a) $\widehat{\mathbb{A}^r} \simeq \mathbb{A}^r$, where $\widehat{\mathbb{A}^r}$ denotes the Pontryagin dual group of $\mathbb{A}^r$ (with respect to the above ind-pro topology.*

[1] For the detailed construction, please refer to Appendix A.

*(b)* $(F^r)^\perp = F^r$*;*

*(c)* $\mathbb{A}^r(g)^\perp = \mathbb{A}^r(\iota_{(\omega_0)} \cdot g^{-1})$*, where* $\iota_{(\omega_0)} \in \mathbb{I} := \mathrm{GL}_1(\mathbb{A})$ *denotes an idelic element corresponding to the invertible sheaf* $\mathcal{O}_X((\omega_0))$*, or better, to the divisor* $D$*.*

*(d) (Topology Duality) Induced by the pairing* $\langle \cdot, \cdot \rangle_{\omega_0}$*, there exists a corresponding duality isomorphism*

$$\mathrm{Hom}_{\mathbb{F}_q}\left(\mathbb{A}^r/(\mathbb{A}^r(g) + F^r), \mathbb{F}_q\right) \simeq \mathbb{A}^r(\iota_{(\omega_0)} \cdot g^{-1}) \cap F^r. \tag{11}$$

*(e) (Adelic Riemann-Roch Theorem)*

$$h^0(F, g) - h^1(F, g) =: \chi(F, g) = \deg(g) - r(g-1). \tag{12}$$

*Here as usual* $\deg(g) := \deg(\det(g))$.[2]

Obviously, (d) is simply the Serre duality

$$H^1(X, \mathcal{E}(g)) \simeq H^0(X, \mathcal{E}(g^{-1})((\omega_0))^\vee \tag{13}$$

and (e) is nothing but the Riemann-Roch theorem

$$\chi(X, \mathcal{E}(g)) = \deg(\mathcal{E}(g)) - r(g-1). \tag{14}$$

This theorem was first outlined by the author as a by-product of an adelic cohomology theory for arithmetic curves in a paper on 'Geometry of Numbers' and now Chapter 2 of [13]. A detailed proof can be found in Sugahara's thesis [10]. By saying so, we also should mention that most of the theorem can be proved using Chevalley's preparatitions in any standard text books on adeles for function fields, say, [4], [7], [9].

## 2.2 Rank $r$ Algebraic Geometry Codes

Let $X$ be an integral regular projective curve on $\mathbb{F}_q$. Denote by $F/\mathbb{F}_q$ its field of rational functions and $\mathbb{A}$ the adelic ring associated to $F$. Let $D = \sum_{i=1}^n p_i$ be a degree $n$ divisor on $X$. This implies, in particular, that $p_i$, $i = 1, \ldots, n$ are mutually distinct $\mathbb{F}_q$-rational points of $X$. Fix a $D$- special rational differential $\omega_0$ on $X$, namely, $\omega_0 \in \Omega_F$ is a non-trivial rationals differential on $X$ such that

$$\mathrm{ord}_{p_i}(\omega_0) = -1 \qquad \text{and} \qquad \mathrm{res}_{p_i}(\omega_0) = 1 \qquad \forall i = 1, \ldots, n. \tag{15}$$

Let $g = (g_p) \in \mathrm{GL}_r(\mathbb{A})$. Then $g_{p_i}\mathcal{O}_{p_i}^r \subset F_{p_i}^r$ is a full rank $\mathcal{O}_{p_i}$-lattice. Since $\mathcal{O}_{p_i}$ is a PID, there exists $n_{ij} \in Z$, $i = 1, \ldots, r$, such that

$$n_{i1} \leq n_{i2} \leq \ldots \leq n_{ir} \qquad \text{and} \qquad g_{p_i}\mathcal{O}_{p_i}^r \simeq \mathrm{diag}\left(\pi_{p_i}^{n_{i1}}, \ldots, \pi_{p_i}^{n_{ir}}\right)\mathcal{O}_{p_i}^r. \tag{16}$$

Here $\mathrm{diag}(a_1, \ldots, a_r)$ denotes the diagonal matrix with diagonal components $a_1, \ldots, a_r$. In other words, there exists $M_i, N_i \in \mathrm{GL}_r(\mathcal{O}_{p_i})$ such that

$$M g_{p_i} N = \mathrm{diag}(\pi_{p_i}^{n_{i1}}, \ldots, \pi_{p_i}^{n_{ir}}). \tag{17}$$

[2]Certainly, $\deg(g) = \deg(\mathcal{E}(g))$ and, by definition, $\deg(\det \mathcal{E}(g)) = \deg(\mathcal{E}(\det(g)))$.

It is well known that $(n_{i1}, \dots n_{ir})$ depends only on $g$ and does not depend on the choices of the local parameter $\pi_{p_i}$ used. For our use, we call $(n_{i1}, \dots n_{ir})$ the $p_i$-*multiple orders*. Easily,

$$\sum_{j=1}^{r} n_{ij} = \mathrm{ord}_{p_i} \det(g). \tag{18}$$

**Condition 1.** A element $g \in \mathrm{GL}_r(\mathbb{A})$ is called *$D$-balanced* if its $p_i$-(multiple) orders satisfy the conditions

$$(n_{i1}, \dots n_{ir}) = (0, \dots, 0) \qquad \forall i = 1, \dots, n. \tag{19}$$

**Lemma 3.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$.*

*(1) $g$ is $D$-special if and only if $g_{p_i} \in \mathrm{GL}_r(\mathcal{O}_{p_i})$ for all $i = 1, \dots, n$. Here $g = (g_{kj})_{k,j} = ((g_{kj,p})_p)$.*

*(2) If $g$ is $D$-special, then $\big((f_j(p_1)), \dots, (f_j(p_n))\big)$ makes sense for each element $f = (f_j) \in H^0(F, g)$.*

*Proof.* (1) If $g_p \in \mathrm{GL}_r(\mathcal{O}_{p_i})$, then $g_{kj,p} \in \mathcal{O}_{p_i}$. This implies that $n_{ji} \geq 0$ for all $j$. On the other hand, since $\det(g_p) \in \mathcal{O}_{p_i}^*$, $\mathrm{ord}_{p_i}(\det(g)) = 0$. Hence, by (18), $n_{ji} \geq 0$ for all ($i$ and) $j$.

Conversely, if $(n_{i1}, \dots n_{ir}) = (0, \dots, 0)$, by (17), we have $\mathrm{diag}(\pi_{p_i}^{n_{i1}}, \dots, \pi_{p_i}^{n_{ir}})$ and hence also $g_{p_i}$ belong to $\mathrm{GL}_r(\mathcal{O}_{p_i})$.

(2) Since $g_{p_i} \in \mathrm{GL}_r(\mathcal{O}_{p_i})$, $\det(g_{p_i}) \in \mathcal{O}_{p_i}^*$, and hence $g_{p_i}^{-1}$ belong to $\mathrm{GL}_r(\mathcal{O}_{p_i})$ as well. This implies that $g_{p_i}(p_i) \in \mathrm{GL}_r(\mathbb{F}_q)$. Therefore, note only

$$(g_{p_1}(f_j)(p_1), \dots, g_{p_n}(f_j)(p_n)) = (g_{p_1}(p_i)\,(f_j(p_1)), \dots, g_{p_n}(p_n)\,(f_j(p_n))) \tag{20}$$

is well-defined, but the morphism

$$g_{p_i}: \begin{array}{ccc} \mathbb{F}_q^{rn} & \longrightarrow & \mathbb{F}_q^{rn} \\ (g(p_i)(f_j(p_1)), \dots, g(p_n)(f_j(p_n)) & \longmapsto & ((f_j(p_1)), \dots, (f_j(p_n))) \end{array} \tag{21}$$

makes sense. ☐

Now we are ready to introduce the first main definition.

**Definition 4.** Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-special. The *rank $r$ algebraic geometry codes associated to $g$ with respect to $D$* is defined by the codewords space

$$C_{F,r}(D, g) := \Big\{ \big((f_j(p_1)), \dots, (f_j(p_n))\big) \in \mathbb{F}_q^{rn} \,|\, (f_j) \in H^0(F, g) \Big\}. \tag{22}$$

Obviously, the length of the codewords in $C_r(F, g(-D))$ is $rn$. To understand these types of new codes, we consider the evaluation morphism

$$\mathrm{ev}_{D,g}: \begin{array}{ccc} H^0(F, g) & \longrightarrow & C_{F,r}(D, g) \\ (f_j) & \longmapsto & \big((f_j(p_1)), \dots, (f_j(p_n))\big). \end{array} \tag{23}$$

By definition, $\mathrm{ev}_{D,g}$ is surjective. To determine its kernel $\mathrm{Ker}(\mathrm{ev}_{D,g})$, we first note that for $(f_j) \in H^0(F, g)$,

$$\big((f_j(p_1)), \dots, (f_j(p_n))\big) = ((0), \dots, (0)) \Longleftrightarrow g(f_j) \in \pi_{p_i} \mathcal{O}_{p_i}^r \qquad \forall i = 1, \dots, n. \tag{24}$$

Set now $\iota_D = (\iota_{D_p})_p \in I$ be the idle associated to the divisor defined by

$$\iota_{D_p} = \begin{cases} \pi_{p_i} & p = p_i \ (i = 1, \dots, n) \\ 1 & p \notin \{p_1, \dots, p_n\} \end{cases} \tag{25}$$

