Adelic Extension Classes, Atiyah Bundles and Non-Commutative Codes

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Abstract

This paper consists of three components. In the first, we give an adelic interpretation of the classical extension class associated to extension of locally free sheaves on curves. Then, in the second, we use this construction on adelic extension classes to write down explicitly adelic representors in $\mathrm{GL}_r(\mathbb{A})$ for Atiyah bundles I_r on elliptic curves. All these works make sense over any base fields. Finally, as an application, for $m \geq 1$, we construct the global sections of $I_r(mQ)$ in local terms and apply it to obtain rank r MDS codes based on the codes spaces $C_{F,r}(D, I_r(mQ))$ introduced in [6].

1 Locally Free Sheaves in Adelic Language

Let X be a integral, regular, projective curve of genus g over a finite field $\mathbb{F}_q$. Denote by F the field of rational functions of X, by $\mathbb{A}$ the adelic ring of F, and by $\mathcal{O}$ the integer ring of $\mathbb{A}$.

Let $\mathcal{M}_{X,r}$ be the moduli stack of locally free sheaves of rank r on X. It is well known that there is a natural identification among elements of $\mathcal{M}_{X,r}$ and the adelic quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O})$:

$$\phi : \mathcal{M}_{X,r} \simeq \mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O}). \tag{1}$$

Indeed, if $\mathcal{E}$ is a rank r locally free sheaf on X, the fiber E_η of $\mathcal{E}$ over the generic point η of X is an F-linear space of dimension r. Among its $\mathrm{GL}_r(F)$-equivalence class, fix an F-linear isomorphism

$$\xi : E_\eta \longrightarrow F^r. \tag{2}$$

Then there exists a dense open subset U of X such that ξ induces a tribulation of $\mathcal{E}$ on U

$$\xi_U : \mathcal{E} \simeq \mathcal{O}_U^{\oplus r}. \tag{3}$$

To go local, let x be a closed point of X, and denote its formal neighborhood by $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$, where $\widehat{\mathcal{O}}_x$ is the x-adic completion of the local ring $\mathcal{O}_{X,x}$. Denote by $\widehat{F}_x$ the fraction field of $\widehat{O}_x$. Since X is separable, the natural inclusion $F \hookrightarrow \widehat{F}_x$ induces a morphism

$$\widehat{\iota}_x : \mathrm{Spec}(\widehat{\mathcal{O}}_x) \longrightarrow X. \tag{4}$$

Set $\widehat{\mathcal{E}}_x := \widehat{\iota}_x^* \mathcal{E}$ the pull-back of $\mathcal{E}$ on $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$. Then the fiber $\widehat{E}_{x,\eta}$ of $\widehat{\mathcal{E}}_x$ over the generic point admits a natural $\widehat{F}_x$-linear space of dimension r. Furthermore,

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since $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$ is affine, for the rank r locally free sheaf $\widehat{\mathcal{E}}_x$, there exists a free $\widehat{\mathcal{O}}_x$-module $E_x^{\sim}$ of rank r such that

$$\widehat{\mathcal{E}}_x \simeq \widetilde{E_x^{\sim}} \qquad \text{and} \qquad E_x^{\sim} \otimes_{\widehat{\mathcal{O}}_x} \widehat{F}_x = \widehat{E}_{x,\eta}, \tag{5}$$

where $\widetilde{E_x^{\sim}}$ denotes the locally free sheaf associated to $E_x^{\sim}$. Therefore, induced from the free $\widehat{\mathcal{O}}_x$-module structure on $\mathbb{E}_x$, there is a natural isomorphism

$$\widetilde{\xi}_x : \widehat{E}_{x,\eta} = E_x^{\sim} \otimes_{\widehat{\mathcal{O}}_x} \widehat{F}_x \simeq \widehat{F}_x^r. \tag{6}$$

On the other hand, induced from $F \hookrightarrow \widehat{F}_x$, there is a natural identification $E_\eta \otimes_F \widehat{F}_x = \widehat{E}_{x,\eta}$. Hence, induced from ξ in (2) and the natural inclusion $F \hookrightarrow \widehat{F}_x$, there is a natural isomorphism

$$\widehat{\xi}_x : \widehat{E}_{x,\eta} = E_\eta \otimes_F \widehat{F}_x \simeq \widehat{F}_x^r. \tag{7}$$

Consequently, for the isomorphisms $\widetilde{\xi}_x$ and $\widehat{\xi}_x$ in (6) and (7) respectively, there exists a unique element $\widehat{g}_x \in \mathrm{GL}_r(\widehat{F}_x)/\mathrm{GL}_r(\widehat{\mathcal{O}}_x)$ such that the following diagram is commutative

$$
\begin{array}{ccc}
E_\eta \otimes_F \widehat{F}_x & = \widehat{E}_{x,\eta} = & E_x^{\sim} \otimes_{\widehat{\mathcal{O}}_x} \widehat{F}_x \\
\widehat{\xi}_x \downarrow \simeq & & \simeq \downarrow \widetilde{\xi}_x \\
\widehat{F}_x^r & \overset{g_x}{\underset{\simeq}{}} & \widehat{F}_x^r.
\end{array}
\tag{8}
$$

In this way, we obtain an element $(g_x) \in \prod_{x \in X} \mathrm{GL}_x(\widehat{F}_x)$. Furthermore, this element (g_x) belongs to $\mathrm{GL}_r(\mathbb{A})$. Indeed, by (2), for $x \in U$,

$$\widetilde{E_x^{\sim}} \simeq \widehat{\mathcal{O}}_x^r. \tag{9}$$

This implies that, for such x, g_x may be choosen to be the identical matrix $I_r := \mathrm{diag}(1, 1, \ldots, 1)$. We denote the adelic class of (g_x) constructed above by $g_{\mathcal{E}}$ for later use.

Conversely, for a class in $\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(\mathbb{A}) / \mathrm{GL}_r(\mathcal{O})$, there exists a dense open subset U of X such that g_x are identity matrix for all $x \in U$. This yields an trivial locally free sheaf $\mathcal{O}_U^{\oplus r}$ on U. On the other hand, for a closed point x of $X \smallsetminus U$, the image $g_x^{-1}(\widehat{\mathcal{O}}_x^{\oplus r})$ is a full rank $\widehat{\mathcal{O}}_x$-module contained in $\widehat{F}_x^r$ since $g_x \in \mathrm{GL}_x(\widehat{F}_x)$, and hence induces a locally free sheaf $\widetilde{g_x^{-1}(\widehat{\mathcal{O}}_x^{\oplus r})}$ of rank r on $\mathrm{Spec}(\widehat{\mathcal{O}}_x)$. All these gives the locally free sheaf $\mathcal{E}_{(g_x)}$ of rank r on X by the so-called fpqc-gluing using the components of (g_x). It is not difficult to see that the adelic class $g_{\mathcal{E}_{(g_x)}}$ coincides with (that of) (g_x).

2 Extensions of Locally Free Sheaves in Adelic Language

Over the curve X, let

$$\mathbb{E}: \qquad 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \to 0 \tag{10}$$

be a short exact sequence of locally free sheaves. For $i = 1, 2, 3$, denote by r_i the rank of $\mathcal{E}_i$, and denote by $g_i \in \mathrm{GL}_{r_i}(\mathbb{A})$ the adelic class associated to $\mathcal{E}_i$ introduced in §1. Our aim of this section is to write down g_2 in terms of g_1 and g_2 and, more importantly, the extension class associated to $\mathbb{E}$.

2.1 Classical Approach

In this subsection, we give a classical description of extension. The main references is [2].

Applying the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$ to $\mathbb{E}$ leads to a long exact sequence of sheaves

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1). \quad (11)$$

Following Grothendieck, the extension $\mathbb{E}$, up to isomorphism, is uniquely determined by the δ-image of the identity element $\mathrm{Id}_{\mathcal{E}_3}$ of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3)$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1)$. Being working over the curve X, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) \simeq H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1), \quad (12)$$

where, for a locally free sheaf $\mathcal{E}$, we denote its its dual sheaf by $\mathcal{E}^\vee$. Note that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3) \simeq H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3), \quad (13)$$

(11) is equivalent to the long exact sequence

$$0 \to H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) \to H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_2) \to H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3) \xrightarrow{\delta} H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1). \quad (14)$$

In addition, there are natural an isomorphism and a decomposition

$$\mathcal{E}_3^\vee \otimes \mathcal{E}_3 = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3) \simeq \mathcal{O}_X \oplus \mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3), \quad (15)$$

respectively. Here $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3)$ denotes the sheaf of endmorphisms of $\mathcal{E}_3$ and $\mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3)$ denotes the sub-sheaf of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3)$ resulting from the so-called trace zero endmorphisms. Furthermore, since $H^0(X, \mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}_3)) = 0$, the morphism δ in (11) is equivalent to the induced morphism

$$\delta : H^0(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1), \quad (16)$$

and the extension $\mathbb{E}$, up to equivalence, is uniquely determined by $\delta(1)$, the δ-image of the unit element 1 of $H^0(X, \mathcal{O}_X)$. Here, as usual, if

$$\mathbb{E}' : \qquad 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_2' \longrightarrow \mathcal{E}_3 \to 0 \quad (17)$$

is another extension of $\mathcal{E}_3$ by $\mathcal{E}_1$, and there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_3 & \to & 0 \\
 & & \| & & \phi \downarrow \simeq & & \| & & \\
0 & \to & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{E}_3 & \to & 0
\end{array} \quad (18)$$

then ϕ is called an equivalence between two extensions $\mathbb{E}$ and $\mathbb{E}'$ of of $\mathcal{E}_3$ by $\mathcal{E}_1$. Normally, we denote an equivalence by $\phi : \mathbb{E} \simeq \mathbb{E}'$.

