Murmurations and Sato-Tate Conjectures for High Rank Zetas of Elliptic Curves

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Murmurations in Arithmetic Geometry and Related Topics
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Murmuration at Kanmon Straits

Figure: Kanmon Straits: Murmuration

Stability

- X: (conn. reg. proj.) curve of genus g over $\mathbb{F}_q$
- $\mathcal{E}$: rank n vec. bundle over $X/\mathbb{F}_q$
- $\det \mathcal{E}$: determinant line bundle on $X/\mathbb{F}_q$
- $s \not\equiv 0$: non-zero rational section of $\det \mathcal{E}$
- $(s) = \text{zeros} - \text{poles} = \sum_k a_k p_k$
- $\deg(\mathcal{E}) := \deg(\det \mathcal{E}) = \sum_k a_k \deg(p_k)$: degree of $\mathcal{E}$
- $\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{n}$: Mumford's μ-slope of $\mathcal{E}$
- $\mathcal{E}$ is called (Mumford) semi-stable if $\forall$ subbundle $\mathcal{E}'$ of $\mathcal{E}$

$$\mu(\mathcal{E}') \leq \mu(\mathcal{E})$$

Rank n Zetas

Definition (Non-Abelian Zeta: Weng)

Fixed $n \in \mathbb{Z}_{\geq 1}$. For a conn. reg. proj. curve X/F_q, define its rank n non-abelian zeta function $\widehat{\zeta}_{X/\mathbb{F}_q,n}(s)$ by

$$\widehat{\zeta}_{X/\mathbb{F}_q,n}(s) := \sum_{\mathcal{E}} \frac{q^{h^0(X,\mathcal{E})} - 1}{\#\mathrm{Aut}(\mathcal{E})} (q^{-s})^{\chi(X,\mathcal{E})}, \quad \forall \quad \Re(s) > 1$$

where $\mathcal{E}$: rank n semi-stable vec. bdls of degrees $\in \mathbb{Z}_{\geq 0} n$

Example (Naturality in n = 1)

$$\widehat{\zeta}_{X/\mathbb{F}_q,1}(s) = \widehat{\zeta}_{X/\mathbb{F}_q}(s) := (q^{-s})^{-(g-1)} \cdot \zeta_{X/\mathbb{F}_q}(s)$$

w/ $\zeta_{X/\mathbb{F}_q}(s) := \sum_{D \geq 0} \frac{1}{N(D)^s}$, Artin zeta of $X/\mathbb{F}_q$

Zeta Facts

Theorem (Zeta Facts: Weng)

$\widehat{\zeta}_{X/\mathbb{F}_q,n}(s)$ satisfies

(1) Rationality: $\exists$ deg 2g polynomial $P_{X/\mathbb{F}_q,n}(T) \in \mathbb{Q}[T]$ s.t.

$$\widehat{\zeta}_{X/\mathbb{F}_q,n}(s) =: \widehat{Z}_{X/\mathbb{F}_q,n}(T) = \frac{P_{X/\mathbb{F}_q,n}(T)}{(1-T)(1-QT)}$$

w/ $t := q^{-s}, T := t^n$ and $Q = q^n$

(2) Functional Equation: $\widehat{\zeta}_{X/\mathbb{F}_q,n}(1-s) = \widehat{\zeta}_{X/\mathbb{F}_q,n}(s)$

(3) Residue in Geometry: $\mathrm{Res}_{s=1}\widehat{\zeta}_{X/\mathbb{F}_q,n}(s) = \beta_{X/\mathbb{F}_q,n}(0)$
w/ α- and β- invariants in rank n degree d of $X/\mathbb{F}_q$:

$$\alpha_{X/\mathbb{F}_q,n}(d) := \sum_{\mathcal{E}} \frac{q^{h^0(X,\mathcal{E})}-1}{\#\mathrm{Aut}(\mathcal{E})}, \quad \beta_{X/\mathbb{F}_q,n}(d) := \sum_{\mathcal{E}} \frac{1}{\#\mathrm{Aut}(\mathcal{E})}$$

Riemann Hypothesis

$$\widehat{\zeta}_{X/\mathbb{F}_q,n}(s) = 0 \Longrightarrow \Re(s) = \frac{1}{2}.$$

This is equivalent to

$P_{X/\mathbb{F}_q,n}(T) \in \mathbb{Q}[T]$ admits no real zeros.