In other words, $\iota_D = (\pi_p^{\mathrm{ord}_p(D)})_p \in \mathbb{I} = \mathrm{GL}_1(\mathbb{A})$, which clearly also characterizes the divisor $D$. Then, by definition of $H^0(F, \cdot)$ again, we have

$$\begin{aligned} \mathrm{Ker}(\mathrm{ev}_{D,g}) &= \big\{(f_j) \in F^r : g_{p_i}(f_j) \in \pi_{p_i} \mathcal{O}_{p_i}^r \ \forall i = 1, \dots, n\big\} \\ &= \{(f_j) \in F^r : g(f_j) \in \iota_D \mathcal{O}^r\} \\ &= \big\{(f_j) \in F^r : \iota_D^{-1} g(f_j) \in \mathcal{O}^r\big\} \\ &= \mathbb{A}^r(\iota_D^{-1} g) \cap F^r \\ &= H^0(F, \iota_D^{-1} g) \end{aligned} \tag{26}$$

By an abuse of notation, in the sequel, we denote $\iota_D^{-1} g$ simply by $g(-D)$. Then what we have just said proves the following

**Theorem 5.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-special. There exists a canonical short exact sequence*

$$0 \to H^0(F, g(-D)) \longrightarrow H^0(F, g) \stackrel{\mathrm{ev}_{D,g}}{\longrightarrow} C_{F,r}(D, g) \to 0, \tag{27}$$

*In particular, the dimension of $C_{F,r}(D, g)$ is equal to*

$$k_{D,g} = h^0(F, g) - h^0(F, g(-D)). \tag{28}$$

**Example 1.** *Let $E$ be an effective divisor on $X$ such that the characterizing idele $\iota_E := (\pi_p^{\mathrm{ord}_p(E)}) \in \mathrm{GL}_1(A)$ is $D$-balanced. Obviously, this latest condition is equivalent to the condition that the supports of $D$ and $E$ are mutually disjoint. Moreover, it is not too difficult to see that*

$$C_{F,1}(D, \iota_E) = C_L(D, E). \tag{29}$$

# 3 Rank $r$ Differential Codes

Before we introduce the codewords space which is dual to the rank $r$ algebraic geometry codes $C_{F,r}(D, g)$, we reexamine how $C_\Omega(D, E)$ is introduced. In that case, we use the space $\Omega_F(E - D)$ which is defined by

$$\Omega_F(E - D) := \big\{\omega \in \Omega_F \mid (\omega) \geq E - D\big\} \tag{30}$$

To go further, we fix an $F$-linear isomorphism

$$\begin{array}{cccc} \phi: & \Omega_F & \longrightarrow & F \\ & \omega & \longmapsto & \frac{\omega}{\omega_0}. \end{array} \tag{31}$$

In terms of $\phi$, we have

$$\begin{aligned} \Omega_F(E - D) =& \big\{\phi(\omega) \cdot \omega_0 \in \Omega_F \mid \phi(\omega) + (\omega_0) \geq E - D\big\} \\ =& \big\{f\, \omega_0 \in \Omega_F \mid f + (\omega_0) + D - E \geq 0\big\} \\ =& \big\{f\, \omega_0 \in \Omega_F \mid \iota_{(\omega_0)+D}\, f \in \iota_E\, \mathcal{O}\big\}. \end{aligned} \tag{32}$$

Motivated by this, we introduce the following

**Definition 6.** The *rank r rational differential space* $\Omega_F^r(g(-D))$ is defined by

$$\begin{aligned}\Omega_F^r(g(-D)) :=& \Big\{(\omega_j) \in \Omega_F^r \,|\, \iota_{(\omega_0)+D}\,(\phi(\omega_j)) \in g\,\mathcal{O}^r\Big\}\\ =& \Big\{\omega_0\,(f_j) \in \Omega_F^r \,|\, \iota_{(\omega_0)+D}\,(f_j) \in g\,\mathcal{O}^r\Big\}.\end{aligned} \tag{33}$$

In particular, if $g$ is $D$-balanced, since $\mathrm{ord}_{p_i}(\omega_0\iota_D) = 0$ and $\omega_0$ is $D$-special, we certainly get

$$\phi(\omega_j)(p_i) = \mathrm{res}_{p_i}(\omega_0\phi(\omega_j)) = \omega_{j,P_i}(1). \tag{34}$$

With this, we are ready to introduce the next main definition.

**Definition 7.** Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-balanced. The *rank r differential codewords space associated to $D$ and $g$* is defined by

$$C_{\Omega,r}(D,g) := \Big\{\big((\omega_{j,p_1}(1)),\ldots,(\omega_{j,p_n}(1))\big)\,\Big|\,(\omega_j) \in \Omega_F^r(g(-D))\Big\}. \tag{35}$$

To see the dimension of this codewords space, we introduce the following morphism

$$\begin{array}{rccc}\mathrm{ev}_{D,g}^{\perp}: & \Omega_F^r(g(-D)) & \longrightarrow & C_{\Omega,r}(D,g)\\ & (\omega_j) & \longmapsto & \big((\omega_{j,p_1}(1)),\ldots,(\omega_{j,p_n}(1))\big).\end{array} \tag{36}$$

Directly from the definition, $\mathrm{ev}_{D,g}^{\perp}$ is surjective. Hence to obtain the dimension of the codewords space, it suffices to determine its kernel $\mathrm{Ker}(\mathrm{ev}_{D,g}^{\perp})$. Since $(\omega_j) \in \mathrm{Ker}(\mathrm{ev}_{D,g}^{\perp})$ if and only if $(\omega_{j,p_1}(1)) = 0$ for all $i = 1,\ldots n$, which, by (34), is equivalent to the condition that $\big(\phi(\omega_j)(p_i)\big) = (0,\ldots,0)$. Therefore,

$$\begin{aligned}\mathrm{Ker}(\mathrm{ev}_{D,g}^{\perp}) =& \Big\{(\omega_j) \in \Omega_F^r \,|\, \iota_{(\omega_0)+D}\,(\phi(\omega_j)) \in \iota_D\, g\,\mathcal{O}^r\Big\}\\ =& \Omega_F^r(g(-D+D))\\ =& \Omega_F^r(g)).\end{aligned} \tag{37}$$

This then proves the first part of the following

**Theorem 8.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-balanced.*

*(1) There is a short exact sequence of $\mathbb{F}_q$-linear spaces*

$$0 \to \Omega_F^r(g) \longrightarrow \Omega_F^r(g(-D)) \stackrel{\mathrm{ev}_{D,g}^{\perp}}{\longrightarrow} C_{\Omega,r}(D,g) \to 0. \tag{38}$$

*(2) The dimension $k_{D,g}^{\perp}$ of the codes $C_{\Omega,r}(D,g)$ is equal to*

$$h^1(F,g(-D)) - h^1(F,g). \tag{39}$$

*(3) The dimensions of $C_{F,r}(D,g)$ and $C_{\Omega,r}(D,g)$ satisfy*

$$k_{D,g} + k_{D,g}^{\perp} = n\,r. \tag{40}$$

*Proof.* What left is the proof of (2) and (3). Assume that (2) holds. Then by Theorem 5, we have, by the Riemann-Roch theorem,

$$\begin{aligned} k_{D,g}+k_{D,g}^{\perp} &= \big(h^0(F,g)-h^0(F,g(-D))\big)+\big(h^1(F,g(-D))-h^1(F,g)\big)\\ &= \big(h^0(F,g)-h^1(F,g)\big)-\big(h^0(F,g(-D))-h^1(F,g(-D))\big)\\ &=\big(\deg(g)-r(g-1)\big)-\big(\deg(g)-r(g-1)\big)\\ &=\deg(g)-\deg(g(-D))\\ &=n\,r \end{aligned} \tag{41}$$

since, we have, by definition,

$$\begin{aligned} \deg(g(-D)) &= \deg\big(\det(g(-D))\big)=\deg\big(\det(\iota_D^{-1}g)\big)\\ &=\deg\big(\det(g)\cdot\iota_D^{-r}\big)\\ &=\deg\big(\det(g)\big)+\deg\big(\iota_D^{-r}\big)\\ &=\deg\big(\det(g)\big)-r\deg\big(\iota_D\big)\\ &=\deg(g)-r\deg(D). \end{aligned} \tag{42}$$

This proves (3) assuming (2).

Finally, we prove (2). For this, we use the isomorphism obtained from the duality theorem

$$\Omega_F^r(g)\simeq \mathrm{Hom}_{\mathbb{F}_q}\big(\mathbb{A}^r/\big(\mathbb{A}^r(g(-D))+F^r\big),\mathbb{F}_q\big)=H^1(F,g)^\vee. \tag{43}$$

This then completes the proof. □

**Example 2.** *Let $E$ be an effective divisor on $X$ such that the characterizing idele $\iota_E := (\pi_p^{\mathrm{ord}_p(E)}) \in \mathrm{GL}_1(A)$ is $D$-balanced, i.e. the supports of $E$ and $D$ are mutually disjoint. Then, easily from the definitions of both sides below,*

$$C_{\Omega,1}(D,\iota_E)=C_\Omega(D,E). \tag{44}$$

# 4 Duality between AG and Differential Codes in Rank $r$

Theorem refmthm2(3) suggests that, similar to canonical AG and differential codes, there is also a natural duality between rank $r$ AG and differential codes. This is indeed the case, as to be exposed in this section.