2.2 Locally Description

In this subsection, we give a more concrete local description of the boundary map

$$\delta : H^0(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_3) \to H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1). \tag{19}$$

For this purpose, we first recall the adelic interpretation of $H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1)$. Denote by $g_\mathcal{E} \in \mathrm{GL}_r(\mathbb{A})$ be an adelic element associated to a rank r locally free sheaf $\mathcal{E}$. Then

$$H^1(X, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) = \mathbb{A}^{r_1 r_3} / (\mathbb{A}^{r_1 r_3}(\mathcal{E}_3^\vee \otimes \mathcal{E}_1) + F^{r_1 r_3}), \tag{20}$$

where

$$\mathbb{A}^{r_1 r_3}(\mathcal{E}_3^\vee \otimes \mathcal{E}_1); = \left\{ a \in \mathbb{A}^{r_1 r_3} : g_{\mathcal{E}_3^\vee \otimes \mathcal{E}_1} a \in \mathcal{O}^{r_1 r_3} \right\}. \tag{21}$$

It is not difficult to see that this space is isomorphic to

$$\prod_{x \in X}' \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta}) \Big/ \Big(\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^\sim, \widehat{E}_{1,x}^\sim) + \mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta}) \Big). \tag{22}$$

Here $\prod'$ denotes the restrict product of $\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta})$ with respect to $\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^\sim, \widehat{E}_{1,x}^\sim)$. Hence, we should find a natural morphism from $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3)$ to (22) which gives the boundary map (11) for the extension classes.

By applying $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$, or the same $\mathcal{E}_3^\vee \otimes$, to the extension $\mathbb{E}$, we obtain a short exact sequence of locally free sheaves

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \to 0, \tag{23}$$

since the the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_3, \cdot)$ and $\mathcal{E}_3^\vee \otimes$ are left and right exactness, respectively. Furthermore, by applying the derived functor of $\Gamma(X, \cdot)$ to (23), we arrive at the long exact sequence (11), namely,

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_2) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_3) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1). \tag{24}$$

To understand this boundary map, we next recall that how the boundary map in the following well-known snake lemma is constructed.

Lemma 1 (Snake Lemma). *Let R be a commutative ring. Assume that*

$$\begin{array}{ccccccccc}
0 & \to & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \to & 0 \\
 & & \phi_1 \downarrow & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\
0 & \to & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \to & 0
\end{array} \tag{25}$$

is a commutative diagram of R-modules with exact rows. Then, for the kernel and cockerel of ϕ_i, there is a long exact sequence

$$0 \to \mathrm{Ker}(\phi_1) \to \mathrm{Ker}(\phi_2) \to \mathrm{Ker}(\phi_3) \xrightarrow{\delta} \mathrm{Coker}(\phi_1) \to \mathrm{Coker}(\phi_2) \to \mathrm{Coker}(\phi_3) \to 0.$$

In particular, the boundary mapping δ is defined by

$$\begin{array}{rcl}
\delta : & \mathrm{Ker}(\phi_3) & \longrightarrow & \mathrm{Coker}(\phi_1) \\
 & a_3 & \mapsto & \phi_2(a_2) \in B_1 \bmod \phi_1(A_1),
\end{array} \tag{26}$$

where $a_2 \in A_2$ is a left of the element $a_3 \in A_3$.[1]

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[1] Certainly, δ is we–defined. Indeed, since $a_3 \in \mathrm{Ker}(\phi_3)$ implies that a_3 has the zero image in $cKer(\phi_3)$. This implies that the element $\phi_2(a_2)$ of $\mathrm{Coker}(\phi_2)$ belongs to the sub-module $\mathrm{Coker}(\phi_1)$.

To apply this lemma, we now consider the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{1,x}^{\sim}) & \to & \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{2,x}^{\sim}) & \to & \mathrm{End}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}) & \overset{\delta}{\to} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1;x,\eta}) & \to & \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{2;x,\eta}) & \to & \mathrm{End}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\overset{\delta}{\to} & & \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x}) & \to & \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{2,x}) & \to & \mathrm{End}(\overline{E}_{3,x}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0.
\end{array}
$$
(27)

Here, for $i = 1, 2, 3$,

$$
\mathrm{Hom}(\overline{E}_{i,x}, \overline{E}_{j,x}) := \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{i;x,\eta}, \widehat{E}_{j;x,\eta})\big/\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{i,x}^{\sim}, \widehat{E}_{j,x}^{\sim}).
$$

Obviously, by (22), we have

$$
\prod_{x \in X}' \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x})\big/\mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta}) \simeq
$$
$$
\prod_{x \in X}' \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta})\Big/\Big(\prod_{x \in X}' \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{1,x}^{\sim}) + \mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta})\Big).
$$
(28)

Accordingly, in the senate lemma, we set a_2 to be the identity morphism Id of $\mathrm{End}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim})$, by the Snake Lemma, we obtain an element $\kappa_x = \delta(\mathrm{Id})$ in $\mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x})$. Therefore, applying the natural quotient morphism

$$
\prod_{x \in X}' \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x}) \longrightarrow \prod_{x \in X}' \mathrm{Hom}(\overline{E}_{3,x}, \overline{E}_{1,x})\big/\mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta}) \quad (29)
$$

we obtain an element $([\kappa_x]) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1)$, which is nothing but the extension class for the extension $\mathbb{E}$ of $\mathcal{E}_3$ by $\mathcal{E}_1$, by (28), (22), (20) and (12). This then completes the proof of the following

Theorem 2. *The natural bijections*

$$
\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_3, \mathcal{E}_1) \simeq H^1(X, \mathcal{E}_3^{\vee} \otimes \mathcal{E}_1)
$$

$$
\Phi \downarrow \simeq \qquad \qquad \simeq \downarrow
$$

$$
\prod_{x \in X}' \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta})\Big/\Big(\prod_{x \in X}' \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{1,x}^{\sim}) + \mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta})\Big)
$$

such that

$$
\Phi(\delta(\mathrm{Id})) = \big([\kappa_x]\big)_{x \in X}. \tag{30}
$$

Proof. Indeed, the commutative diagram of bijections are direct consequence of (22), (20) and (12). And the relation (30) comes direct from the construction of κ_x. □

We end this section by an effective construction of the inverse map of Φ.

Let $s_\eta = (s_{x,\eta})$ be be an element of $\mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1,;x,\eta})$.

To clarifying the structures involved, first we *assume that s_η is regular for all but one closed point $x \in X$.* That is to say, there exists one and only one closed point $x_0 \in X$ such that $s_{x,\eta}$ are regular when $x \neq x_0$. Choose an open neighborhood U_0 of x_0 in X such that $s_{x_0,\eta}$ is regular over $U_0 \smallsetminus \{x_0\}$. Shrinking U_0 if necessary, we may assume that U_0 is affine. Denote its affine ring by A_{U_0}. Within the $r_1 + r_3$-dimensional $\widehat{F}_x$-linear space $\widehat{E}_{1;x_0,\eta} \oplus \widehat{E}_{3;x_0,\eta}$, construct an A_{U_0}-module generated by $E_{1,U_0}^\sim \oplus \{0\}$ and $\big\{ (s_{x_0,\eta}(b), b) : b \in E_{3,U_0}^\sim \big\}$, where, for $i = 1, 3$, $E_{i,U_0}^\sim = \Gamma(U_0, \mathcal{E}_i|_{U_0})$ so that $\mathcal{E}_i|_{U_0} \simeq \widetilde{E_{i,U_0}^\sim}$. In other words, this new A_{U_0}-module contained in $E_{1;x,\eta} \oplus E_{3;x,\eta}$ is given by

$$E_{1,U_0}^\sim \rtimes_{x_0,s_{x_0,\eta}} E_{3,U_0}^\sim := \big\{ (a + s_\eta(b), b) : a \in E_{1,U_0}^\sim,\ b \in E_{3,U_0}^\sim \big\}. \tag{31}$$

Obviously, this A_{U_0}-module is free and hence induces a locally free sheaf which we denote by $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0}$. That is,

$$\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0} := \widetilde{E_{1,U_0}^\sim \rtimes_{x_0,s_{x_0,\eta}} E_{3,U_0}^\sim}. \tag{32}$$

By our construction, obviously,

(i) $\mathcal{E}_1|_{U_0}$ is a locally free $\mathcal{O}_{U_0}$ sub-sheaf of $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0}$ such that

$$\big(\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0} \big) \big/ \big(\mathcal{E}_1|_{U_0} \big) \simeq \mathcal{E}_3|_{U_0}. \tag{33}$$

(ii) Since s_η is regular over $U \smallsetminus \{p\}$,

$$\big(\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0} \big) \big|_{U_0 \smallsetminus \{x_0\}} = \mathcal{E}_1|_{U_0} \oplus \mathcal{E}_3|_{U_0}. \tag{34}$$

In particular, it is possible to glue the locally free sheaves $\mathcal{E}_1|_{U_0} \rtimes_{x_0,s_{x_0,\eta}} \mathcal{E}_3|_{U_0}$ on U_0 and $(\mathcal{E}_1 \oplus \mathcal{E}_3)|_{X \smallsetminus \{x_0\}}$ on $X \smallsetminus \{x_0\}$ over $U_0 \smallsetminus \{x_0\}$. Denote the resulting locally sheaf by $\mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3$. Obviously, there is a natural short exact sequence

$$\mathbb{E}_{s_\eta} : 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3 \longrightarrow \mathcal{E}_3 \to 0. \tag{35}$$

Moreover, it is not too difficult to see that s_η is equivalent to $([\kappa_x])_{x \in X}$ associated to $\mathbb{E}_{s_\eta}$ constructed before Theorem 2.

Now we are ready to treat the general case. *Assume that, as we may, there exist closed points $x_1, \ldots, x_n$ such that*

(1) for $x \notin \{x_1, \ldots, x_n\}$, $s_{x,\eta}$ is regular on X,

(2) for each $i = 1, \ldots, n$, $s_{x_i,\eta}$ is regular for all but one closed point $x_i \in X$

Similarly as in a single closed point case above, for each i, choose an affine open neighborhood U_i of x_i in X such that $U_i \cap U_j = \emptyset$ if $i \neq j$, and on each U_i, we can construct a locally free sheaf $\mathcal{E}_1|_{U_i} \rtimes_{x_i,s_{x_i,\eta}} \mathcal{E}_3|_{U_i}$. Consequently, we may glue these locally free sheaves on the U_i's with the free sheaf $(\mathcal{E}_1 \oplus \mathcal{E}_3)|_{X \smallsetminus \{x_1,\ldots,x_n\}}$ on $X \smallsetminus \{x_1, \ldots, x_n\}$ on the overlaps $U_i \smallsetminus \{x_i\}$. In this way, we obtain a locally free sheaf $\mathcal{E}_1 \rtimes_{s_\eta} \mathcal{E}_3$ of rank $r_1 + r_3$ on X, which is an extension of $\mathcal{E}_3$ by $\mathcal{E}_1$. In other words, we obtain a short exact sequence of locally free sheaves on X

$$\mathbb{E}_{s_\eta} : 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \rtimes_{s\eta} \mathcal{E}_3 \longrightarrow \mathcal{E}_3 \to 0. \tag{36}$$

In addition, by our construction, it is not difficult to see that $(s_{i,\eta})$ is equivalent to the element $([\kappa_x])$ of $\mathbb{E}_{s_\eta}$ constructed before Theorem 2. In this we have constructed the inverse of Φ. In particular, if all the $s_{x,\eta}$'s are regular, then $s_{x,\eta}$ is equivalent to zero and the associated extension is trivial.