Theorem (Current State)

The RH holds when

(i) $n = 1$: Classical, due to Hasse-Weil

(ii) $X = E$ elliptic curve: Weng-Zagier

(iii) $n = 2$: H. Yoshida,

(iv) $n = 3$: Weng

Number field analogue established in a weak form for $F = \mathbb{Q}$, $n \geq 2$ by Lagrias-Suzuki (n=2), Suzuki (n=3), Ki (n=4,5), and in general, by myself based on Ki-Komori-Suzuki.

Set

$$\widehat{\nu}_k := \begin{cases} \widehat{\zeta}^*_{X/\mathbb{F}_q}(1) & k=1 \\ \widehat{\zeta}_{X/\mathbb{F}_q}(k)\cdot\widehat{\nu}_{k-1} & k\geq 2 \end{cases}$$

and

$$B_k(x) := \sum_{p=1}^{k} \sum_{\substack{k_1,\dots,k_p>0 \\ k_1+\dots+k_p=k}} \frac{\widehat{\nu}_{k_1}\cdots\widehat{\nu}_{k_p}}{(1-q^{k_1+k_2})\cdots(1-q^{k_{p-1}+k_p})}\cdot\frac{1}{1-q^{k_p}x}$$

Theorem (Special Uniformity: Mozgovoy-Reineke, Weng-Zagier)

We have, for $(G,P)=(SL_n,P_{n-1,1})$,

$$\widehat{\zeta}_{X/\mathbb{F}_q,n}(s) = \widehat{\zeta}^{SL_n}_{X/\mathbb{F}_q}(s) := \widehat{\zeta}^{(G,P)}_{X/\mathbb{F}_q}(s)$$

$$= q^{\binom{n}{2}(g-1)} \sum_{k=0}^{n-1} B_k(q^{ns-k}) B_{n-k-1}(q^{k+1-ns}) \widehat{\zeta}_{X/\mathbb{F}_q}(ns-k).$$

In particular, for $X = E$ an elliptic curve, for simplicity, set

$$\alpha_n = \alpha_{E/F_q,n}(0) \qquad \text{and} \qquad \beta_n = \beta_{E/\mathbb{F}_q,n}(0).$$

Then

$$\widehat{\zeta}_{E/\mathbb{F}_q,n}(s) = \alpha_n + \beta_n \cdot \frac{(Q-1)T}{(1-T)(1-QT)} = \frac{P_{E/\mathbb{F}_q;n}(T)}{(1-T)(1-QT)}$$

and

$$P_{E/\mathbb{F}_q;n}(T) = \alpha_{X/\mathbb{F}_q,n}(0)\Big(1 - a_{E/\mathbb{F}_q,n}T + QT^2\Big)$$

w/

$$a_{E/F_q,n} := (Q+1) - (Q-1)\frac{\beta_n}{\alpha_n}.$$

1. $\mathbb{E}$: (reg. int.) elliptic curve over $\mathbb{Q}$
2. p_i: the i-th prime integer ($i \geq 1$) e.g. $p_1 = 2, p_2 = 3, \ldots$
3. $E/\mathbb{F}_{p_i}$: the p_i-reduction of $\mathbb{E}$
4. $N_1, N_2 \in \mathbb{Z}_{>0}$: satisfying $N_1 \leq N_2$
5. $\mathcal{E}_r[N_1, N_2]$: set of elliptic curves $\mathbb{E}/\mathbb{Q}$ of arithmetic rank r with the conductor in the interval $[N_1, N_2]$.[1]

Definition (Rank n murmuration Function)