We start with the non-degenerate pairing

$$\begin{array}{cccc} \langle\cdot,\cdot\rangle: & (\mathbb{F}_q^r)^n\times(\mathbb{F}_q^r)^n & \longrightarrow & \mathbb{F}_q\\ & \big((a_{ji})_j)_i,((b_{ji})_j)_i\big) & \longmapsto & \sum_{i,j=1}^{n,r} a_{ji}b_{ji}. \end{array} \tag{45}$$

Obviously, this $\langle\cdot,\cdot\rangle$ induces a natural pairing between subspaces $C_{F,r}(D,g)$ and $C_{\Omega,r}(D,g)$ of $(\mathbb{F}_q^r)^n$ as follows:

$$\begin{array}{cccc} \langle\cdot,\cdot\rangle_{D,g}: & C_{F,r}(D,g)\times C_{\Omega,r}(D,g) & \longrightarrow & \mathbb{F}_q\\ & \Big(\big((f_j(p_i))_j\big)_i,\big((\omega_{j,p_i}(1))_j\big)_i\Big) & \longmapsto & \displaystyle\sum_{i,j=1}^{n,r} f_j(p_i)\cdot\omega_{j,p_i}(1). \end{array} \tag{46}$$

**Theorem 9.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-balanced.*

*(1) The natural pairing $\langle\cdot,\rangle_{D,g}$ degenerates completely. That is, the image of $\langle\cdot,\rangle_{D,g}$ consists of only the zero element.*

*(2) $C_{\Omega,r}(D,g)$ is the dual codes of $C_{F,r}(D,g)$. That is to say,*

$$C_{\Omega,r}(D,g)^{\perp} = C_{F,r}(D,g). \tag{47}$$

*Proof.* Since (1) and (2) are equivalent, we only need to prove (2). By Theorem 8(3), we have

$$\dim_{\mathbb{F}_q} C_{F,r}(D,g) + \dim_{\mathbb{F}_q} C_{\Omega,r}(D,g) = n\,r. \tag{48}$$

As a direct consequence, it suffices to show that

$$C_{\Omega,r}(D,g)^{\perp} \supseteq C_{F,r}(D,g). \tag{49}$$

Let $\big((\omega_{j,p_1})(1),\dots,(\omega_{j,p_n})(1)\big)$ be an element of $C_{F,r}(D,g)$ with $(\omega_j) \in \Omega_F^r(g(-D))$. Since $\Omega_F^r(g(-D)) = \mathrm{Hom}_{\mathbb{F}_q}\left(\mathbb{A}^r/\big(\mathbb{A}^r(g(-D)) + F^r\big), \mathbb{F}_q\right)$, we have

$$(\omega_j)\big(\mathbb{A}^r(g(-D))\big) = \{0\} \qquad \text{and} \qquad (\omega_j)\big(F^r\big) = \{0\}. \tag{50}$$

The second equality is equivalent to the residue formula. Indeed, if we write $\omega_j = \omega_0 \cdot \phi(\omega_j)$, then, for any $(f_j) \in F^r$,

$$(\omega_j)(f_j) = \sum_{j=1}^{n} \omega_j(f_j) = \sum_{j=1}^{n} \omega_0(f_j\phi(\omega_j)) = \omega_0\Big(\sum_{j=1}^{n} f_j\phi(\omega_j)\Big) = 0, \tag{51}$$

by the residue formula. So we must deduce (49) from the first relation in (50). Take then an element $(a_j) \in \mathbb{A}^r(g(-D))$. By definition, $(a_j) \in \iota_D^{-1} g(a_j) \in \mathcal{O}^r$. This means that

$$\pi_{p_i}(a_{j,p_i}) \in g_{p_i}^{-1}\mathcal{O}^r \qquad \text{and} \qquad (a_{j,q})g_q \in \mathcal{O}_q^r \;\; \forall q \neq p_i,\; \forall i = 1,\dots,n, \tag{52}$$

since $g$ is $D$-balanced. In other words, for any $(a_j) \in \mathbb{A}^r$ satisfying (52), we have $(\omega_j)(a_j) = 0$. To go further, we take an element $\big((f_j(p_1)),\dots,(f_j(p_n))\big) \in C_{\Omega,r}(D,g)$ for a certain $(f_j) \in H^0(F,g)$. Then, by (34)

$$\begin{aligned} &\Big\langle \big((f_j(p_1)),\dots,(f_j(p_n))\big), \big((\omega_{j,p_1})(1),\dots,(\omega_{j,p_n})(1)\big)\Big\rangle \\ &= \sum_{i,j=1}^{n,r} f_j(p_i)\omega_{j,p_i})(1) = \sum_{i,j=1}^{n,r} f_j(p_i)\phi(\omega_j)(p_i) \\ &= \sum_{i,j=1}^{n,r} (f_j\phi(\omega_j))(p_i) = \sum_{i,j=1}^{n,r} \Big(f_j\frac{\omega_j}{\omega_0}\Big)(p_i). \end{aligned} \tag{53}$$

But $\omega_0$ is $D$-special means that

$$\mathrm{ord}_{p_i}(\omega_0) = -1 \qquad \text{and} \qquad \mathrm{res}_{P_i}(\omega_0) = 1 \qquad \forall i = 1,\dots,n. \tag{54}$$

This implies that

$$\Big(f_j\frac{\omega_j}{\omega_0}\Big)(p_i) = \omega_{j,p_i}\Big(\frac{f_j}{\pi_{p_i}}\Big). \tag{55}$$

Therefore,

$$\begin{aligned}
&\big\langle\big(((f_j(p_1)),\dots,(f_j(p_n))\big),\big((\omega_{j,p_1})(1),\dots,(\omega_{j,p_n})(1)\big)\big\rangle\\
&=\sum_{i,j=1}^{n,r}\omega_{j,p_i}\Big(\frac{f_j}{\pi_{p_i}}\Big)=\sum_{i=1}^{n}(\omega_{j,p_i})\Big(\frac{f_j}{\pi_{p_i}}\Big)\\
&=\sum_{\substack{q:\,q\neq p_i\\ i=1,\dots,n}}\omega_{j,q}(f_j(q))+\sum_{i=1}^{n}(\omega_{j,p_i})\Big(\frac{f_j}{\pi_{p_i}}\Big)\\
&=(\omega_j)(\iota_D^{-1}(f_j)).
\end{aligned}\tag{56}$$

Here, in the equality above the last, we have used the fact that for $(\omega_j)\in\Omega_{F,r}(g(-D))$ and $(f_j)\in H^0(F,g)$, $\omega_{j,q}(f_j(q))=0$. Thus, by using (52), we get $\iota_D^{-1}(f_j)\in\mathbb{A}^r(g(-D))$. Therefore, by the fact that $(\omega_j)(\mathbb{A}(g(-D)))=\{0\}$ for $(\omega_j)\in\Omega_F^r(g(-D))$, we have $(\omega_j)(\iota_D^{-1}(f_j))$. This means that, for all elements $(\omega_j)\in\Omega_F^r(g(-D))$ and $(f_j)\in\mathbb{A}^r(g)$,

$$\big\langle\big(((f_j(p_1)),\dots,(f_j(p_n))\big),\big((\omega_{j,p_1})(1),\dots,(\omega_{j,p_n})(1)\big)\big\rangle=0\tag{57}$$

by (56). This establishes (49) and hence completes the proof of the theorem. □

As a direct consequence of the proof of the theorem above, we have the following

**Corollary 10.** *Let $g\in\mathrm{GL}_r(\mathbb{A})$ be $D$-balanced. Then*

$$C_{\Omega,r}(D,g)=C_{F,r}(D,\iota_{(\omega_0)+D}\,g^{-1}).\tag{58}$$

*where $(\omega_0)$ is a $D$-special rational differential.*

# 5 Estimate Dimensions of Rank $r$ AG Codes

It is proved in Theorems 5 and 8, we have determined the dimensions $k_{D,g}$ and $k_{D,g}^{\perp}$ for the rank $r$ algebraic geometry codes $C_{F,r}(D,g)$ and $C_{\Omega,r}(D,g)$, respectively. Namely, for $D$-balanced $g\in\mathrm{GL}_r(\mathbb{A})$,

$$\begin{aligned}
k_{D,g}&=h^0(F,g)-h^0(F,g(-D)),\\
k_{D,g}^{\perp}&=h^1(F,g(-D))-h^1(F,g).
\end{aligned}\tag{59}$$

As for classical AG codes, this is not good enough. Indeed, there, with a conditions that $\deg(E)<\deg(D)$ and $\deg(E)>2g-2$, by the vanishing results

$$h^0(F,\iota_{E-D})=0\qquad\text{and}\qquad h^1(X,\iota_E)=0,\tag{60}$$

we obtain

$$k_{D,\iota_E}=h^0(F,\iota_E)\qquad\text{and}\qquad k_{D,\iota_E}^{\perp}=h^1(F,\iota_{E-D}),\tag{61}$$

respectively, provided that the supports of $D$ and $E$ are disjoint.

By contrast, these is no such vanishing result for cohomologies of rank $r$ settings.

**Example 3.** Let $g = \mathrm{diag}(\iota_{D_1}, \iota_{D_2})$. Then

$$h^i(F, g) = h^i(F, \iota_{D_1}) + h^i(F, \iota_{D_2}) \qquad i = 0, 1. \tag{62}$$

There is *no simple yet general vanishing result* for $h^i(F, g)$ depending only on the degree $d$ of $g$, since the degrees of the divisors $D_1$ and $D_2$ can be changed freely as long as they satisfy $\deg(D_1) + \deg(D_2) = d$. Hence, even $h^i(F, \iota_{D_j})$ admit vanishing properties, but not $h^i(F, g)$.

This is not the end of our theory. Much better, in algebraic geometry, there is a powerful and general vanishing theory based on the stability.

**Definition 11.** *(1)* **(Mumford [5])** A locally free sheaf $\mathcal{E}$ on $X$ is called *semi-stable*, resp. *stable*, if for any proper subsheaf $\mathcal{F}$ of $\mathcal{E}$,

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}) \quad \textit{rasp.} \quad \mu(\mathcal{F}) < \mu(\mathcal{E}) \tag{63}$$

Here as usual, the $\mu$-slope is defined by

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})}. \tag{64}$$

(2) An element $g \in \mathrm{GL}(\mathbb{A})$ is called *semi-stable*, resp. *stable*, if its associated locally free sheaf $\mathcal{E}(g)$ is semi-stable, resp. stable.