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2.3 Adeles and Locally Free Sheaf from Extension

As in (10), let

$$\mathbb{E} : 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \to 0, \tag{37}$$

be a short exact sequence of locally free sheaves on X. For each $i = 1, 2, 3$, let $g_i = (g_{i,x}) \in \mathrm{GL}_{r_i}(F)\backslash\mathrm{GL}_{r_i}(\mathbb{A})/\mathrm{GL}_{r_i}(\mathcal{O})$ be the adelic elements associated to $\mathcal{E}_i$ introduced in §1.

Proposition 3. *Let* $([\kappa_x]) \in \bigoplus_{x \in X} \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, E_{1;x,\eta})/\mathrm{Hom}_{\widehat{\mathcal{O}}_x}(E_{3,x}^{\sim}, E_{2,x}^{\sim})$ *be the extension class of Theorem 2 associated to the extension* $\mathbb{E}$*. Then, the adelic element* $g_2 = (g_{2,x}) \in \mathrm{GL}_{r_i}(F)\backslash\mathrm{GL}_{r_i}(\mathbb{A})/\mathrm{GL}_{r_i}(\mathcal{O})$ *of the locally free sheaf* $\mathcal{E}_2$ *is given by*

$$g_{2,x} = \begin{pmatrix} g_{1,x} & \kappa_x \\ 0 & g_{3,x} \end{pmatrix} \qquad \forall x \in X. \tag{38}$$

Here κ_x *are viewed as elements of the spaces* $M_{r_1 \times r_3}(\widehat{F}_x)$ *of* $r_1 \times r_3$*-matrices with entries in* $\widehat{F}_x$.

Proof. This is a direct consequence of the proof of Theorem 2, particularly, the construction of Φ^{-1} at the end of the previous subsection. Indeed, by (31), we have $g_{2,x}$ is a upper triangular matrices with diagonal blocks $g_{1,x}$ and $g_{3,x}$ and with κ_x as the right-upper block as stated in the proposition. Finally, the reason that $([\kappa_x]) \in M_{r_1 \times r_3}(\mathbb{A})$ comes from the fact that the restrict product $\prod'$ is used in the quotient space $\dfrac{\prod'_{x \in X} \mathrm{Hom}_{\widehat{F}_x}(\widehat{E}_{3;x,\eta}, \widehat{E}_{1;x,\eta})}{\prod'_{x \in X} \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{E}_{3,x}^{\sim}, \widehat{E}_{1,x}^{\sim}) + \mathrm{Hom}_F(E_{3,\eta}, E_{1,\eta})}.$ $\qquad \square$

3 Adelic Interpretations of Atiyah Bundles

3.1 Atiyah Bundles over Elliptic Curves

In the sequel, let X be an elliptic curve over $\mathbb{F}_q$. Then the vector bundle over X are classified by Atiyah in [1] using the so-called indecomposable bundles. After Mumford introduced the slope stability ([3]), it is know that an indecomposable bundle over elliptic curves is semi-stable, and it is stable if and only if the rank and the degree of the bundle is mutually prime. (For a proof, see e.g. Appendix A of Tu [5].)

Theorem 4 (Atiyah[1]). *The semi-stable bundles on an elliptic curve* $X/\mathbb{F}_q$ *are classified as follows.*

(1) *The bundle* I_n *defined inductively as the unique non-trivial extension of* I_{n-1} *by* $\mathcal{O}_X$*. In particular,* $I_r^{\vee} \simeq I_r$.

(2) *Assume that* $(r, d) = 1$*. Then for each line bundle* L *in the Picard group* $\mathrm{Pic}^d(X)$ *on* X*, parametrizing degree* d *line bundles on* X*, there exists a unique stable bundle* $W_r(d; L)$ *of rank* r *and degree* d *such that*

$$\det W_r(\lambda) = \lambda. \tag{39}$$

In particular, for any line bundle $L_1 \in \mathrm{Pic}^{d_1}(X)$,

$$W_r(d; L) \otimes L_1 \simeq W_r(d + rd_1; L \otimes L_1^{\otimes r}). \tag{40}$$

(3) *Every indecomposable semi-stable vector bundle on X is isomorphic to $I_r \otimes W_{r'}(\lambda)$.*

(4) *Every semi-stable bundle on X of slope k/n for mutually prime k and n is a direct sum of bundles $I_{nm_1} \otimes W_n(L)$ for suitable m's and L's in $\mathrm{Pic}^k(X)$. Here, as usual, the slope is defined as the degree divided by the rank (of the bundle).*

By (40), up to tensor by line bundles, it suffices to consider $W_r(d; L)$ such that $0 < d < r$ and $(r, d) = 1$.

In the sequel of the paper, we focus on the Atiyah bundles $I_r \;\; (r \geq 2)$. The case for $W_r(d; L)$ will be discussed elsewhere.

3.2 Adelic Expressions for of Atiyah Bundle I_r

We first give a detailed description for the extension class associated to I_r.

3.2.1 Cases of I_2 and I_3

To start with, we consider I_2, constructed as the unique non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. Indeed, since $H^1(X, \mathcal{O}_X^\vee \otimes \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X)^\vee \simeq \mathbb{F}_q$, there exists one and only one non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$, determined by the image of the identity morphism $\mathrm{Id} : \mathcal{O}_X \to \mathcal{O}_X$ in .

$$\prod\nolimits_{x \in X}' \mathrm{Hom}_{\widehat{F}_x}(\widehat{F}_x, \widehat{F}_x) / (\prod\nolimits_{x \in X}' \mathrm{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{\mathcal{O}}_x, \widehat{\mathcal{O}}_x) + \mathrm{Hom}_F(F, F)). \qquad (41)$$

Since this space is nothing but

$$\mathbb{A}/(\mathcal{O} + F) = H^1(X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X^\vee \otimes \mathcal{O}_X)^\vee \simeq \mathbb{F}_q. \qquad (42)$$

to calculate the extension class $[\kappa_x]$, it suffices to analyze the quotient space $\mathbb{A}/(\mathcal{O} + F)$. For this, we fix an $\mathbb{F}_q$-rational point Q of X.

Lemma 5. *The extension class in $\mathbb{A}/(\mathcal{O} + F)$ for I_2 is given by*

$$\kappa_{I_2, x} = \begin{cases} \pi_Q^{-1} & x = Q \\ 0 & x \neq Q. \end{cases} \qquad (43)$$

Proof. For the rational point Q of X, by the vanishing of $H^1(X, \mathcal{O}_X(Q))$,

$$\mathbb{A} = \mathbb{A}(Q) + F. \qquad (44)$$

Therefore, by the first and the second isomorphism theorems

$$\begin{aligned}
\mathbb{A}/(\mathcal{O} + F) &= (\mathbb{A}(Q) + F)/(\mathbb{A}(0) + F) \simeq \mathbb{A}(Q)/(A(0) + \mathbb{A}(Q) \cap F) \\
&\simeq \frac{\mathbb{A}(Q)/A(0)}{(A(0) + \mathbb{A}(Q) \cap F)/\mathbb{A}(0)} \simeq \frac{\mathbb{A}(Q)/A(0)}{(A(0) + \mathbb{A}(Q) \cap F)/\mathbb{A}(0)} \\
&\simeq \frac{\mathbb{A}(Q)/A(0)}{(\mathbb{A}(Q) \cap F)/(\mathbb{A}(0) \cap F)}.
\end{aligned} \qquad (45)$$

Here, in the last step above, we have used the fact that

$$H^0(X, \mathcal{O}_X) = \mathbb{A}(0) \cap F = \mathbb{A}(Q) \cap F = H^0(X, \mathcal{O}_X(Q)).$$

Denote by π_Q the local parameter of the local field $(\widehat{F}_Q, \widehat{\mathcal{O}}_Q)$ associated to Q. We have

$$\mathbb{A}(Q)/\mathbb{A}(0) \simeq (\pi_Q^{-1}\widehat{\mathcal{O}}_Q)/\widehat{\mathcal{O}}_Q, \tag{46}$$

since

$$\mathbb{A}(0) = \prod_{x \in X} \widehat{\mathcal{O}}_x \qquad \text{and} \qquad \mathbb{A}(Q) = \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x \times \{a \in \widehat{F}_Q : \pi_Q a \in \widehat{\mathcal{O}}_Q\}$$

Therefore,

$$\mathbb{A}/(\mathcal{O} + F) \simeq (\pi_Q^{-1}\widehat{\mathcal{O}}_Q)/(\pi_Q^{-1+1}\widehat{\mathcal{O}}_Q) \simeq (\pi_Q^{-1}\widehat{\mathcal{O}}_Q)/\widehat{Q}_Q \simeq \pi_Q^{-1}\mathbb{F}_q. \tag{47}$$

This verifies the assertion in the lemma. $\qquad\square$

Corollary 6. *An adelic representor $g_{I_2} = (g_{I_2,x})$ for the Atiyah bundle I_2 in $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O})$ may be chosen as*

$$g_{I_2,x} = \begin{cases} \begin{pmatrix} 1 & \pi_Q^{-1} \\ 0 & 1 \end{pmatrix} & x = Q \\[2mm] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & x \neq Q. \end{cases} \tag{48}$$

Proof. This is a direct consequence of Proposition 3 and Lemma 5. $\qquad\square$

We end this discussion on I_2 by calculating $H^1(X, I_2)$. Induced from the non-split exact sequence $0 \to \mathcal{O}_X \to I_2 \to \mathcal{O}_X \to 0$ is a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to &H^0(X, \mathcal{O}_X) \to H^0(X, I_2) \to H^0(X, \mathcal{O}_X) \\ &\to H^1(X, \mathcal{O}_X) \to H^1(X, I_2) \to H^1(X, \mathcal{O}_X) \to 0 \end{aligned} \tag{49}$$

Since $H^0(X, \mathcal{O}_X) \simeq H^0(X, I_2) \simeq \mathbb{F}_q$ and $H^0(X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X)$, we have

$$H^1(X, I_2) \simeq H^1(X, \mathcal{O}_X) \simeq \mathbb{F}_q.$$

Next, we consider the Atiyah bundle I_3. This is constructed as a non-trivial extension

$$0 \to I_2 \longrightarrow I_3 \longrightarrow \mathcal{O}_X \to 0. \tag{50}$$

Since $H^1(X, I_2) \simeq \mathbb{F}_q$, I_3 is the unique non-trivial extension of $\mathcal{O}_X$ by I_2.