The rank n average value $f_{r,n}(i)$ is defined by:

$$f_{r,n}(i) := \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \times \sum_{E \in \mathcal{E}_r[N_1,N_2]} \begin{cases} a_{E/\mathbb{F}_{p_i},1} & n = 1 \\ a_{E/\mathbb{F}_{p_i},2} + p_i - 1 & n = 2 \\ \frac{1}{n-1} \cdot \left(a_{E/\mathbb{F}_{p_i},n} + (n-1)p_i + (n-5)\right) & n \geq 3 \end{cases}$$

[1] Here as in the rank one case, for each isogeny class of elliptic curves $\mathbb{E}/\mathbb{Q}$, only a single representative elliptic curve is selected in $\mathcal{E}_r[N_1, N_2]$.

[Repeated]

In particular, for $X = E$ an elliptic curve, for simplicity, set

$$\alpha_n = \alpha_{E/F_q,n}(0) \qquad \text{and} \qquad \beta_n = \beta_{E/\mathbb{F}_q,n}(0).$$

Then

$$\widehat{\zeta}_{E/\mathbb{F}_q,n}(s) = \alpha_n + \beta_n \cdot \frac{(Q-1)T}{(1-T)(1-QT)} = \frac{P_{E/\mathbb{F}_q;n}(T)}{(1-T)(1-QT)}$$

and

$$P_{E/\mathbb{F}_q;n}(T) = \alpha_{X/\mathbb{F}_q,n}(0)\Big(1 - a_{E/\mathbb{F}_q,n}T + QT^2\Big)$$

w/

$$a_{E/F_q,n} := (Q+1) - (Q-1)\frac{\beta_n}{\alpha_n}.$$

The Riemann hypothesis holds for $\zeta_{E/\mathbb{F}_q,n}$ implies

$$-1 \leq \frac{1}{2\sqrt{Q}} \cdot a_{E/\mathbb{F}_q,n} \leq 1.$$

Since cosin function is strictly decreasing in the interval $[0,\pi]$, accordingly, introduce the rank n argument $\theta_{E/\mathbb{F}_q,n}$ of $E/\mathbb{F}_q$ by

$$\theta_{E/\mathbb{F}_q,n} := \mathsf{arccos}\left(\frac{1}{2\sqrt{Q}} \cdot a_{E/\mathbb{F}_q,n}\right) \in [0,\pi]. \tag{1}$$

Definition (Rank n Big Delta Distribution)

$$\Delta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n} := \begin{cases} \sqrt{q}\,\mathsf{cos}\,\theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},2} + \frac{1}{2}\big(\sqrt{p_i} - \frac{1}{\sqrt{p_i}}\big) & \text{for } n=2 \\ \frac{\sqrt{p_i^{n-1}}}{n-1}\big(\frac{\pi}{2} - \theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n}\big) + \frac{1}{2}\big(\sqrt{p_i} + \frac{n-5}{(n-1)\sqrt{p_i}}\big) & \text{for } n \geq 3 \end{cases} \tag{2}$$

Secondary Structures of Rank n-Zeta Zeros

3 new aspects emerged from the secondary structures of rank n zeta zeros of elliptic curves $\mathbb{E}/\mathbb{Q}$:

1. 1^{st}: $\theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n} \to \frac{\pi}{2} \quad (p_i \to \infty)$
2. 2^{ed}: $(\theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n} - \frac{\pi}{2})$ is too tiny to be detected. Hence a suitable huge magnification, namely, $\frac{\sqrt{p_i^{n-1}}}{n-1}$, should be introduced.
3. 3^{rd}: There is a blowing-up within $\frac{\sqrt{p_i^{n-1}}}{n-1}(\theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n} - \frac{\pi}{2})$. Accordingly, the term $\frac{1}{2}(\sqrt{p_i} + \frac{n-5}{(n-1)\sqrt{p_i}})$ should be added. In particular, for $n \geq 3$,

$$\Delta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n} := \frac{\sqrt{p_i^{n-1}}}{n-1}\Big(\frac{\pi}{2} - \theta^{\mathbb{E}}_{E/\mathbb{F}_{p_i},n}\Big) + \frac{1}{2}\Big(\sqrt{p_i} + \frac{n-5}{(n-1)\sqrt{p_i}}\Big)$$

Theorem (Shi-Weng)

Fix a natural number $n \geq 2$.