It is not too difficult to prove the following:

**Lemma 12.** *Let $\mathcal{E}$ be a semi-stable local free sheaf on $X$ of rank $r$.*

*(1) If $\deg(\mathcal{E}) > r(2g-2)$, then $H^1(X, \mathcal{E}) = \{0\}$.*

*(2) If $\deg(\mathcal{E}) < 0$, then $H^0(X, \mathcal{E}) = \{0\}$.*

*Proof.* Indeed, if $\deg(\mathcal{E}) < 0$ and $H^0(X, \mathcal{E}) \neq \{0\}$, there exists a non-trivial global section $s$ of $\mathcal{E}$. Hence, we obtain an injection $\mathcal{O}_X \stackrel{s}{\hookrightarrow} \mathcal{E}$. By the semi-stability condition on $\mathcal{E}$,

$$\deg(\mathcal{O}_X) \leq \frac{\deg(\mathcal{E})}{r}. \tag{65}$$

In particular, $\deg(\mathcal{E}) \geq 0$. This proves (2) and hence also (1) by the duality theorem. □

As a direct consequence, using the duality and the Riemann-Roch theorem, we obtain the following

**Corollary 13.** *Let $g \in \mathrm{GL}_r(\mathbb{A})$ be $D$-balanced. Assume that $g$ is semi-stable.*

*(1) If $\deg(g(-D)) < 0$, namely, $\deg(g) < r\deg(D)$, then*

$$k_{D,g} = h^0(F, g) \geq \deg(g) - r(g-1). \tag{66}$$

*(2) If $\deg(g) > 2r(g-1)$ then*

$$k_{D,g}^{\perp} = h^1(F, g(-D)) \geq r\big(\deg(D) + (g-1)\big) - \deg g. \tag{67}$$

*(3) If $r\deg(D) > \deg(g) > 2r(g-1)$ then*

$$k_{D,g} = \deg(g) - r(g-1) \qquad \text{and} \qquad k_{D,g}^{\perp} = r\big(\deg(D) + (g-1)\big) - \deg g. \tag{68}$$

# 6 Masses of Semi-Stable Locally Free Sheaves

To have a rich theory for rank $r$ algebraic geometry codes, there is a problem to find how many $\mathbb{F}_q$-rational semi-stable locally free sheaves of rank $r$ and degree $d$. To answer this, (not really as usual), we denote by $\mathcal{M}_{X,r}(d)$ the moduli stack of $\mathbb{F}_q$-rational semi-stable locally free sheaves of rank $r$ and degree $d$.

**Definition 14** ([1]). For each pair $(n,d)$, we define the *$\beta$-invariant* $\beta(r,d)$ by

$$\beta_{r,d} := \sum_{[\mathcal{E}]\in\mathcal{M}_{X,r}(d)} \frac{1}{|\mathrm{Aut}(\mathcal{E})|}. \tag{69}$$

Then, staring from the fact that the Tamagawa number of $\mathrm{SL}_r(\mathbb{A})$ is one, using parabolic reduction, Harder-Narasimhan can calculate $\beta_{n,d}$ for all $d$. For example, with additional works of Desale-Ramanan and Zagier, we have the following well-known formula for $\beta_{r,0}$.

**Theorem 15** ([1], see also [12] ). *For any integer $\alpha$,*

$$\beta_{r,r\alpha} = \sum_{k=1}^{r}(-1)^k \sum_{\substack{n_1,\dots,n_k\in\mathbb{Z}_{>0}\\ n_1+\dots+n_k=r}} \frac{\prod_{i=1}^{k}\widehat{v}_{X,n_i}}{\prod_{j=1}^{k-1} q^{n_j+n_{j+1}}-1}. \tag{70}$$

*Here $\widehat{v}_{X,n} := \widehat{\zeta}_X(n)$ with $\widehat{\zeta}_X(s) = q^{(g-1)s}\zeta_X(s)$ the (complete) Artin zeta function for $X$.*[3]

Back to elements $g\in\mathrm{GL}_r(A)$, we mention that Lafforgue obtains an analytic characterization for $g$ to be semi-stable using Arthur's analytic truncation, a fundamental tool introduced by Arthur in his study of trace formula. For details, please refer to §V.1 of [3]. This result is further generalized by the author to general reductive groups ([13]).

# 7 Minimal Distances of Rank $r$ AG Codes

To begin with, we first recall the following well-known

**Definition 16.** *Let $C$ be a linear code and let $a$, $b$ two codewords of $C$.*

*(1) The (Hamming) distance $d(a,b)$ between $a$ and $b$ is the minimum number of coordinate positions in which they differ.*

*(2) The (Hamming) weight $w(a)$ of $a$ is the number of coordinate positions which are non-zero.*

*(3) The minimal distance $d(C)$ of $C$ is the weight of the smallest weight non-zero codewords. That is,*

$$d(C) = \min_{a,b\in C, a\neq b} d(a,b) = \min_{a\in C, a\neq 0} w(a). \tag{71}$$

Our main aim in this section is to study the minimal distance of the rank $r$ algebraic geometry codes $C_{F,r}(D,g)$ and hence also for $C_{\Omega,r}(D,g)$ for $D$ balanced element $g \in\in \mathrm{GL}_r(\mathbb{A})$.

[3] We here set $\widehat{\zeta}_X(1) = \mathrm{res}_{s=1}\widehat{\zeta}_X(s)$.

**Lemma 17.** *Let $f = (f_j) \in H^0(F,g)$. Then the weight of the codeword $\big((f_j(p_1)),\dots(f_j(p_n))\big) \in C_{F,r}(D,g)$ is given by*

$$w\big((f_j(p_1)),\dots(f_j(p_n))\big) = nr - \sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1}. \tag{72}$$

*where, for $a,b \in \mathbb{R}$, $\delta_{a\geq b}$ is defined to be 0 if $a<b$ and 1 if $a \geq b$*

*Proof.* This comes directly from the definition of the weight of a codeword, since

$$f_j(p_i) = 0 \qquad \text{if and only if} \qquad \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1}. \qquad \square$$

Denote by $d_{D,g}$, resp. $d^{\perp}_{D,g}$, be the minimal distance of $C_{F,r}(D,g)$. Then

$$\begin{aligned} d_{D,g} &= \min\Big\{nr - \sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} \,\Big|\, f=(f_j)\in H^0(F,g)\Big\} \\ &= nr - \max\Big\{\sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} \,\Big|\, f=(f_j)\in H^0(F,g)\Big\}. \end{aligned} \tag{73}$$

To go further, recall that there is a natural upper bound

$$k_{D,g} + d_{D,g} \leq rn + 1 \tag{74}$$

coming from the singleton bound for linear codes. Our aim next is to obtain a general lower bound for $k_{D,g} + d_{D,g}$. Thus, if we assume that in addition that $g$ is semi-stable and $\deg(g(-D)) < 0$, then $k_{D,g} \geq \deg(g) - r(g-1)$ by Corollary 13. So it suffices to find a universal lower bound for $d_{D,g} + \deg(g)$. For this, we reexamine the condition that $f=(f_j)\in H^0(F,g)$, i.e.

$$g_p(f_j) \in \mathcal{O}_p^r \qquad \forall p. \tag{75}$$

Since $\mathcal{O}_p$ is a PID, it makes sense for us to talk about the multiple order $(n_{p1},\dots,n_{pr})$ of $g$ at $p$. That is, there exists $M_p$, $N_p \in \mathrm{GL}_r(\mathcal{O}_p)$ such that

$$g_p = M_p \mathrm{diag}\big(\pi_p^{n_{p1}},\dots,\pi_p^{n_{pr}}\big) N_p. \tag{76}$$

Note that here the condition $n_{p1}\leq \dots \leq n_{pr}$ is dropped so that

$$g_p(f_j) \in \mathcal{O}_p^r \qquad \text{if and only if} \qquad \mathrm{ord}_p(N_{pk}(f_j)) + n_{pk} \geq 0 \quad \forall k=1,\dots,r. \tag{77}$$

Here $N_{pk}$ denotes the $k$-th cow if $N_p$. This implies that

$$\deg(g) = \sum_p \sum_{j=1}^{r} n_{pj}\deg(p) \quad \text{and} \quad \deg(g) + \sum_{k=1}^{r}\sum_p \mathrm{ord}_p\big(N_{pk}(f_j)\big)\deg(p) \geq 0, \tag{78}$$

as these summations involves only finitely many $p$.