To determine the associated extension class κ_{I_3}, we first write $H^1(F, g_{I_2})$ as the quotient space $\mathbb{A}^2/(\mathbb{A}^2(g_{I_2}) + F^2)$. Since $I_2(Q) := I_2 \otimes \mathcal{O}_X(Q)$ is semi-stable of positive degree, $H^1(F, g_{I_2(Q)}) = \{0\}$ by the vanishing theorem for semi-stable bundles. This implies that

$$\mathbb{A}^2 = \mathbb{A}^2(g_{I_2(Q)}) + F^2 \tag{51}$$

Thus, similarly to (45), by the first and the second isomorphism theorem,

$$\begin{aligned} H^1(F, g_{I_2}) =& (\mathbb{A}^2(g_{I_2(Q)}) + F^2)/(\mathbb{A}^2(g_{I_2}) + F^2) \\ \simeq& \frac{\mathbb{A}^2(g_{I_2(Q)})/\mathbb{A}^2(g_{I_2})}{(\mathbb{A}^2(g_{I_2(Q)}) \cap F^2)/(\mathbb{A}^2(g_{I_2}) \cap F^2)} \end{aligned} \tag{52}$$

Lemma 7. *We have*

$$\mathbb{A}^2(g_{I_2}) = \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \{(-\pi_Q^{-1}a_1 + a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \widehat{\mathcal{O}}_Q\}$$

$$\mathbb{A}^2(g_{I_2(Q)}) = \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \{(-\pi_Q^{-1}a_1 + a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \pi_Q^{-1}\widehat{\mathcal{O}}_Q\}$$

$$\mathbb{A}^2(g_{I_2}) \cap F^2 = \{(a_2, 0) \in F^2 : a_2 \in \mathbb{F}_q\} \simeq \mathbb{F}_q \times \{0\}$$

$$\mathbb{A}^2(g_{I_2(Q)}) \cap F^2 = \left(\prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \left\{ \begin{array}{c} \pi_Q^{-1}(-\pi_Q^{-1}a_1 + a_2, a_1) : \\ -\pi_Q^{-1}a_1 + a_2 = \pi_Q a_2' \\ a_1 = \pi_Q a_1', \\ a_1', a_2' \in \widehat{\mathcal{O}}_X \end{array} \right\} \right) \bigcap F^2 \simeq \mathbb{F}_q^2$$

(53)

Proof. Our description of the space $\mathbb{A}^2(g_{I_2(Q)})$ is a direct consequence for the space $\mathbb{A}^2(g_{I_2})$, so it suffices to calculate the later. In addition, by Corollary 6 g_{I_2} is determined as

$$g_{I_2,x} = \begin{cases} \begin{pmatrix} 1 & \pi_Q^{-1} \\ 0 & 1 \end{pmatrix}, x = Q \\ \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \neq Q \end{cases}$$

we have

$$\mathbb{A}^2(g_{I_2}) = \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \left\{ (a_2, a_1) \in \widehat{F}_Q^2 : \begin{pmatrix} 1 & \pi_Q^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \in \widehat{\mathcal{O}}_Q^2 \right\}$$

$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \{(-\pi_Q^{-1}a_1 + a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \widehat{\mathcal{O}}_Q\}$$

(54)

since $a_2 + \pi_Q^{-1}a_1 \in \widehat{\mathcal{O}}_Q$ implies that $a_2 = -\pi_Q^{-1}a_1 + a_2'$, where $a_2' := a_2 + \pi_Q^{-1}a_1 \in \widehat{\mathcal{O}}_X$. This proves the first two relations.

To prove the third, let $(f_2, f_1) \in \mathbb{A}^2(g_{I_2}) \cap F^2$. Then

$$\begin{cases} \mathrm{ord}_x(f_1) \geq 0 & \forall x \in X \\ \mathrm{ord}_x(f_2) \geq 0 & \forall x \in X \smallsetminus \{Q\} \\ \mathrm{ord}_x(f_2 + \pi_Q^{-1}f_1) \geq 0 & x = Q \end{cases}$$

(55)

The first implies that $f_1 \in \mathbb{F}_q$, a constant function on X, since rational functions on X admitting no poles are constant functions on X with values in $\mathbb{F}_q$. In addition, if $f_1 \neq 0$, then by the third relation $\mathrm{ord}_Q(f_2) = -1$. This contrdicts with the second relation, since there is no rational function on X which admits a simple pole in the elliptic curve X and admits no other poles. That is to say, $f_1 = 0$. Using this. for f_2, by similar reason as above, we have $f_2 \in \mathbb{F}_q$. This verifies the third assertion in the lemma.

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Finally, we treat th space $\mathbb{A}^2(g_{I_2(Q)}) \cap F^2$. By the first (and the second) assertion(s), we have

$$
\begin{aligned}
&\mathbb{A}^2(g_{I_2(Q)}) \cap F^2 \\
&= \Big(\prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \big\{ (-\pi_Q^{-1}a_1 + a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \pi_Q^{-1}\widehat{\mathcal{O}}_Q \big\} \Big) \cap F^2 \\
&= \left\{ (f_2, f_1) \in F^2 : \begin{array}{l} \operatorname{ord}_x(f_i) \geq 0 \ \ \forall x \neq Q, \\ \operatorname{ord}_Q(f_1) = \operatorname{ord}_Q(\pi_Q^{-1}a_1) \geq -1, \ \ \exists a_1, a_2 \in \widehat{\mathcal{O}}_Q \\ \operatorname{ord}_Q(f_2) = \operatorname{ord}_Q(\pi_Q^{-1}(-\pi_Q^{-1}a_1 + a_2)) \geq -2, \end{array} \right\}
\end{aligned} \tag{56}
$$

Using the same argument as above, we have $f_1 \in \mathbb{F}_q$. This implies that

$$
a_1 \in \pi_Q \widehat{\mathcal{O}}_Q \qquad \text{and} \qquad \operatorname{ord}_Q(f_2) \geq -1. \tag{57}
$$

Thus, $f_2 \in \mathbb{F}_q$, by the same argument as above. This proves the second equation of the forth assertion in the lemma. In addition, $\pi_Q^{-1}(-\pi_Q^{-1}a_1 + a_2) \in \widehat{\mathcal{O}}_Q$. Hence, if we set $a_1 = \pi_Q a_1'$ with $a_1' \in \widehat{\mathcal{O}}_Q$, then

$$
\pi_Q^{-1}\widehat{\mathcal{O}}_Q \ni \pi_Q^{-1}(-\pi_Q^{-1}a_1 + a_2) = \pi_Q^{-1}(-a_1' + a_1). \tag{58}
$$

This proves the first equation of the forth assertion in the lemma. $\qquad \square$

With this lemma, we may continue (52) to obtain

$$
\begin{aligned}
H^1(F, g_{I_2}) &= \frac{\dfrac{\big\{ (-\pi_Q^{-1}a_1+a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \pi_Q^{-1}\widehat{\mathcal{O}}_Q \big\}}{\big\{ (-\pi_Q^{-1}a_1+a_2, a_1) \in \widehat{F}_Q^2 : a_1, a_2 \in \widehat{\mathcal{O}}_Q \big\}}}{\dfrac{\left(\prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \left\{ \begin{array}{l} \pi_Q^{-1}(-\pi_Q^{-1}a_1 + a_2, a_1) : \\ -\pi_Q^{-1}a_1 + a_2 = \pi_Q a_2' \\ a_1 = \pi_Q a_1', \\ a_1', a_2' \in \widehat{\mathcal{O}}_X \end{array} \right\} \right) \cap F^2}{\big\{ (a_2, 0) \in F^2 : a_2 \in \mathbb{F}_q \big\}}} \\[2em]
&\simeq \frac{(-\pi_Q^{-1}, 1)\pi_Q^{-1}\mathbb{F}_q + (1,0)\pi_Q^{-1}\mathbb{F}_q}{\left(\prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^2 \times \left\{ \begin{array}{l} \pi_Q^{-1}(-\pi_Q^{-1}a_1, a_1) : \\ -\pi_Q^{-1}a_1 \in \pi_Q \widehat{\mathcal{O}}_X \\ a_1 = \pi_Q a_1', a_1' \in \widehat{\mathcal{O}}_X \end{array} \right\} \right) \cap F^2} \\[2em]
&\simeq \frac{(-\pi_Q^{-1}, 1)\pi_Q^{-1}\mathbb{F}_q + (1,0)\pi_Q^{-1}\mathbb{F}_q}{(-\pi_Q^{-1}, 1)\pi_Q^{-1}\mathbb{F}_q} \simeq (\pi_Q^{-1}, 0)\mathbb{F}_q.
\end{aligned} \tag{59}
$$

This then proves the first part of the following

Proposition 8. *For the Atiyah bundle I_3, we have*

(1) The extension class of $\mathcal{O}_X$ by I_2 associated to I_3 is given by

$$
\kappa_{I_3, x} = \begin{cases} (\pi_Q^{-1}, 0) & x = Q \\ (0, 0) & x \neq Q. \end{cases} \tag{60}
$$

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(2) *An adelic representative* $g_{I_3} = (g_{I_3,x})$ *for the Atiyah bundle* I_3 *in the quotient space* $\mathrm{GL}_3(F)\backslash\mathrm{GL}_3(\mathbb{A})/\mathrm{GL}_3(\mathcal{O})$ *may be chosen as*

$$
g_{I_3,x} = \begin{cases} \begin{pmatrix} 1 & \pi_Q^{-1} & 0 \\ 0 & 1 & \pi_Q^{-1} \\ 0 & 0 & 1 \end{pmatrix} & x = Q \\[1em] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & x \neq Q. \end{cases} \tag{61}
$$

Proof. It suffices to prove the second part. However, this is a direct consequence of the first part and Proposition 3. □

3.2.2 General Cases of I_r

With the cases for I_2 and I_3 treated, we are now ready to treat I_r for general r, for which we have the following one of our main result of this paper.