(1) (Rank n Murmurations) Fixed $r \in \mathbb{N}$. For families of a regular (integral) elliptic curves $\mathbb{E}/\mathbb{Q}$'s, when plotting the points $(i, f_{r,n}(i))$ $i \geq 1$ in the sufficiently large rang, the murmuration phenomenon appear in exactly the same way as the one associated to the $(i, f_{r,1}(i))$'s (of the same families).

(2) (Rank n Sato-Tate Conjecture) Let $\mathbb{E}/\mathbb{Q}$ be a non CM elliptic curve. For $\alpha, \beta \in \mathbb{R}$ satisfying $0 \leq \alpha < \beta \leq \pi$, we have

$$\lim_{N\to\infty} \frac{\#\{p \leq N : p : \ \text{prime}, \ \cos\alpha \geq \Delta_{E/\mathbb{F}_p,n} \geq \cos\beta\}}{\#\{p \leq N : p : \ \text{prime}\}} = \frac{2}{\pi}\int_\alpha^\beta \sin^2\theta d\theta.$$

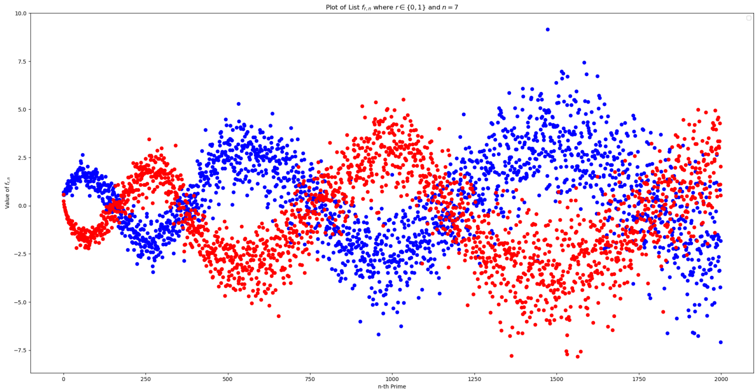

Figure: Plot of $f_{r,n}(i)$ where $r \in 0, 1$ and $n = 7$, for elliptic curves with conductor in [7500, 10000]. $f_{0,n}(i)$ is in blue and $f_{1,n}(i)$ is in red.

Secondary Structures of Distributions of Rank n Zeta Zeros

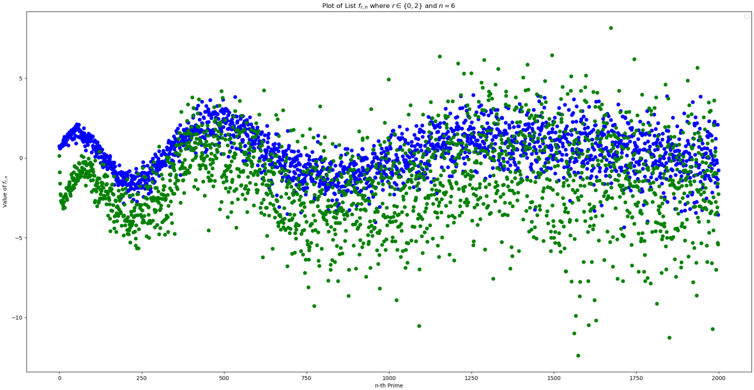

Figure: Plot of $f_{r,n}(i)$ where $r \in 0, 2$ and $n = 6$, for elliptic curves with conductor in $[5000, 10000]$. $f_{0,n}(i)$ is in blue and $f_{2,n}(i)$ is in green.