Choose now $f_0 = (f_{0j}) \in H^0(F,g)$ such that

$$\sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_{0j})\geq 1} = \max\Big\{\sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} \,\Big|\, f=(f_j)\in H^0(F,g)\Big\}. \tag{79}$$

Then

$$\deg(g)+\sum_{i,j=1}^{n,r}\operatorname{ord}_{p_i}(f_{0j})+\sum_{j=1}^{r}\sum_{p\notin\{p_1,\dots,p_n\}}\operatorname{ord}_p(f_{0j})\deg(p)\geq 0 \tag{80}$$

In fact much better can be done. To see this, introduce the following

**Definition 18.** Let $f=(f_j)\in H^0(F,g)$ be a global section.

(1) The *margin adelic element* $\chi(f,D)\in \mathrm{GL}_r(O)$ *of* $f$ *with respect to* $D$ is defined by

$$\chi(f,D)_p:=\begin{cases}\operatorname{diag}(1,\dots,1) & p\notin\{p_1,\dots,p_n\}\\ N_p^{-1}\operatorname{diag}\big(\pi_p^{-\delta_{\operatorname{ord}_p(f_1)\geq 1}},\dots,\pi_p^{-\delta_{\operatorname{ord}_p(f_r)\geq 1}}\big) & p\in\{p_1,\dots,p_n\}.\end{cases}$$

(2) The *logarithmic transform* $g_{\log(D,f)}$ *of* $g$ *with respect to* $(D,f)$ is defined by $g\cdot\chi(f,D)$. Namely,

$$g_{\log(f,D),p} := \begin{cases} g_p & p\notin\{p_1,\dots,p_n\}\\ M_p\operatorname{diag}\big(\pi_p^{-\delta_{\operatorname{ord}_p(f_1)\geq 1}},\dots,\pi_p^{-\delta_{\operatorname{ord}_p(f_r)\geq 1}}\big) & p\in\{p_1,\dots,p_n\}\end{cases} \tag{81}$$

**Lemma 19.** *With the same notation as above, we have*

$$f_0\in H^0\big(F,g\cdot\chi(f_0,D)\big). \tag{82}$$

*In particular, if* $g$ *is semi-stable, then* $g\cdot\chi(f_0,D)$ *is also semi-stable.*

*Proof.* If $p\notin\{p_1,\dots,p_n\}$, the $p$ component $\big(g\cdot\chi(g,D)^{-1}\big)_p$ of $g\cdot\chi(g,D)^{-1}$ is simply $g_p$, Hence,

$$g\cdot\chi(g,D)^{-1}f_0\in\mathcal{O}_p^r. \tag{83}$$

Now assume that $p=p_i$ for a certain $i=1,\dots,n$. Then

$$\begin{aligned}&\big(g\cdot\chi(g,D)^{-1}\big)_p f_0 = g_{\log(f,D),p}f_0\\ &= M_p\operatorname{diag}\big(\pi_p^{-\delta_{\operatorname{ord}_p(f_1)\geq 1}},\dots,\pi_p^{-\delta_{\operatorname{ord}_p(f_r)\geq 1}}\big)f_0\\ &= M_p\big(\pi_p^{-\delta_{\operatorname{ord}_p(f_1)\geq 1}}f_1,\dots,\pi_p^{-\delta_{\operatorname{ord}_p(f_r)\geq 1}}f_j\big)^t\in\mathcal{O}_p^r\end{aligned} \tag{84}$$

since $\operatorname{ord}_p\big(\pi_p^{-\delta_{\operatorname{ord}_p(f_r)\geq 1}}f_j\big)\geq 0$ for all $j=1,\dots,r$. This prove that $f_0\in H^0\big(F,g\cdot\chi(g,D)\big)$. To prove the semi-stablity statement, we note that the correspondences between sub-sheaves of $\mathbb{E}(g)$ and $\mathbb{E}(g\cdot\chi(f,D))$ in terms of their stalks at each points. Hence, if $g$ is semi-stable, so is $g\cdot\chi(f_0,D)$, since $f_0\in H^0\big(F,g\cdot\chi(f_0,D)\big)$. □

**Theorem 20.** *Let* $g\in\mathrm{GL}_r(\mathbb{A})$ *be* $D$*-balanced and semi-stable. Then the minimal distance* $d_{D,g}$ *of the rank* $r$ *algebraic geometry code has the following lower bound.*

$$d_{D,g}\geq nr-\deg(g) \tag{85}$$

*In particular, if* $\deg(g)<rn$, *we have*

$$r(n-(g-1))\leq k_{D,g}+d_{D,g}\leq rn+1 \tag{86}$$

*Proof.* Since $H^0\big(F, g\cdot\chi(g,D)\big)\neq\{0\}$, we have, by (79) and (73),

$$\begin{aligned}0 \leq &\deg\big(g\cdot\chi(g,D)\big) = \deg(g)+\deg(\chi(D,g))\\ =&\deg(g) - \sum_{i,j=1}^{n,r}\delta_{\mathrm{ord}_{p_i}(f_{0j})\geq 1}\\ =&\deg(g) - \max\Big\{\sum_{i,j=1}^{n,r}\delta_{\mathrm{ord}_{p_i}(f_j)\geq 1}\,\big|\, f=(f_j)\in H^0(F,g)\Big\}\\ =&\deg(g)+d_{D,g}-nr\end{aligned} \tag{87}$$

With this, then the last statement is simply that of Corollary 13.(1). □

Certainly, if $g=\iota_E$, this result coincides with the well-known bounds for the classical AG codes $C_L(D,E)$ claiming that

$$n+1-g\leq k_{D,g}+d_{D,g}\leq n+1. \tag{88}$$

# Appendix A
# Adelic Interpretations of Locally Free Sheaves

For each $g=(g_p)\in\mathrm{GL}_r(\mathbb{A})$, we obtain a natural family of lattices $\{g^{-1}(\mathcal{O}_p^r)\}_{p\in X}$. In other words, for each $p\in X$, $g^{-1}(\mathcal{O}_p^r)\subset F_p^r$ is a full rank $\mathcal{O}_p$-lattice in $F_p$. Here, as usual, $(F_p,\mathcal{O}_p,\mathfrak{m}_p.\pi_p)$ denotes the local field $F_p$, the local ring $\mathcal{O}_p$ of integers, the maximal ideal $\mathfrak{m}_p$ and a local parameter $\pi_p$, of $X$ at $p$. In parallel, denote by $(F_p,\mathcal{O}_p,\mathfrak{m}_p.\pi_p)$ the corresponding data after taking the completion.

To obtain an adelic interpretation of locally free sheaves, for each full rank $\mathcal{O}_p$-lattice $\mathcal{M}_p$ of $F_p^r$, we introduce a skyscraper sheaf $\mathcal{M}_p$ on $X$ by

$$U\longmapsto\mathcal{M}_p(U):=\begin{cases}\mathcal{M}_p & p\in U\\ 0 & p\notin U.\end{cases} \tag{89}$$

Accordingly, $\big\{g_p^{-1}(\mathcal{O}_p^r)(U)\big\}_p$ makes sense. In addition, we obtain a sheaf $\mathcal{E}(g)$ on $X$ defined by

$$\mathcal{E}(g):\quad U\longmapsto\mathcal{F}^r(U)\cap\big(\cap_{p\in U}g_p^{-1}(\mathcal{O}_p^r)(U)\big)\,. \tag{90}$$

Here, $\mathcal{F}^r$ denotes the constant sheaf on $X$ associated to $F^r$. We have the following well-known:

**Lemma 21.** *(see e.g. [2]) $\mathcal{E}(g)$ is a rank $r$ locally free sheaf on $X$.*

*Proof.* By definition, $\mathcal{E}(g)_{\overline{P}}=g_p^{-1}(\mathcal{O}_p^r)$. Hence by Ex. 5.7(b) of Ch.2 in [2], it suffices to show that $\mathcal{E}(g)$ is a coherent $\mathcal{O}_X$-sheaf coherent. This is a local problem. Choose then a point $p\in X$. It is not too difficult to prove that, see e.g. Lemma 6.2 of [10], there exist $g_{1,p}\in\mathrm{GL}_r(F)$ and $g_{2,p}\in\mathrm{GL}_n(\mathcal{O}_p)$ such that $g_p^{-1}=g_{1,p}g_{2,p}$. Choose then an affine open neighborhood $U_P$ of $p$ such that for all $q\in U,\ q\neq p,\ g_q\in\mathrm{GL}_r(\mathcal{O}_q)$ and $A\in\mathrm{GL}_r(\mathcal{O}_q)\subset\mathrm{GL}_r(\mathcal{O}_{\overline{q}})$. This is possible since there exists only finitely many $q \in X$ such that $g_p \notin \mathrm{GL}_r(\mathcal{O}_{\overline{q}})$ and $A \notin \mathrm{GL}_r(\mathcal{O}_{\overline{q}})$. Consequently,

$$\begin{aligned}\mathcal{E}(g)|_U =& \mathcal{F}^r|_U \cap \big(\cap_{p\in U} g_p^{-1}(\mathcal{O}_p^r)|_U\big) = \mathcal{F}^r|_U \cap \big(\cap_{p\in U} g_p^{-1}(\mathcal{O}_p^r|_U)\big)\\ =& \mathcal{F}^r|_U \cap \big(\cap_{p\in U} (g_{1,p}g_{2,p})(\mathcal{O}_p^r|_U)\big) = \mathcal{F}^r|_U \cap \big(\cap_{p\in U} g_{1,p}(\mathcal{O}_p^r|_U)\big)\\ =& g_{1,p}(\mathcal{F}^r|_U) \cap \big(\cap_{p\in U} g_{1,p}(\mathcal{O}_p^r|_U)\big) = g_{1,p}\left(\mathcal{F}^r|_U) \cap \big(\cap_{p\in U} \mathcal{O}_p^r|_U\big)\right)\\ =& g_{1,p}(\mathcal{O}_U^r).\end{aligned} \tag{91}$$

Therefore, $\mathcal{E}(g)$ is coherent and hence locally free. □

Denote by $\mathcal{M}_{X,r}$ be the moduli stack of (isomorphism classes of) rank $r$ locally free sheaves on $X$. Then we have the following well-known

**Proposition 22.** *There is a natural bijective correspondence*

$$\begin{array}{rccc}\pi: & \mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O}) & \longrightarrow & \mathcal{M}_{X,r}\\ & [g] & \longmapsto & [\mathcal{E}(g)]\end{array} \tag{92}$$

*Proof.* We first prove that $\pi$ is well-defined. Assume that $g,\, h \in \mathrm{GL}_r(\mathbb{A})$ satisfy $\mathbb{E}(g) = \mathbb{E}(h)$. Then, for each $p \in X$, $g_p^{-1}(\mathcal{O}_p^r) = h_p^{-1}(\mathcal{O}_p^r)$, or equivalently, $(h_p g_p^{-1})(\mathcal{O}_p^r) = \mathcal{O}_p^r$. This implies that $hg^{-1} \in \mathrm{GL}_r(\mathcal{O})$. More generally, assume that $\mathbb{E}(g) \simeq \mathbb{E}(h)$, this induces an isomorphism $\phi_\eta : \mathbb{E}(g)_\eta \simeq \mathbb{E}(h)_\eta$. Since $\mathbb{E}(g)_\eta \simeq F^r$ and $\mathbb{E}(h)_\eta \simeq F^r$, $\phi_K$ is determined by an element $\Phi \in \mathrm{GL}_r(F)$. Obviously, for each $p \in X$,

$$\Phi\left(g_p^{-1}(\mathcal{O}_p^r)\right) \simeq g_p^{-1}(\mathcal{O}_p^r) \simeq h_p^{-1}(\mathcal{O}_p^r).$$

Hence $\pi$ is not only well-defined, but injective.