Theorem 9. *For the Atiyah bundle* I_r*, we have*

(1) *The extension class of* $\mathcal{O}_X$ *by* I_{r-1} *associated to* I_r *is given by*

$$
\kappa_{I_r,x} = \begin{cases} (\pi_Q^{-1}, \overbrace{0,\dots,0}^{r-2}) & x = Q \\[1em] (\overbrace{0,0,\dots,0}^{r-1}) & x \neq Q. \end{cases} \tag{62}
$$

(2) *An adelic representative* $g_{I_r} = (g_{I_r,x})$ *for the Atiyah bundle* I_r *in the quotient space* $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A})/\mathrm{GL}_r(\mathcal{O})$ *may be chosen as*

$$
g_{I_r,x} = \begin{cases} \begin{pmatrix} 1 & \pi_Q^{-1} & 0 & \dots & 0 & 0 \\ 0 & 1 & \pi_Q^{-1} & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \pi_Q^{-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} & x = Q \\[2em] \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} & x \neq Q. \end{cases} \tag{63}
$$

Proof. We prove this theorem by an induction on r. The cases for $r = 2, 3$ are verified in Lemma 5, Corrary 6 and Proposition 8.

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Assume now that the assertions in the theorem are verified for all I_k when $k \leq r - 1$. Since $I_r^\vee \simeq I_r$, we may equally use the exact sequence

$$0 \to I_{r-1} \longrightarrow I_r \longrightarrow \mathcal{O}_X \to 0 \tag{64}$$

This implies that

$$H^1(X, \mathcal{O}_X^\vee \otimes I_{r-1}) \simeq H^1(X, I_{r-1}) \simeq H^0(X, I_{r-1}^\vee)^\vee \simeq H^0(X, I_{r-1})^\vee = \mathbb{F}_q, \tag{65}$$

since $H^0(X, I_{r-1}) \simeq \mathbb{F}_q$ by our induction assumption. Thus it is not surprising that there is one and only one non-trivial extension of $\mathcal{O}_X$ by I_{r-1}. To write down explicitly the extension class $(\kappa_{I_r,x})$ associated to the extension (64), similarly to the case of I_2 and I_3, we first write $H^1(X, I_{r-1})$ as

$$H^1(X, I_{r-1}) \simeq (\mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) + F^{r-1})/(\mathbb{A}^{r-1}(g_{I_{r-1}}) + F^2)$$
$$\simeq \frac{\mathbb{A}^{r-1}(g_{I_{r-1}(Q)})/\mathbb{A}^{r-1}(g_{I_{r-1}})}{(\mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) \cap F^{r-1})/(\mathbb{A}^{r-1}(g_{I_{r-1}}) \cap F^{r-1})} \tag{66}$$

Lemma 10. *We have*

$$\mathbb{A}^{r-1}(g_{I_{r-1}})$$
$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \left(\sum_{i=1}^{r-1} (-\pi_Q)^{i-(r-1)} a_i, \dots, -\pi_Q^{-1} a_1 + a_2, a_1 \right) \in \widehat{F}_Q^2 : \atop a_1, a_2, \dots, a_{r-1} \in \widehat{\mathcal{O}}_Q \right\}$$

$$\mathbb{A}^{r-1}(g_{I_{r-1}(Q)})$$
$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \left(\sum_{i=1}^{r-1} (-\pi_Q)^{i-(r-1)} a_i, \dots, -\pi_Q^{-1} a_1 + a_2, a_1 \right) \in \widehat{F}_Q^2 : \atop a_1, a_2, \dots, a_{r-1} \in \pi_Q^{-1} \widehat{\mathcal{O}}_Q \right\}$$

$$\mathbb{A}^{r-1}(g_{I_{r-1}}) \cap F^{r-1} = \{(a_{r-1}, \overbrace{0, \dots, 0}^{r-2}) \in F^{r-1} : a_{r-1} \in \mathbb{F}_q\} \simeq \mathbb{F}_q \times \{0\}^{r-2}$$

$$\mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) \cap F^{r-1}$$
$$= \left\{ \begin{array}{c} ((-\pi_Q)^{-r+2}, \dots, -\pi_Q^{-1}, 1)\pi_Q^{-1} a_1 \\ +((-\pi_Q)^{-r+3}, \dots, -\pi_Q^{-1}, 1, 0)\pi_Q^{-1} a_2 \\ + \dots + (\pi_Q^{-1}, 1, 0 \dots, 0)\pi_Q^{-1} a_{r-2} \\ +(1, 0 \dots, 0)\pi_Q^{-1} a_{r-1} : \\ a_1, a_2, \dots, a_{r-2}, a_{r-1} \in \mathbb{F}_q \end{array} \right\} \bigcap F^{r-1} \simeq \mathbb{F}_q^{r-1}$$

$$\tag{67}$$

Proof. Our description of the second space $\mathbb{A}^{r-1}(g_{I_{r-1}(Q)})$ is a direct consequence of the structure for the first space $\mathbb{A}^{r-1}(g_{I_{r-1}})$, so it suffices to calculate

the later. In addition, by our inductive assumption on $g_{I_{r-1}}$, we have

$$\mathbb{A}^2(g_{I_2})$$

$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \begin{array}{c} (a_{r-1}, \ldots, a_2, a_1) \in \widehat{F}_Q^{r-1} : \\ \begin{pmatrix} 1 & \pi_Q^{-1} & 0 & \ldots & 0 & 0 \\ 0 & 1 & \pi_Q^{-1} & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \pi_Q^{-1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{r-1} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} \in \widehat{\mathcal{O}}_Q^{r-1} \end{array} \right\}$$

$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \begin{array}{c} (a_{r-1}, \ldots, a_2, a_1) \in \widehat{F}_Q^{r-1} : \\ a_{r-1} + \pi_Q^{-1} a_{r-2}, \ldots, a_2 + \pi_Q^{-1} a_1, a_1 \in \widehat{\mathcal{O}}_Q \end{array} \right\}$$

$$= \prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \begin{array}{c} \left(\sum_{i=1}^{r-1} (-\pi_Q)^{i-(r-1)} a_i, \ldots, -\pi_Q^{-1} a_1 + a_2, a_1 \right) \in \widehat{F}_Q^2 : \\ a_1, a_2, \ldots, a_{r-1} \in \widehat{\mathcal{O}}_Q \end{array} \right\}$$

$$(68)$$

Indeed, if we set $a_2' = a_2 + \pi_Q^{-1} a_1 \in \widehat{\mathcal{O}}_Q$, then $a_2 = a_2' - \pi_Q^{-1} a_1$. Similarly, for $3 \le j \le r-1$, if we set $a_{j-1}' = a_{j-1} + \pi_Q^{-1} a_{j-2} \in \widehat{\mathcal{O}}_Q$, then

$$a_{j-1} = a_{j-1}' - \pi_Q^{-1} a_{j_2} = a_{j-1}' - \pi_Q^{-1} \sum_{i=1}^{j-2} (-\pi_Q)^{i-(j-2)} a_i'$$

$$= \sum_{i=1}^{j-1} (-\pi_Q)^{i-(j-1)} a_i' \tag{69}$$

This completes our proof for the first relation and hence also for the first two relations.

To prove the third, let $(f_{r-1}, \ldots, f_2, f_1) \in \mathbb{A}^2(g_{I_2}) \cap F^{r-1}$. Then

$$\begin{cases} \operatorname{ord}_x(f_1) \ge 0 & \forall x \in X \\ \begin{cases} \operatorname{ord}_x(f_2) \ge 0 & \forall x \in X \smallsetminus \{Q\} \\ \operatorname{ord}_x(f_2 + \pi_Q^{-1} f_1) \ge 0 & x = Q \end{cases} \\ \ldots \ldots \\ \begin{cases} \operatorname{ord}_x(f_{r-1}) \ge 0 & \forall x \in X \smallsetminus \{Q\} \\ \operatorname{ord}_x(f_{r-1} + \pi_Q^{-1} f_{r-2}) \ge 0 & x = Q \end{cases} \end{cases} \tag{70}$$

The first implies that $f_1 \in \mathbb{F}_q$, a constant function on X, since rational functions on X admitting no poles are constant functions on X with values in $\mathbb{F}_q$. In addition, if $f_1 \neq 0$, then by the first in the second group relations, $\operatorname{ord}_Q(f_2) = -1$. This contradicts with the second relation of this group, since there is no rational function on X which admits a simple pole in the elliptic curve X and admits no other poles. That is to say, $f_1 = 0$. Using this, with an induction on $k+1$. Then if $f_k \in \mathbb{F}_q$ but not zero, we have f_k admits a single simple pole on X. This is impossible with the same argument as above. Hence $f_k = 0$ and

$f_{k+1} \in \mathbb{F}_q$. Therefore, $f_1 = f_2 = \ldots = f_{r-2} = 0$ and $f_{r-1} \in \mathbb{F}_q$. This verifies the third assertion in the lemma.