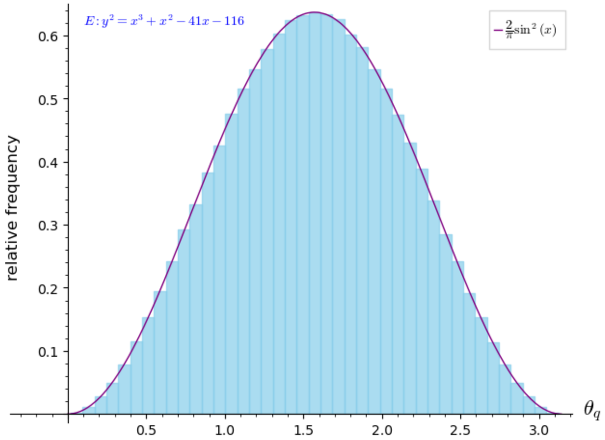

Figure: Sato-Tate distribution of rank 3 zeta function $\zeta_{E/\mathbb{F}_q,3}(s)$ over elliptic curve $\mathbb{E}/\mathbb{Q}: y^2 = x^3 + x^2 - 41x - 116$ and $q \leq N = 10,000,000$.

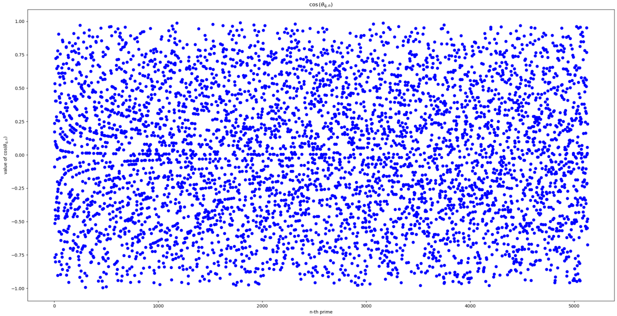

Figure: Plot of $\Delta_{E/\mathbb{F}_q,n}$ over elliptic curve $E : y^2 = x^3 + x^2 - 41x - 116$ and $q \leq N = 50,000$ when $n = 5$.

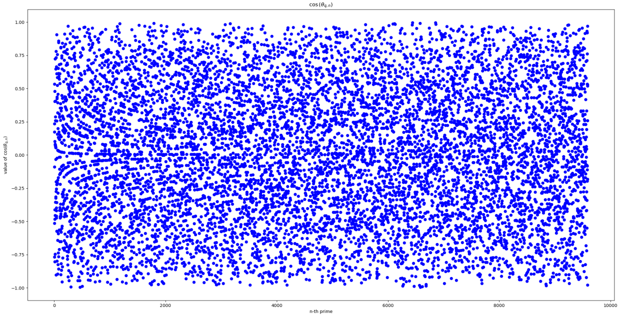

Figure: Plot of $\Delta_{E/\mathbb{F}_q,n}$ over elliptic curve $E: y^2 = x^3 + x^2 - 41x - 116$ and $q \leq N = 100,000$ when $n = 5$.

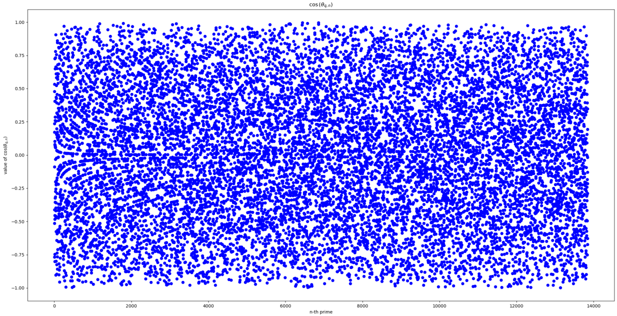

Figure: Plot of $\Delta_{E/\mathbb{F}_q,n}$ over elliptic curve $E : y^2 = x^3 + x^2 - 41x - 116$ and $q \leq N = 150,000$ when $n = 5$.