Next we prove that $\pi$ is an surjection. Let $\mathbb{E}$ be a rank $r$ locally free sheaf on $X$. Then $\mathcal{E}_p \subset F_p$ is a rank $r$ projective $\mathcal{O}_p$-module. Therefore, there exists an element $g_p \in \mathrm{GL}_r(F_p)$ such that $g_p(\mathcal{E}_p) = \mathcal{O}_p^r$. But for all but finitely many $p \in X$, $\mathcal{E}_p \simeq \mathcal{O}_p^r$. This implies that, for such a $p$, $g_p \in \mathrm{GL}_r(\mathcal{O}_p)$. Therefore, $g := (g_p) \in \mathrm{GL}_r(\mathbb{A})$. On the other hand, by definition, $\mathcal{E}(g) \simeq \mathcal{E}$. □

**Example 4.** Let $D = \sum_p n_p\, p$ be a divisor on $X$ and denote by $\mathcal{O}_X(D)$ the invertible sheaf on $X$ associated to $D$. To give an adelic interpretation, we set $g_p = \pi_p^{n_p}$ and $g_D = (g_p)$. This implies that

$$g_p^{-1}\mathcal{O}_p = \pi_p^{-n_p}\mathcal{O}_p \simeq \mathcal{O}_X(D)_p. \tag{93}$$

In other words $\mathcal{E}(g_D) = \mathcal{O}_X(D)$. In addition,

$$\begin{aligned}&H^0\big(X, \mathcal{E}(g_D)\big) = \big\{f \in F : gf \in \mathcal{O}\big\}\\ =&\big\{f \in F : g_p f \in \mathcal{O}_p\ \forall p\big\} = \big\{f \in F : \pi_p^{n_p} f \in \mathcal{O}_p\ \forall p\big\}\\ =&\big\{f \in F : \mathrm{ord}_p\big(\pi_p^{n_p} f\big) \geq 0\ \forall p\big\} = \big\{f \in F : \mathrm{ord}_p(f) + n_p \geq 0\ \forall p\big\}\\ =&\big\{f \in F : (f) + D \geq 0\big\} = H^0\big(X, \mathcal{O}_X(D)\big).\end{aligned} \tag{94}$$

This shows that it is equally easy to use adelic language, instead of locally sheaves.

# Appendix B
# Review of Classical Algebraic Geometry Codes

As in the previous appendix, we claim no credit but accept any possible mistakes for the contents here. In fact, the materials can found in [4], [6], [9] and [11].

## B.1 AG codes in terms of $H^0$

Let $D = p_1 + \ldots + p_n$ be a degree $n$ divisor and let $E$ be a positive divisor. Assume that the $p_i$'s are mutually distinct and that $|E| \cap |D| = \emptyset$, where $|\cdot|$ denotes the support of the divisor. Then by Example 4,

$$g_D = (g_p) \qquad \text{and} \qquad g_E = (\pi_p^{\mathrm{ord}_p(E)}) \tag{95}$$

where $g_p = \begin{cases} \pi_{p_i} & p = p_i \\ 1 & p \notin \{p_1, \ldots, p_n\} \end{cases}$. In addition,

$$\begin{aligned} H^0(X, \mathcal{O}_X(E)) &= \left\{ f \in F : \begin{array}{ll} \pi_p^{\mathrm{ord}_p(E)} f \in \mathcal{O}_p & \forall p \notin \{p_1, \ldots, p_n\}, \\ f \in \mathcal{O}_p & \forall p \in \{p_1, \ldots, p_n\} \end{array} \right\}, \\ H^0(X, \mathcal{O}_X(E-D)) &= \left\{ f \in F : \begin{array}{ll} \pi_p^{\mathrm{ord}_p(E)} f \in \mathcal{O}_p & \forall p \notin \{p_1, \ldots, p_n\}, \\ f \in \pi_p \mathcal{O}_p & \forall p \in \{p_1, \ldots, p_n\} \end{array} \right\}, \end{aligned} \tag{96}$$

Consequently, if $f \in H^0(X, \mathcal{O}_X(E))$, then $f \in \mathcal{O}_{p_i}$ for all $i = 1, \ldots, n$. This implies that $(f(p_1), \ldots, f(p_n))$ makes sense. This then leads to the space

$$C_L(D, E) := \left\{ (f(p_1), \ldots, f(p_n)) \in \mathbb{F}_q^n : f \in H^0(X, \mathcal{O}_X(D)) \right\}. \tag{97}$$

Moreover, the natural morphism

$$\begin{array}{rccc} \phi_{D,E} : & H^0(X, \mathcal{O}_X(D)) & \longrightarrow & C_L(D, E) \\ & f & \longmapsto & (f(p_1), \ldots, f(p_n)) \end{array} \tag{98}$$

is surjective by definition, and its kernel is given by

$$\mathrm{Ker}(\phi_{D,E}) = H^0(X, \mathcal{O}_X(E - D)) \tag{99}$$

by the description of $H^0(X, \mathcal{O}_X(E - D))$ in (96). This gives the short exact sequence

$$0 \to H^0(X, \mathcal{O}_X(E - D)) \to H^0(X, \mathcal{O}_X(E)) \xrightarrow{\phi_{D,E}} C_L(D, E) \to 0. \tag{100}$$

This shows that the description of algebraic geometry code is equally clear in terms of adelic language.

## B.2 Codes in terms of $\Omega$

Let $D$ and $E$ be the same as in Example B.1. As considered in algebraic geometry code, we consider

$$C_\Omega(D, E) := \left\{ (\omega_{p_1}(1), \ldots, \omega_{p_n}(1)) \in \mathbb{F}_q^n : \ \omega \in \Omega_F(E - D) \right\}. \tag{101}$$

Here $\Omega_F(E-D)$ is viewed as a collection of rational differentials $\omega$ such that $(\omega)\geq E-D$. In terms of cohomology theory, we have

$$\Omega_F(E-D)\simeq H^0(X,K_X(D-E)). \tag{102}$$

However, with this expression, it is difficult to see how $(\omega_{p_1}(1),\dots,\omega_{p_n}(1))$ can be defined. To explain this, we use the duality

$$H^0(X,K_X(D-E))\simeq H^1(X,\mathcal{O}_X(E-D))^\vee. \tag{103}$$

Recall that $H^1(X,\mathcal{O}_X(E-D)):=\mathbb{A}/(\mathbb{A}(E-D)+F)$ and hence

$$H^1(X,\mathcal{O}_X(E-D))^\vee:=\mathrm{Hom}_{\mathbb{F}_q}\left(\mathbb{A}/(\mathbb{A}(E-D)+F),\mathbb{F}_q\right). \tag{104}$$

In this language, $\omega\in\Omega_F(E-D)$ if and only if the morphism $\omega:\mathbb{A}\to\mathbb{F}_q$ satisfies the condition that

$$\omega(\mathbb{A}(E-D))=0 \qquad \text{and} \qquad \omega(F)=0. \tag{105}$$

Obviously, $\omega:\mathbb{A}\to\mathbb{F}_q$ induces $\omega_p:F_p\to\mathbb{F}_q$ by restricting $\omega$ to $\mathbb{A}_q=F_p$. Moreover, for each $a\in\mathbb{A}$, there are only finitely many $p$ such that $p\notin\mathcal{O}_p$, thus, for the rest almost all $p$'s, $\omega_p(a_p)=0$. In this way, we have

$$\omega(a)=\sum_p\omega(a_p). \tag{106}$$

Recall that the space $\Omega_F$ of all rational differentials is one dimension over $F$. We may and hence will write $\omega=hd\pi$ for some $h\in F$. Hence $\omega_p(a_p)=\mathrm{res}_p(a_ph)$. Therefore, the second equation that $\omega(F)=0$ is equivalent to

$$\omega(f)=\sum_p\mathrm{res}_p(fh)=0\quad\forall f\in F. \tag{107}$$

This is nothing but the well-known *residue formula.*

To understand the first relation, we make soe preparations. First, since $s\in H^0(X,K_X(D-E))$, we have $(s)+(\omega_0)+(D-E)\geq 0$, where $\omega_0$ denotes a rational section of the canonical sheaf $K_X$. For later use, set $(\omega_0)=W_0$. This is equivalent to $(s\omega_0)\geq E-D$. Since $\omega=s\omega_0$, this implies that

$$\mathrm{res}_p(h\pi_p^{\mathrm{ord}_p(D-E)+n_+})=0\qquad\forall p\in|D|\cup|E|,\ \ \forall n_+\in\mathbb{Z}_{\geq 0}. \tag{108}$$