Finally, we treat th space $\mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) \cap F^{r-1}$. By the first (and the second) assertion(s), we have

$$\mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) \cap F^{r-1}$$

$$= \left(\prod_{x \in X \smallsetminus \{Q\}} \widehat{\mathcal{O}}_x^{r-1} \times \left\{ \begin{array}{c} \left(\sum_{i=1}^{r-1} (-\pi_Q)^{i-(r-1)} a_i, \ldots, -\pi_Q^{-1} a_1 + a_2, a_1 \right) \in \widehat{F}_Q^2 \\ : a_1, a_2, \ldots, a_{r-1} \in \pi_Q^{-1} \widehat{\mathcal{O}}_Q \end{array} \right\} \right) \bigcap F^{r-1}$$

$$= \left\{ \begin{array}{c} (f_{r-1}, \ldots, f_2, f_1) \in F^{r-1} : \\ \mathrm{ord}_x(f_i) \geq 0 \;\; \forall x \neq Q, \quad \forall i = 1, \ldots, r-1 \\ \exists a_1, a_2, \ldots, a_{r-1} \in \widehat{\mathcal{O}}_Q \\ \mathrm{ord}_Q(f_1) = \mathrm{ord}_Q(\pi_Q^{-1} a_1) \geq -1, \\ \mathrm{ord}_Q(f_2) = \mathrm{ord}_Q(\pi_Q^{-1}(-\pi_Q^{-1} a_1 + a_2) \geq -2, \\ \ldots\ldots \\ \mathrm{ord}_Q(f_{r-1}) = \mathrm{ord}_Q(\pi_Q^{-1}(-\pi_Q^{-1} a_{r-2} + a_{r-1}) \geq -(r-1) \end{array} \right\}$$

(71)

Using the same argument as above, we have $\mathrm{ord}_Q(f_1) = 0$ and hence $f_1 \in \mathbb{F}_q$. This implies that

$$a_1 \in \pi_Q \widehat{\mathcal{O}}_Q \qquad \text{and} \qquad \mathrm{ord}_Q(f_2) \geq -1. \tag{72}$$

Similarly, using f_1 with the same reason, we get $f_2 \in \mathbb{F}_q$. With an induction, the same reason implies that all $f_i \in \mathbb{F}_q (1 \leq i \leq r-1)$. This proves the second equation of the forth assertion in the lemma. To prove the first equation, instead of using the same argument as above, we decompose the Q-factor subspace

$$\left\{ \begin{array}{c} \left(\sum_{i=1}^{r-1} (-\pi_Q)^{i-(r-1)} a_i, \ldots, -\pi_Q^{-1} a_1 + a_2, a_1 \right) \in \widehat{F}_Q^2 : \\ a_1, a_2, \ldots, a_{r-1} \in \pi_Q^{-1} \widehat{\mathcal{O}}_Q \end{array} \right\} \text{ of } \mathbb{A}^{r-1}(g_{I_{r-1}(Q)}) \text{ as}$$

$$\left\{ \begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)\pi_Q^{-1} a_1 + ((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)\pi_Q^{-1} a_2 \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)\pi_Q^{-1} a_{r-2} + (1, 0 \ldots, 0)\pi_Q^{-1} a_{r-1} : \\ a_1, a_2, \ldots, a_{r-2}, a_{r-1} \in \widehat{\mathcal{O}}_Q \end{array} \right\}$$

(73)

Essentially with the same argument, by looking the last component $\pi_Q^{-1} a_1$ of $(-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)\pi_Q^{-1} a_1$, to have an element $(f_{r-1}, \ldots, f_2, f_1)$ of F^{r-1} in the intersection, by working with poles and zeros of rational functions over elliptic curves, we conclude that $a_1 \in \pi_Q \widehat{\mathcal{O}}_X$. Based on this, if we set $a_1 = \pi_Q a_1'$, then, by looking at the part

$$((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 0)a_1' + ((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)\pi_Q^{-1} a_2$$

for f_2, we conclude that $\pi_Q^{-1} a_2 + a_1' \in \widehat{\mathcal{O}}_Q$. This implies that $a_2 = \pi_Q a_2' \in \pi_Q \widehat{\mathcal{O}}_Q$. Thus inductively, for $a_k = \pi_Q a_k' \in \pi_Q \widehat{\mathcal{O}}_Q$, we have for the element

$$((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-k}, 0 \ldots, 0)a_k' + ((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-k+1}, 0 \ldots, 0)\pi_Q^{-1} a_{k+1}$$

associated to f_{k+1}, we conclude that

$$a_{k+1} = \pi_Q a'_{k+1} \in \pi_Q \widehat{\mathcal{O}}_Q \qquad \forall 1 \le i \le r-1. \tag{74}$$

This proves the first equation of the forth assertion in the lemma. $\qquad\square$

With this lemma, we now continue the calculation in (66) to obtain

$$H^1(X, I_{r-1})$$

$$\simeq \frac{\left\{\begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)a_1 \\ +((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)a_2 \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)a_{r-2} \\ +(1, 0 \ldots, 0)a_{r-1} : \\ a_1, a_2, \ldots, a_{r-1} \in \pi_Q^{-1}\widehat{\mathcal{O}}_Q \end{array}\right\} \Bigg/ \left\{\begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)a_1 \\ +((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)a_2 \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)a_{r-2} \\ +(1, 0 \ldots, 0)a_{r-1} : \\ a_1, a_2, \ldots, a_{r-1} \in \widehat{\mathcal{O}}_Q \end{array}\right\}}{\left\{\begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)a_1 \\ +((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)a_2 \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)a_{r-2} \\ +(1, 0 \ldots, 0)a_{r-1} : \\ a_1, a_2, \ldots, a_{r-1} \in \pi_Q^{-1}\mathbb{F}_q \end{array}\right\} \Bigg/ (\mathbb{F}_q \times \{0\}^{r-2})}$$

$$\simeq \frac{\left(\begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)\pi_Q^{-1}\mathbb{F}_q \\ +((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)\pi_Q^{-1}\mathbb{F}_q \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)\pi_Q^{-1}\mathbb{F}_q \\ +(1, 0 \ldots, 0)\pi_Q^{-1}\mathbb{F}_q \end{array}\right)}{\left(\begin{array}{c} ((-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1)\pi_Q^{-1}\mathbb{F}_q \\ +((-\pi_Q)^{-r+3}, \ldots, -\pi_Q^{-1}, 1, 0)\pi_Q^{-1}\mathbb{F}_q \\ + \ldots + (\pi_Q^{-1}, 1, 0 \ldots, 0)\pi_Q^{-1}\mathbb{F}_q \end{array}\right)} \simeq (\pi_Q^{-1}\mathbb{F}_q, \overbrace{0 \ldots, 0}^{r-2})$$

$$\tag{75}$$

Therefore, $\kappa_{I_{r-1}}$ is given as in the theorem.

This completes the proof of the Theorem. $\qquad\square$

4 Elements in $H^0(F, g_{I_r}(mQ))$

4.1 Loca Conditions

To start with, we here give a very important property for global sections in $H^0(F, g_{I_r}(mQ))$.

Since $g_{I_r}(mQ)$ is semi-stable of degree mr, by the Riemann-Roch theorem and the vanishing theorem (for semi-stable bundles, we have

$$\dim_{\mathbb{F}_q} H^0(F, g_{I_r}(mQ)) = mr. \tag{76}$$

Let $\mathbf{f} = (f_r, \ldots, f_1) \in H^0(F, g_{I_r}(mQ))$. Then, by definition, we have the following characterizing condition

$$(g_{I_r}(mQ))\mathbf{f}^t \in \mathcal{O}^r. \tag{77}$$

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Hence by Theorem 9, $(f_r, \ldots, f_1)$ satisfies the conditions

$$
\begin{cases}
\begin{pmatrix}
1 & \pi_Q^{-1} & 0 & \ldots & 0 & 0 \\
0 & 1 & \pi_Q^{-1} & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & \pi_Q^{-1} \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
f_r \\ f_{r-1} \\ \vdots \\ f_1
\end{pmatrix}
\in \pi_Q^{-m}\widehat{\mathcal{O}}_Q^r & x = Q \\[4em]
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\left(\begin{pmatrix}
f_r \\ f_{r-1} \\ \vdots \\ f_1
\end{pmatrix}\right)
\in \widehat{O}_x^r & x \neq Q
\end{cases}
\tag{78}
$$

That is to say, for $x \neq Q$, $f_j \in \widehat{\mathcal{O}}_x \ \forall \ 1 \leq j \leq r$. And, at the point Q, we have

$$
\begin{cases}
f_r & = -\pi_Q^{-1}f_{r-1} + \pi_Q^{-m}f_r' \quad \in -\pi_Q^{-1}f_{r-1} + \pi_Q^{-m}\widehat{\mathcal{O}}_Q \\
f_{r-1} & = -\pi_Q^{-1}f_{r-2} + \pi_Q^{-m}f_{r-1}' \in -\pi_Q^{-1}f_{r-2} + \pi_Q^{-m}\widehat{\mathcal{O}}_Q \\
\ldots & \ldots\ldots \\
f_3 & = -\pi_Q^{-1}(-\pi_Q^{-1}\pi_Q^{-m}f_1' \ + \pi_Q^{-m}f_2') \ + \pi_Q^{-m}f_3' \\
& = \ \pi_Q^{-m}((-\pi_Q)^{-2}f_1' + (-\pi_Q^{-1})f_2') \\
f_2 & = -\pi_Q^{-1}f_1 \ + \pi_Q^{-m}f_2' \quad \in -\pi_Q^{-1}f_1 \ + \pi_Q^{-m}\widehat{\mathcal{O}}_Q \\
f_1 & = \qquad\qquad \pi_Q^{-m}f_1' \quad \in \qquad\qquad \pi_Q^{-m}\widehat{\mathcal{O}}_Q
\end{cases}
\tag{79}
$$

for some $f_r', \ldots, f_2', f_1' \in \widehat{\mathcal{O}}_Q$. We obtain

$$
\begin{cases}
f_r & = \pi_Q^{-m}\sum_{i=1}^{r-1}(-\pi_Q)^{-r+i}f_i' \\
f_{r-1} & = \pi_Q^{-m}\sum_{i=1}^{r-2}(-\pi_Q)^{-r-1+i}f_i' \\
\ldots & \ldots\ldots \\
f_2 & = \pi_Q^{-m}(-\pi_Q^{-1}f_1' + f_2') \\
f_1 & = \pi_Q^{-m} \qquad f_1'
\end{cases}
\tag{80}
$$

Consequently,

$$
\begin{aligned}
& (f_r, f_{r-1}\ldots, f_2, f_1) \\
= & \pi_Q^{-m}\left(\sum_{i=1}^{r-1}(-\pi_Q)^{-r+i}f_i', \ \sum_{i=1}^{r-2}(-\pi_Q)^{-r-1+i}f_i', \ldots, -\pi_Q^{-1}f_1' + f_2', f_1'\right) \\
= & \pi_Q^{-m}\begin{pmatrix}
\left((-\pi_Q)^{-r+1}, (-\pi_Q)^{-r+2}, \ldots, -\pi_Q^{-1}, 1\right)f_1' \\
+ \left(-\pi_Q\right)^{-r+2}, (-\pi_Q)^{-r+3}, \ldots, 1, 0\right)f_2' + \ldots \\
+ (-\pi_Q, 1, \ldots, 0, 0)f_{r-1}' + (1, 0, \ldots, 0)f_r'
\end{pmatrix}
\end{aligned}
\tag{81}
$$

for some $f_r', f_{r-1}', \ldots, f_2', f_1' \in \widehat{\mathcal{O}}_Q$.

4.2 Cases I_2 and I_3

First we consider case for I_2.