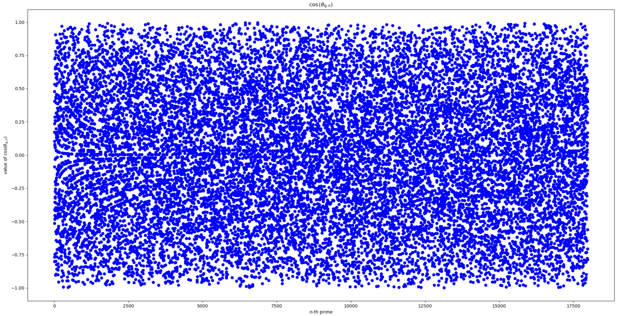

Figure: Plot of $\Delta_{E/\mathbb{F}_q,n}$ over elliptic curve $E : y^2 = x^3 + x^2 - 41x - 116$ and $q \leq N = 200,000$ when $n = 5$.

[Repeated]
In particular, for X = E an elliptic curve, for simplicity, set

$$\alpha_n = \alpha_{E/F_q,n}(0) \qquad \text{and} \qquad \beta_n = \beta_{E/\mathbb{F}_q,n}(0).$$

Then

$$\widehat{\zeta}_{E/\mathbb{F}_q,n}(s) = \alpha_n + \beta_n \cdot \frac{(Q-1)T}{(1-T)(1-QT)} = \frac{P_{E/\mathbb{F}_q;n}(T)}{(1-T)(1-QT)}$$

and

$$P_{E/\mathbb{F}_q;n}(T) = \alpha_{X/\mathbb{F}_q,n}(0)\Big(1 - a_{E/\mathbb{F}_q,n}T + QT^2\Big)$$

w/

$$a_{E/F_q,n} := (Q+1) - (Q-1)\frac{\beta_n}{\alpha_n}.$$

Theorem

(i) [Counting Miracle: (X = E: Zagier-Weng; X general: K. Sugahara and Mozgovoy-Reineke)]

$$\alpha_{X/\mathbb{F}_q,n+1}(0) = q^{n(g-1)}\beta_{X/\mathbb{F}_q,n}(0) \quad \forall \quad n \geq 0$$

(ii) [Semi-Stable Mass: Harder-Narasimhan, Laumon-Rapoport, Zagier, Weng]

$$\beta_{X/F_q,n}(0) = \sum_{p=1}^{n} \sum_{\substack{k_1,\dots,k_p>0 \\ k_1+\dots+k_p=n}} \frac{\widehat{\nu}_{k_1}\cdots\widehat{\nu}_{k_p}}{(1-q^{k_1+k_2})\cdots(1-q^{k_{p-1}+k_p})}$$

(iii) [2-step Structural Recurssion: Zagier-Weng]
For $n \geq 1$, $\beta_{-1} := 0$ and $\beta_0 := 1$,

$$(q^n-1)\beta_n = (q^n+q^{n-1}-a_{E/\mathbb{F}_q,1})\beta_{n-1} - (q^{n-1}-q)\beta_{n-2}$$

Example (n = 1)

When $n = 1$, we have

$$\begin{aligned}(q^1 - 1)\beta_{E/\mathbb{F}_q,1} =& (q^1 + q^{1-1} - a_{E/\mathbb{F}_q,1})\beta_{1-1} - (q^{1-1} - q)\beta_{1-2} \\ =& q + 1 - a_{E/\mathbb{F}_q,1} = \#E(\mathbb{F}_q).\end{aligned}$$

Accordingly,

$$\zeta_{E,1}(s) = \beta_0 + \beta_{E/\mathbb{F}_q,1} \cdot \frac{(q^1 - 1)t^1}{(1 - t^1)(1 - q^1t^1)} = \frac{1 - a_{E/\mathbb{F}_q,1}t + qt^2}{(1 - t)(1 - qt)}$$

i.e. the classical Hasse-Weil zeta $\zeta_{E/\mathbb{F}_q}(s)$.