Secondly, we go back to the definition of $\mathbb{A}(E-D)$:

$$\begin{aligned}\mathbb{A}(E-D)=&\big\{a\in\mathbb{A}:(a)+E-D\geq 0\big\}\\ =&\left\{a\in\mathbb{A}:\begin{array}{ll}a_p\in\mathcal{O}_p^* & p\notin|E|\cup|D|\\ a_p\in\pi^{-\mathrm{ord}_p(E)}\mathcal{O}_p & p\in|E|\\ a_p\in\pi_p\mathcal{O}_p & p\in|D|\end{array}\right\}.\end{aligned} \tag{109}$$

Now, to see $\omega(\mathbb{A}(E-D))=0$, we make the following calculation.

$$\begin{aligned}&\sum_p\mathrm{res}_p\omega(\mathbb{A}(E-D))\\ =&\sum_{p\notin|E|\cup|D|}\mathrm{res}_p(h\mathcal{O}_p^*)+\sum_{p\in|E|}\mathrm{res}_p(h\pi^{-\mathrm{ord}_p(E)}\mathcal{O}_p)+\sum_{p\in|D|}\mathrm{res}_p(h\pi_p\mathcal{O}_p)\\ =&\sum_{p\notin|E|\cup|D|}\mathrm{res}_p(h)+\sum_{p\in|E|}\mathrm{res}_p(h\pi^{-\mathrm{ord}_p(E)}\mathcal{O}_p)+\sum_{p\in|D|}\mathrm{res}_p(h\pi_p\mathcal{O}_p)\\ =&\sum_{p\notin|E|\cup|D|}\mathrm{res}_p(h).\end{aligned} \tag{110}$$

by (108). On the other hand, $(\omega) \geq E - D$ implies that for $p \notin |E| \cup |D|$, $\mathrm{ord}_p(h) \geq 0$. Therefore,

$$\sum_p \mathrm{res}_p \omega(\mathbb{A}(E-D)) = \sum_{p \notin |E| \cup |D|} \mathrm{res}_p(h) = 0. \tag{111}$$

This proves the following well-known

**Corollary 23** (Duality Theorem)**.** *With the same notation as above,*

$$\begin{aligned} &H^0(X, K_X(D-E)) \simeq F \cap \mathbb{A}(W_0 + D - E)) \\ &\simeq \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{A}/(\mathbb{A}(E-D)+F), \mathbb{F}_q) \simeq H^1(X, \mathcal{O}_X(E-D))^\vee. \end{aligned} \tag{112}$$

Now we analysis the space $C_\Omega(D,E)$. For $\omega \in \Omega_F(E-D)$, as above, viewing it as a morphism $\mathbb{A} \to \mathbb{F}_q$, we obtain a morphism $\omega_p : F_p \to \mathbb{F}_p$. In particular, if $p \in |D|$, we have $\omega_p(\pi_p \mathcal{O}_p) = 0$. Therefore, $\omega(\mathcal{O}_p) = \omega_p(\mathbb{F}_q) = \mathbb{F}_q \omega_p(1)$. This implies that the map

$$\begin{array}{cccc} \varphi_{D,E}: & \Omega_F(E-D) & \longrightarrow & C_\Omega(D,E) \\ & \omega & \longmapsto & (\omega_{p_1}(1), \ldots, \omega_{p_n}(1)) \end{array} \tag{113}$$

is surjective. To see the kernel of $\varphi_{D,E}$, we need to see for which $\omega \in \Omega_F(E-D)$, $\omega_p(1) = 0$ for all $p \in |D|$. For this, we use the duality theorem to see that

$$\begin{aligned} \Omega_F(E-D) =& H^1(X, \mathcal{O}_X(E-D))^\vee \\ =& \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{A}/(\mathbb{A}(E-D)+F), \mathbb{F}_q) \\ =& H^0(X, K_X(D-E)). \end{aligned} \tag{114}$$

In other words, as mentioned above, $\omega \in \Omega_F(E-D)$ if and only if the $\omega$-image is zero on both $F$ and the space

$$\mathbb{A}(E-D) = \left\{ a \in \mathbb{A} : \begin{array}{ll} a_p \in \mathcal{O}_p^* & p \notin |E| \cup |D| \\ a_p \in \pi_p^{-\mathrm{ord}_p(E)} \mathcal{O}_p & p \in |E| \\ a_p \in \pi_p \mathcal{O}_p & p \in |D| \end{array} \right\}. \tag{115}$$

In particular,

$$\omega_p(\mathcal{O}_p) = \omega_p(\mathbb{F}_q + \pi \mathcal{O}_p) = \omega_p(\mathbb{F}_q) = \mathbb{F}_q\, \omega_p(1), \qquad \forall p \in |D| \tag{116}$$

since $\omega$ is $\mathbb{F}_q$-linear. With a similar discussion, if $\omega$ in the kernel of $\varphi_{D,E}$, then not only the $\omega$-image of both $F$ and $\mathbb{A}(E-D)$ is zero, $\omega_p(\mathcal{O}_P) = \{0\}$. That is to say, the $\omega$-image is zero on both $F$ and the space

$$\left\{ a \in \mathbb{A} : \begin{array}{ll} a_p \in \mathcal{O}_p^* & p \notin |E| \cup |D| \\ a_p \in \pi_p^{-\mathrm{ord}_p(E)} \mathcal{O}_p & p \in |E| \\ a_p \in \mathcal{O}_p & p \in |D| \end{array} \right\}, \tag{117}$$

which is nothing but $\mathbb{A}(E)$. This then establish the following

**Theorem 24.** *There is a short exact sequence*

$$0 \to \Omega_F(E) \to \Omega_F(E-D) \xrightarrow{\varphi_{D,E}} C_\Omega(D,E) \to 0. \tag{118}$$

As a direct consequence of the above discussion, we also obtain the following technical result, which plays a central role in the classical approaches of AG codes.

**Corollary 25.** *(Proposition 1.7.3 of [9]) Let $\omega \neq 0$ be a Weil differential of $F/\mathbb{F}_q$ and $p \in X$ be a closed point. Then*

$$v_p(\omega) = \max\{r \in \mathbb{Z} : \omega_p(f) = 0 \ \forall f \in F \text{ s.t. } v_p(f) \geq -r\}. \tag{119}$$

*In particular, if $v_p(\omega) \geq -1$, then $\omega_p(1) = 0$ if and only if $v_p(\omega) \geq 0$.*

*Proof.* In fact, if we set $E = n\,p + E'$ with $|E| = \{p\} \cup |E'|$, then, by the above discussion, for the place $p$ concern, $\omega \in \Omega(E)$ if and only if $\omega_p(\pi^{-n}\mathcal{O}_p) = 0$. □

## B.3 Ω codes are AG codes

We use the same notation as in the previous subsubsections. By the duality theorem, the exact sequence in Theorem 24 becomes the following

$$0 \to H^0(X, K_X(-E)) \to H^0(X, K_X(D-E)) \stackrel{\varphi_{D,E}^\vee}{\to} C_\Omega(D,E) \to 0. \tag{120}$$

So the point is to see whether $(\omega_{p_1}(1), \ldots, \omega_{p_n}(1))$ for $\omega \in \Omega_F(E-D)$ can be written as $(f(p_1), \ldots, f(p_n)$ for $f \in H^0(X, K_X(D-E))$ under the correspondence

$$\begin{array}{cccc} \Phi: & \Omega_F(E-D) & \longrightarrow & H^0(X, K_X(D-E)) \\ & \omega & \longmapsto & \omega/\omega_0 =: f \end{array} \tag{121}$$

where $\omega_0$ is a suitable rational differential on $X$. To obtain a 'proper' $\omega_0$, we first see what are the properties which $\omega_0$ should satisfies.

Set $H = (\omega_0) + E - D$. Then for $p \in |D|$,

$$\begin{aligned} \mathbb{F}_q\omega_p(1) =& \omega_p(\mathbb{F}_p) = \omega_p(\mathbb{F}_p + \pi_p\mathcal{O}_p) \\ =& \omega_p(\mathcal{O}_p) = (f\omega_0)_p(\mathcal{O}_p) \end{aligned} \tag{122}$$

By definition,

$$\mathrm{ord}_p(f) + \mathrm{ord}_p(\omega_0) = \mathrm{ord}_p(f\omega_0) = \mathrm{ord}_p(\omega) \geq -1. \tag{123}$$

In addition, $\omega_p(1) = 0$ if and only if $f(P) = 0$. But $\omega_p(1) = 0$ means that as far as the point $p$ concern, $\omega_p(\mathcal{O}_p) = 0$, hence $\mathrm{ord}_p(\omega_p) \geq -1$. Similarly, $f(p) = 0$ means that $\mathrm{ord}_p(f) \geq 1$. Thus, in this case, by (123),

$$0 = \mathrm{ord}_p(f) + \mathrm{ord}_p(\omega_0) \geq 1 + \mathrm{ord}_p(\omega_0) \tag{124}$$

That is,

$$\mathrm{ord}_p(\omega_0) \geq -1. \tag{125}$$

This means

$$\omega_{0,p}(\pi_p\mathcal{O}_p) = 0 \qquad \text{and} \qquad \omega_{0,p}(\mathcal{O}_p) = \mathbb{F}_q\omega_{0,p}(1). \tag{126}$$

This means that $\mathrm{ord}_p(\omega) \geq 0$ iff $\mathrm{ord}_p(f) \geq 1$. This implies that

$$-1 \leq \mathrm{ord}_p(\omega_p) \qquad \text{and} \qquad |H| \cap |D| = \emptyset. \tag{127}$$

On the other hand, if $\mathrm{ord}_p(\omega_0) \geq 0$, then $\omega_0(\mathcal{O}_p) = 0$. This would implies that $f\omega_0(\mathcal{O}_p) \equiv 0$ since $f(p)$ is well-defined. This cannot happen since we assume that our code is not trivial. All this then implies the following

**Lemma 26.** *The rational differential $\omega_0$ should satisfy*

$$\mathrm{ord}_p(\omega_0) = -1. \tag{128}$$

By twisting a certain $D$-unit, we always can assume that $\omega_0(1) = 1$.