When $m = -1$, we have, at Q,

$$(f_2, f_1) = \pi_Q^{-1}\left((-\pi_Q^{-1}, 1)f_1' + (1, 0)f_2'\right) \qquad \exists f_1', f_2' \in \widehat{\mathcal{O}}_Q \tag{82}$$

This implies that $f_1' = \pi_Q f_1'' \in \pi_Q\widehat{\mathcal{O}}_Q$ since otherwise f_1 is regular at all $x \neq Q$ and admits a simple pole at Q. This is impossible, since X is an elliptic curve. Therefore,

$$(f_2, f_1) = \left((-\pi_Q^{-1}, 1)f_1'' + (\pi_Q^{-1}, 0)f_2'\right) = (\pi_Q^{-1}(f_2' - f_1''), f_1'') = (f_2'', f_1'') \tag{83}$$

for some $f_1'', f_2'' \in \widehat{\mathcal{O}}_Q$, since similarly, $f_2' - f_1'' \in \pi_Q\mathcal{O}_Q$. All these then complete a proof of the following

Lemma 11. If $(f_1, f_2) \in H^0(F, g_{I_2(Q)})$, we have

$$f_1, f_2 \in \mathbb{F}_q \qquad \text{and} \qquad (f_2, f_1) = (f_2'', f_1'') \quad \text{at } Q \tag{84}$$

for some $f_1'', f_2'' \in \mathbb{F}_q \subset \widehat{\mathcal{O}}_Q$ defined above.

Next, we treat the case $m = 2$ for $I_2(2Q)$. This time, we go back to the condition

$$f_1 \in \pi_Q^{-2}\widehat{\mathcal{O}}_Q \qquad \text{and} \qquad f_2 + \pi_Q^{-1}f_1 \in \pi_Q^{-2}\widehat{\mathcal{O}}_Q \tag{85}$$

From the first, we see that $\mathrm{ord}_Q(f_1) = 0, -2$. Moreover,

(a) If $\mathrm{ord}_Q(f_1) = 0$, $\mathrm{ord}_Q(f_2) = 0, -2$. This gives a three (=one+two) dimensional subspace in $H^0(F, g_{I_2(2Q)})$

(b) If $\mathrm{ord}_Q(f_1) = -2$, $\mathrm{ord}_Q(f_2) = -3$. This gives an one dimensional subspace in $H^0(F, g_{I_2(2Q)})$.

Now we are ready to study $H^0(F, g_{I_2(mQ)})$ for general m by an induction on m. Assume that when $m = k$, there is a basis of $H^0(F, g_{I_2(kQ)})$ constructed from the table

$\mathrm{ord}_Q(f_1)\backslash\mathrm{ord}_Q(f_2)$	0	-2	...	-k+1	-k	-k-1
0	O	O	...	O	O	X
-2	O	O	...	O	O	X
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
-k+1	O	O	...	O	O	X
-k	X	X	...	X	X	O

Table 1: Pole orders occurring for $(f_2, f_1) \in H^0(F, g_{I_2(kQ)})$

induced from the relations

$$f_1 \in \pi_Q^{-k}\widehat{\mathcal{O}}_Q \qquad \text{and} \qquad f_2 + \pi_Q^{-1}f_1 \in \pi_Q^{-k}\widehat{\mathcal{O}}_Q. \tag{86}$$

Here in the table above, O (resp. X), means that the values for $(\mathrm{ord}_Q(f_1), \mathrm{ord}_Q(f_2))$ occurs (resp. does not occur). In particular, the total dimension $H^0(F, g_{I_2(kQ)})$

is given by $(k-1) + k + 1 = 2k$. Then for $(f_2, f_1) \in H^0(F, g_{I_2((k+1)Q)})$, with the same discussion, we see that

$$\begin{aligned}
\text{ord}_Q(f_1) &= 0, -2, -3, \ldots, -(k+1) \\
\text{ord}_Q(f_2) &= 0, -2, -3, \ldots, -(k+1), -(k+1) - 1.
\end{aligned} \tag{87}$$

Moreover, from Table 1, not only we should enlarge the table, but to recheck the cases involved. It is not difficult to deduces the following

$\text{ord}_Q(f_1)\backslash\text{ord}_Q(f_2)$	0	-2	$\ldots$	-k+1	-k	-(k+1)	-(k+1)-1
0	O	O	$\ldots$	O	O	O	X
-2	O	O	$\ldots$	O	O	O	X
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-k+1	O	O	$\ldots$	O	O	O	X
-k	O	O	$\ldots$	O	O	O	X
-(k+1)	X	X	$\ldots$	X	X	X	O

Table 2: Pole orders occurring for $(f_2, f_1) \in H^0(F, g_{I_2((k+1)Q)})$

In particular, the total dimension $H^0(F, g_{I_2((k+1)Q)})$ is given by $k + (k+1) + 1 = 2(k+1)$.

With I_2 done, next we check I_3. The difference is that one more relation is added, namely

$$f_3 + \pi_Q^{-1} f_2 \in \pi_Q^{-m} \widehat{\mathcal{O}}_Q. \tag{88}$$

For this, with carefully case-by-case checking, we arrive at the following

$\text{ord}_Q(f_2)\backslash\text{ord}_Q(f_3)$	0	-2	$\ldots$	-m+1	-m	-m-1	-m-2
0	O	O	$\ldots$	O	O	X	X
-2	O	O	$\ldots$	O	O	X	X
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-m+1	O	O	$\ldots$	O	O	X	X
-m	X	X	$\ldots$	X	X	O	X
-m-1	X	X	$\ldots$	X	X	X	O

Table 3: Pole orders occurring for (f_3, f_2) of $(f_3, f_2, f_1) \in H^0(F, g_{I_3(mQ)})$

In particular, this implies that the dimension of $H^0(F, g_{I_3(mQ)})$ is given by

$$(m-1) + (m-1) + m + 1 + 1 = 3m. \tag{89}$$

4.3 Case for I_r

Continuing the discussion in the previous subsubsection, in the case for I_4, one more condition should be added, namely

$$f_4 + \pi_Q^{-1} f_3 \in \pi_Q^{-m} \widehat{\mathcal{O}}_Q. \tag{90}$$

For this, with carefully case-by-case checking, we arrive at the following

$\mathrm{ord}_Q(f_3)\backslash\mathrm{ord}_Q(f_4)$	0	-2	...	-m+1	-m	-m-1	-m-2	-m-3
0	O	O	...	O	O	X	X	X
-2	O	O	...	O	O	X	X	X
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	
-m+1	O	O	...	O	O	X	X	X
-m	X	X	...	X	X	O	X	X
-m-1	X	X	...	X	X	X	O	X
-m-2	X	X	...	X	X	X	X	O

Table 4: Pole orders occurring for (f_4, f_3) of $(f_4, \ldots, f_1) \in H^0(F, g_{I_4(mQ)})$

In particular, this implies that the dimension of $H^0(F, g_{I_4(mQ)})$ is given by

$$(m-1) + (m-1) + (m-1) + m + 1 + 1 + 1 = 4m. \tag{91}$$

Therefore, for general I_r, we should have

$o(f_{r-1})\backslash o(f_r)$	0	-2	...	-m+1	-m	-m-1	...	-m-(r-2)	-m- (r-1)
0	O	O	...	O	O	X	...	X	X
-2	O	O	...	O	O	X	...	X	X
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
-m+1	O	O	...	O	O	X	...	X	X
-m	X	X	...	X	X	O	...	X	X
-m-1	X	X	...	X	X	X	...	X	X
-m-2	X	X	...	X	X	X	...	X	X
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
-m-(r-3)	X	X	...	X	X	X	...	O	X
-m-(r-2)	X	X	...	X	X	X	...	X	O

Table 5: Pole orders occurring for (f_r, f_{r-1}) of $(f_r, \ldots, f_1) \in H^0(F, g_{I_r(mQ)})$

Here $o(f) := \mathrm{ord}_Q(f)$. In particular, this implies that the dimension of $H^0(F, g_{I_r(mQ)})$ is given by

$$\overbrace{(m-1) + \ldots + (m-1)}^{r-1} + m + \overbrace{1 + \ldots + 1}^{r-1} = rm. \tag{92}$$

5 Rank r Codes $C_F(D, g_{I_r}(mQ))$

5.1 Some General Results

Let $p_1, p_2, \ldots, p_n$ be mutually distinct $\mathbb{F}_q$-rational points of X. Set $D = p_1 + p_2 + \ldots, +p_n$ be the associated divisor on X. The, for any $m \geq 1$, using the constructions in [6], we obtain a rank r code $C_F(D, g_{I_2}(mQ))$, defined as the linear space

$$\left\{ \begin{array}{c} (f_2(p_1), f_1(p_1); f_2(p_2), f_1(p_2); \ldots; f_2(p_n), f_1(p_n)) : \\ (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \end{array} \right\} \tag{93}$$

By Table 1, $\mathrm{ord}_Q(f_1) \geq -m$, $\mathrm{ord}_Q(f_2) \geq -m-1$ and it is possible to have

$$\mathrm{ord}_Q(f_1) = -m \qquad \text{and} \qquad \mathrm{ord}_Q(f_2) = -m-1 \tag{94}$$

Now we are ready to state the next main theorem of this paper.

Proposition 12. *Let $(n, m) \in \mathbb{Z}_{\geq 0}^2$. Assume that $p_1, \ldots, p_n$, Q are $\mathbb{F}_q$-rational points on the elliptic curve X with additive operation $\oplus$ satisfy the following conditions.*

(1) Q is the zero element of the group $(X(\mathbb{F}_q), \oplus)$,

(2) $p_1 \oplus \ldots \oplus p_m \oplus p_{m+1} = Q$

(3) $p_m \oplus p_{m+1} \in \{p_{m+2}, \ldots, p_n\}$

Set $D = p_1 + p_2 + \ldots + p_n$ be the divisor on X associated to the p_i's. Then for the D-balanced, semi-stable $g_{I_2}(mQ) \in \mathrm{GL}_2(\mathbb{A})$, the dimension and the minimal distance of the rank r code space $C_{F,r}(D, g_{I_r}(mQ))$ are given by

$$k_{D,g_{I_2}(mQ)} = 2m \qquad \text{and} \qquad d_{D,g_{I_2}(mQ)} = 2(n-m) - 1, \tag{95}$$

respectively. In particular, we have

$$k_{D,g_{I_2}(mQ)} + d_{D,g_{I_2}(mQ)} = 2n - 1 = \ell_{D,g_{I_2}(mQ)} - 1. \tag{96}$$

Proof. We begin with the following well-known

Lemma 13. *Let $\sum_{i=1}^s n_i P_i$ be a divisor on the elliptic curve X with Q as its zero element. Then*

$$\sum_{i=1}^s n_i P_i = (f) \tag{97}$$

for a certain rational function f if and only if

(1) $\sum_{i=1}^s n_i \deg(P_i) = 0$, and

(2) $\bigoplus_{i=1}^s [n_i] P_i = Q$.