Example (n = 2)

Similarly, when $n = 2$, we have

$$\begin{aligned}(q^2-1)\beta_2 =&(q^2+q^{2-1}-a_{E/\mathbb{F}_q,1})\beta_{2-1}-(q^{2-1}-q)\beta_{2-2}\\ =&\frac{(q^2+q-a_{E/\mathbb{F}_q,1})(q+1-a_{E/\mathbb{F}_q,1})}{q-1}.\end{aligned}$$

$$\begin{aligned}\zeta_{E,2}(s) =&\beta_{E/\mathbb{F}_q,1}+\beta_2\cdot\frac{(q^2-1)t^2}{(1-t^2)(1-q^2t^2)}\\ =&\frac{q+1-a_{E/\mathbb{F}_q,1}}{q-1}\times\frac{1-(a_{E/\mathbb{F}_q,1}-q+1)T+QT^2}{(1-T)(1-QT)}\end{aligned}$$

Obviously, $\alpha_2 = (q+1-a_{E/\mathbb{F}_q,1})/(q-1) = \beta_1$ is a constant depending merely on the elliptic curve $E/\mathbb{F}_q$ and, in particular,

$$a_{E,1} = a_{E/\mathbb{F}_q,1} = q+1-\#E(\mathbb{F}_q) \quad\text{and}\quad a_{E,2} = a_{E/\mathbb{F}_q,1}-q+1.$$

Theorem (Asymptotic behavior of $a_{E/\mathbb{F}_q,n}$: Shi-Weng)

We have

$$a_{E/\mathbb{F}_q,1} = a_{E/\mathbb{F}_q}, \qquad a_{E/\mathbb{F}_q,2} = 1 + a_{E/\mathbb{F}_q,1} - q \quad \text{and}$$

$$a_{E/\mathbb{F}_q,n} = (5-n) + (n-1)a_{E/\mathbb{F}_q,1} - (n-1)q + O\Big(\frac{1}{\sqrt{q}}\Big) \quad (n \geq 3)$$

Recall that

$$f_{r,n}(i) := \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \times \sum_{E\in\mathcal{E}_r[N_1,N_2]} \begin{cases} a_{E/\mathbb{F}_{p_i},1} & n=1 \\ a_{E/\mathbb{F}_{p_i},2} + q - 1 & n=2 \\ \frac{1}{n-1}\cdot\big(a_{E/\mathbb{F}_{p_i},n} + (n-1)p_i + n - 5\big) & n \geq 3 \end{cases}$$

$$-1 \leq \frac{1}{2\sqrt{Q_n}} \cdot a_{E/\mathbb{F}_q,n} \leq 1.$$

$$\theta_{E/\mathbb{F}_q,n} := \arccos\left(\frac{1}{2\sqrt{Q_n}} \cdot a_{E/\mathbb{F}_q,n}\right) \in [0,\pi].$$

$$\Delta_{E/\mathbb{F}_q,n} := \begin{cases} \sqrt{q}\cos\theta^{E}_{E/\mathbb{F}_q,2} + \frac{1}{2}\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right) & \text{for} \quad n = 2 \\ \frac{\sqrt{q^{n-1}}}{n-1}\left(\frac{\pi}{2} - \theta_{E/\mathbb{F}_p,n}\right) + \frac{1}{2}\left(\sqrt{q} + \frac{n-5}{(n-1)\sqrt{q}}\right) & \text{for} \quad n \geq 3 \end{cases}$$

Essentially, our functionals $f_{r,n}$ and Δ_n transform asymptotically the a-invariants $a_{E/\mathbb{F}_q,n}$ in rank n into that for $a_{E/\mathbb{F}_q} = a_{E/\mathbb{F}_q,1}$ in rank one, for which the murmurations and the classical Sato-Tate are carefully studied by He-Lee-Oliver-Pozdnyakov and established by Taylor and his collaborators (Clozel, Harris, Shepherd-Barron), respectively.

Thank You

Thank You

Fukuoka, November 13, 2024