**Definition 27.** *A non-zero rational differential $\omega_0$ is call $D$-special if*

$$\mathrm{ord}_p(\omega_0) = -1 \qquad \text{and} \qquad \omega_{0,p}(1) = 1. \tag{129}$$

**Lemma 28.** *There always exists non-trivial $D$-special rational differentials.*

*Proof.* It suffices to prove the existence of rational differential $\omega_0$ such that $\mathrm{ord}_p(\omega_0) = -1$ for all $D$. To see this, we first note that, by the Riemann-Roch theorem, $h^0(X, K_X(D-E)) - h^0(X, K_X(-E))) = \deg(K_X(D)) - (g-1) - (g-1) = n$. Therefore by the duality theorem, within the exact sequence

$$0 \to \Omega_F(E) \stackrel{\iota_{D,E}}{\to} \Omega_F(E-D) \to \mathrm{Coker}(\iota_{D,E}) \to 0, \tag{130}$$

the quotient space $\mathrm{Coker}(\iota_{D,E})$ is not trivial. □

From now on, $\omega_0$ is always taken to be $D$-special.

Now we are ready to continue the calculation in (122). That is, for $f \in H^0(X, \mathcal{O}_X(H))$ and $p \in |D|$,

$$\begin{aligned} \omega_p(1)\,\mathbb{F}_q =& \omega_p(\mathcal{O}_p) = (f\omega_0)(\mathcal{O}_p) \\ =& f(p) \cdot \omega_0(\mathcal{O}_p) = f(p) \cdot \mathbb{F}_q \omega_{0.p}(1) \\ =& f(p)\,\mathbb{F}_q \qquad \forall p \in |D|. \end{aligned} \tag{131}$$

In particular, we have

$$\omega_p(1) = f(p) \qquad \forall p \in D. \tag{132}$$

This then implies the following

**Proposition 29.** *There is a natural identification*

$$C_\Omega(D, E) = C_L(D, H). \tag{133}$$

*In particular, all $\Omega$-codes are $AG$ codes.*

## B.4 The duality between $C_L(D,E)$ and $\Omega_F(D,E)$

Induced from the natural dual pairing

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle_{\omega_0}: & \mathbb{A}\times\mathbb{A} & \longrightarrow & \mathbb{F}_q \\ & (a,b) & \longmapsto & \sum_p \omega_{0,p}(a_p b_p). \end{array} \tag{134}$$

It is well-known that $F^\perp = F$, or equivalently,

$$\omega_0(h) = 0 \qquad \forall h \in F. \tag{135}$$

Surely, this is the famous residue formula.

Now we consider the non-degenerating pairing

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle: & \mathbb{F}_q^n \times \mathbb{F}_q^n & \longrightarrow & \mathbb{F}_q \\ & (x,y) & \longmapsto & \sum_{i=1}^n x_i y_i. \end{array} \tag{136}$$

Under this pairing, we consider the image of $C_L(D,E)\times\Omega_F(D,E)$.

$$\begin{array}{rccc}\langle\cdot,\cdot\rangle: & C_L(D,E)\times\Omega_F(D,E) & \longrightarrow & \mathbb{F}_q\\ & \big((f(p_1),\dots,f(p_n)),(\omega_{p_1}(1),\dots,\omega_{p_1}(1))\big) & \longmapsto & \sum_{i=1}^n f(p_i)\,\omega_{p_i}(1).\end{array} \tag{137}$$

By definition, note that $\mathrm{ord}_{p_i}(\omega/\omega_0)\geq 0$, we have

$$\sum_{i=1}^n f(p_i)\,\omega_{p_i}(1)=\sum_{i=1}^n\omega_{p_i}(f(p_i))=\omega(f)=\omega_0(\omega/\omega_0\cdot f)=0. \tag{138}$$

by (135). This then proves the following:

**Proposition 30.** *We have*

$$C_L(D,E)^\perp=\Omega_F(D,E). \tag{139}$$

## B.5 Invariants of AG Codes $C_L(D,E)$ and $\Omega_F(D,E)$

By the exact sequences (100) and (118), we have

$$\begin{aligned}\dim_{\mathbb{F}_q}C_L(D,E)=&h^0(X,\mathcal{O}_X(E))-h^0(X,\mathcal{O}_X(E-D))\\ \dim_{\mathbb{F}_q}\Omega_F(D,E)=&h^1(X,\mathcal{O}_X(E-D))-h^1(X,\mathcal{O}_X(E)).\end{aligned} \tag{140}$$

This implies, from the Riemann-Roch theorem, that

$$\begin{aligned}&\dim_{\mathbb{F}_q}C_L(D,E)+\dim_{\mathbb{F}_q}\Omega_F(D,E)\\ &=\chi(X,\mathcal{O}_X(E))-\chi(X,\mathcal{O}_X(E-D))=\deg(D)\\ &=n.\end{aligned} \tag{141}$$

This characterizes the dimensions of the codes spaces $C_L(D,E)$ and $\Omega_F(D,E)$.

To see the weights of them, we first recall that the weight $d$ of codes $C$ is the biggest number such that $w(a)\geq d$ for all codewords $a\in C$. That is to say, $d$ is the biggest (natural) number such that for any codeword $a$, the number of its non-zero components is at least $d$.

Let $d_{D,E}$ be the weight of $C_L(D,E)$ which we assume to be non-trivial. By definition, we may find $f\in H^0(X\mathcal{O}_X(E))$ such that $\mathrm{wt}\big(\phi_{D,E}(f)\big)=d_{D,E}$. This means that there are exactly $n-d_{D,E}$ points $p_{i_1},\dots,p_{i_{n-d}}\in|D|$ such that $f(p_{i_j})=0$ for $j=1,\dots,n-d_{D,E}$. This implies that $f\in H^0(X,\mathcal{O}_X(E-\sum_{j=1}^{n-d_{D,E}}p_{i_j}))$. In particular,

$$0\leq\deg(E-\sum_{j=1}^{n-d_{D,E}}p_{i_j})=\deg(E)-n+d_{D,E}. \tag{142}$$

This the implies the following

**Proposition 31.** *(1) The invariants of $C_L(D,E)$ is given by*

$$\big(n,h^0(X,\mathcal{O}_X(E))-h^0(X,\mathcal{O}_X(E-D)),\geq n-\deg(E)\big) \tag{143}$$

*(1) The invariants of $\Omega_F(D,E)$ is given by*

$$\big(n,h^1(X,\mathcal{O}_X(E-D))-h^1(X,\mathcal{O}_X(E)),\geq\deg(E)-2g+2\big) \tag{144}$$

*Proof.* It suffices to prove (2). The statements for the length and the dimension are obvious. To see the lower bound for the weight, we use Proposiion 29. Hence the proof for (1) before the proposition implies that the minimal distance of the codes $\Omega_F(D,E)$ is $\geq n-\deg(H) = n-\deg((\omega_0)+E-n) = \deg(E)-2g-2$. □

For linear codes of types $(n,k,d)$, the so-called *Singleton Bound* is refer to the condition

$$n+1 \geq k+d. \tag{145}$$

In terms of AG codes $C_L(D,E)$ and $\Omega_F(D,E)$, this is equivalent to

$$\begin{aligned} n+1 \geq & h^0(X,\mathcal{O}_X(E)) - h^0(X,\mathcal{O}_X(E-D)) + d_{D,E} \\ \geq & \Big(h^0(X,\mathcal{O}_X(E)) - \deg(E)\Big) - h^0(X,\mathcal{O}_X(E-D)) + n \end{aligned} \tag{146}$$

and

$$\begin{aligned} n+1 \geq & h^1(X,\mathcal{O}_X(E-D)) - h^1(X,\mathcal{O}_X(E)) + d'_{D,E} \\ \geq & h^0(X,K_X(D-E)) - h^0(X,K_X(-E)) + \deg(E) - 2g + 2 \end{aligned} \tag{147}$$

respectively.

(1) Assume that $\deg(E-D) \leq 0$, i.e. $\deg(E) < n$, then, by the vanishing theorem $h^0(X,\mathcal{O}_X(E-D) = \{0\}$ and (146) becomes

$$n+1 \geq h^0(X,\mathcal{O}_X(E))+d_{D,E} \geq \Big(h^0(X,\mathcal{O}_X(E))-\deg(E)\Big)+n \geq n-g+1 \tag{148}$$

In particular, if $\deg(E) > 2g-2$, then

$$k + d_{D,E} = n-g+1. \tag{149}$$

(2) Assume that $\deg(E) > 2g-2$, then $h^1(X,\mathcal{O}_X(E)) = \{0\}$, and (147) becomes

$$\begin{aligned} n+1 \geq & h^1(X,\mathcal{O}_X(E-D)) + d'_{D,E} \\ \geq & h^0(X,K_X(D-E)) + \deg(E) - 2g + 2 \end{aligned} \tag{150}$$

This implies that

$$k^{\perp}_{D,E} = h^0(X,K_X(D-E)) \geq (2g-2)+n-\deg(E)-(g-1) = n+g-1-\deg(E). \tag{151}$$

In particular, if $\deg(E) < n$, then by the discussion using $H$, we get

$$k^{\perp}_{D,E} = n+g-1-\deg(E). \tag{152}$$

All these are surely nothing but the discussions on the invariants $(n, k_{D,E}, d_{D,E})$ and $(n, k^{\perp}_{D,E}, d^{\perp}_{D,E})$ for the AG codes $C_L(D,E)$ and $\Omega_F(D,E)$, respectively, in [9].

In the main text, we will introduce a high rank version for the codes above guided by the discussion in this appendix.

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