Here, for a closed point $P \in X$ and an integer n, we write $[n]P$ for $\overbrace{P \oplus \ldots \oplus P}^{n}$.

Consequently, by the condition (1) and (3) in the theorem, it is possible to choose an element $f_1 \in F$ such that

$$(f_{1,0}) = p_1 + p_2 + \ldots + (p_m \oplus p_{m+1}) - mQ. \tag{98}$$

Similar, by the condition (1) and (2) in the theorem, it is possible to choose an element $f_2 \in F$ such that

$$(f_{2,0}) = p_1 + p_2 + \ldots + p_m + p_{m+1} - (m+1)Q. \tag{99}$$

Obviously,

$$(f_{2,0}f_{1,0}) \in H^0(F, g_{I_r}(mQ)). \tag{100}$$

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Therefore,

$$\sum_{i,j=1}^{n,r} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} = m + (m+1) = 2m + 1. \tag{101}$$

On the other hand, by Table 1, we have

$$\max\left\{ \sum_{i,j=1}^{n,2} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} : (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \right\} \leq m + (m+1) = 2m+1 \tag{102}$$

Therefore,

$$\max\left\{ \sum_{i,j=1}^{n,2} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} : (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \right\} = 2m + 1. \tag{103}$$

Hence, by the fact that the length of the code $C_{F,r}(D, g_{I_r}(mQ))$ is nr, from Lemma 17 of [6], we conclude that the minimal distance of $C_{F,r}(D, g_{I_r}(mQ))$ is given by

$$\begin{aligned} d_{D,g_{I_2}(mQ)} &= 2n - \max\left\{ \sum_{i,j=1}^{n,2} \delta_{\mathrm{ord}_{p_i}(f_j)\geq 1} : (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \right\} \\ &= 2n - (2m+1) = 2(n-m) - 1 \end{aligned} \tag{104}$$

Next, we calculate the dimension $k_{D,g_{I_2}(mQ)}$ of our rank 2 codes. Note that $\deg(g_{I_2}(mQ - D)) = 2(m - n) < 0$, by the vanishing theorem,

$$H^0(F, g_{I_2}(mQ - D)) = \{0\}. \tag{105}$$

This implies that

$$k_{D,g_{I_2}(mQ)} = h^0(F, g_{I_2}(mQ)) - h^0(F, g_{I_2}(mQ - D)) = 2m - 0 = 2m, \tag{106}$$

Therefore, by (115) and (106),

$$k_{D,g_{I_2}(mQ)} + d_{D,g_{I_2}(mQ)} = (2(n-m) - 1) + 2m = 2n - 1. \tag{107}$$

This completes the our proof. $\square$

With similar arguments, we have the following

Proposition 14. *Let $(n, m) \in \mathbb{Z}_{\geq 0}^2$. Assume that $p_1, \ldots, p_n, Q$ are $\mathbb{F}_q$-rational points on the elliptic curve X with additive operation $\oplus$ satisfy the following conditions.*

(1) Q is the zero element of the group $(X(\mathbb{F}_q), \oplus)$,

(2) $p_1 \oplus \ldots \oplus p_m \oplus p_{m+1} \oplus p_{m+2} = Q$

(3) $p_m \oplus p_{m+1} \oplus p_{m+2}, p_{m+1} \oplus p_{m+2} \in \{p_{m+3}, \ldots, p_n\}$

Set $D = p_1 + p_2 + \ldots + p_n$ be the divisor on X associated to the p_i's. Then for the D-balanced, semi-stable $g_{I_3}(mQ) \in \mathrm{GL}_3(\mathbb{A})$, the dimension and the minimal distance of the rank 3 code space $C_{F,r}(D, g_{I_r}(mQ))$ are given by

$$k_{D,g_{I_3}(mQ)} = 3m \qquad \text{and} \qquad d_{D,g_{I_3}(mQ)} = 3(n-m) - 3, \qquad (108)$$

respectively. In particular, we have

$$k_{D,g_{I_3}(mQ)} + d_{D,g_{I_3}(mQ)} = 3n - 3 = \ell_{D,g_{I_2}(mQ)} - 3. \qquad (109)$$

More generally, we have the following

Theorem 15. *Let $(n,r,m) \in \mathbb{Z}_{\geq 0}^3$ satisfying $n \geq m + 2r - 2$. Assume that $p_1, \ldots, p_n, Q$ are $\mathbb{F}_q$-rational points on the elliptic curve X with additive operation $\oplus$ satisfy the following conditions.*

(1) Q is the zero element of the group $(X(\mathbb{F}_q), \oplus)$,

(2) $p_1 \oplus \ldots \oplus p_m \oplus p_{m+1} \oplus \ldots \oplus p_{m+r-1} = Q$

(3) $p_m \oplus p_{m+1} \oplus \ldots \oplus p_{m+r-2} \oplus p_{m+r-1}, \ldots, p_{m+r-2} \oplus p_{m+r-1} \in \{p_{m+r}, \ldots, p_n\}$

Set $D = p_1 + p_2 + \ldots + p_n$ be the divisor on X associated to the p_i's. Then for the D-balanced, semi-stable $g_{I_r}(mQ) \in \mathrm{GL}_r(\mathbb{A})$, the dimension and the minimal distance of the rank r code space $C_{F,r}(D, g_{I_r}(mQ))$ are given by

$$k_{D,g_{I_r}(mQ)} = rm \qquad \text{and} \qquad d_{D,g_{I_r}(mQ)} = r(n-m) - \frac{r(r-1)}{2}, \qquad (110)$$

respectively. In particular, we have

$$k_{D,g_{I_r}(mQ)} + d_{D,g_{I_r}(mQ)} = rn - \frac{r(r-1)}{2} = \ell_{D,g_{I_2}(mQ)} - \frac{r(r-1)}{2}. \qquad (111)$$

5.2 MDS Codes $C_F(D, g_{I_2}(mQ))$

The previous subsection gives some general results for $C_F(D, g_{I_r}(mQ))$. It is then a natural question when $C_F(D, g_{I_r}(mQ))$ becomes MDS. To simplify our discussions, we assume $r = 2$.

By definition, for MDS codes $C_F(D, g_{I_2}(mQ))$,

$$k_{D,g_{I_2}(mQ)} + d_{D,g_{I_2}(mQ)} = \ell_{D,g_{I_2}(mQ)} + 1. \qquad (112)$$

On the other hand, by the discussion in §5.1,

$$k_{D,g_{I_2}(mQ)} = 2m \qquad \text{and} \qquad \ell_{D,g_{I_2}(mQ)} = 2n. \qquad (113)$$

Hence, for MDS codes,

$$d_{D,g_{I_r}(mQ)} = 2(n-m) + 1. \qquad (114)$$

Recall that, by Lemma 17 of [6],

$$d_{D,g_{I_2}(mQ)} = 2n - \max\left\{ \sum_{i,j=1}^{n,2} \delta_{\mathrm{ord}_{p_i}(f_j) \geq 1} : (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \right\} \qquad (115)$$

All these imply that, for MDS codes $C_F(D, g_{I_2}(mQ))$,

$$\max\left\{ \sum_{i,j=1}^{n,2} \delta_{\text{ord}_{p_i}(f_j) \geq 1} : (f_2, f_1) \in H^0(F, g_{I_2}(mQ)) \right\} = 2m - 1. \qquad (116)$$

Our aim in this subsection is to find a global section $(f_2, f_1) \in H^0(F, g_{I_2}(mQ))$ such that

$$\sum_{i=1}^{n} \delta_{\text{ord}_{p_i} \geq 1}(f_1) = m - 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \delta_{\text{ord}_{p_i} \geq 1}(f_2) = m. \qquad (117)$$

Recall that, from the discussion in §5.1, for $(f_2, f_1) \in H^0(F, g_{I_2(mQ)})$,

$$\text{ord}_Q(f_1) \geq -m \qquad \text{and} \qquad \text{ord}_Q(f_2) \geq -(m+1). \qquad (118)$$

and there is only one $\mathbb{F}_q$-subspace $(f_{2,0}, f_{1,0})\mathbb{F}_q$ of $H^0(F, g_{I_2(mQ)})$ such that

$$\text{ord}_Q(f_1) = -m \qquad \text{and} \qquad \text{ord}_Q(f_2) = -(m+1). \qquad (119)$$

For example, if the following conditions are satisfied, the corresponding codes $C_F(D, g_{I_2}(mQ))$ is MDS.

(0) Q is the zero moment of the elliptic curve $X/\mathbb{F}_q$,

(1) $(f_1) = p_1 + \ldots + p_{m-2} + 2p_{m-1} - mQ$,

(2) $(f_2) = p_1 + \ldots + p_{m-1} + 2p_m - (m+1)Q$,

(3) $p_{m-1} = [2]p_m$,

(4) $p_1 \oplus p_1 \oplus \ldots \oplus p_{m-2} \oplus [2]p_{m-1} = p_1 \oplus p_1 \oplus \ldots \oplus p_{m-2} \oplus p_{m-1} \oplus [2]p_m = Q$.

Obviously, if (3) and (4) are satisfied, then so is (1) and (2) by lemma 13. For example, if $X(\mathbb{F}_q)$ contains a cyclic subgroup of order 4 generated by p_m, then (3) and (4) are satisfied by taking $p_1, \ldots, p_{m-2}$ satisfying

$$p_1 \oplus \ldots \oplus p_{m-2} = Q. \qquad (120)$$

Motivated by this, we may also take any cyclic factor of $X/\mathbb{F}_q$ generated by p_m, then assume that $p_m, [\alpha_1]p_m = p_{m-1}, [\alpha_2]p_{m_1} = p_{m-2}\ldots$ to obtain rank r MDS codes $C_F(D, g_{I_r}(mQ))$. In any cases, there are many many ways to obtain MDS codes using $C_F(D, g_{I_r}(mQ))$.

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