Riemann Hypothesis for Non-Abelian Zeta Functions of Curves over Finite Fields

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Abstract

In this paper, we develop some basic techniques towards the Riemann hypothesis for higher rank non-abelian zeta functions of a regular projective curve of genus g over a finite field $\mathbb{F}_q$. In particular, as an application of the Riemann hypothesis for rank n zeta functions, we obtain some explicit bounds on the fundamental non-abelian α- and β-invariants of $X/\mathbb{F}_q$ in terms of X and n, q and g:

$$\alpha_{X,\mathbb{F}_q;n}(mn) = \sum_V \frac{q^{h^0(X,V)}-1}{\#\mathrm{Aut}(V)} \quad \text{and} \quad \beta_{X,\mathbb{F}_q;n}(mn) := \sum_V \frac{1}{\#\mathrm{Aut}(V)} \quad (0 \leq m \leq (g-1))$$

where V runs through all rank n semi-stable $\mathbb{F}_q$-rational vector bundles of degree mn over X. Finally, we demonstrate that the related bounds in lower ranks in turn play a central role in establishing the Riemann hypothesis for rank three zetas, following H. Yoshida's approach to verify rank two Riemann hypothesis.

Contents

1 Special uniformity of zetas

The special uniformity for zeta functions of curves over finite fields is conjectured in [14] and established in [17], with the help of the result in [9]. In this section, we recall some basic constructions involved.

1.1 Non-abelian zeta function of a curve over a finite field

First, for a fixed positive integer $n \geq 1$, the *rank n non-abelian zeta function* of a projective regular curve X over $\mathbb{F}_q$ is defined in [14][1] by

$$\widehat{\zeta}_{X,\mathbb{F}_q;n}(s) = \sum_{m=0}^{\infty} \sum_{V} \frac{q^{h^0(X,\mathcal{E})-1}}{\#\mathrm{Aut}\,\mathcal{E}} (q^{-s})^{\chi(X,\mathcal{E})} \qquad (\Re(s) > 1) \tag{1}$$

where $\mathcal{E}$ (in the second summation) runs through rank n semi-stable vector bundles of degree mn. This definition is a modification of an old one in [12] in which $\mathcal{E}$ is allowed to take all rank n semi-stable vector bundles of degree m. Even this original definition in [12] would yield a rational function satisfying the standard functional equation, it fails to satisfy the Riemann hypothesis. To remedy this, motivated by Drinfeld's work [3] on counting two-dimensional irreducible representations of the fundamental group of a curve over a finite field, we introduce a restriction on the degrees of $\mathcal{E}$, that is, the degrees of V are required to be divided by n, the rank of $\mathcal{E}$. By using the Riemann-Roch theorem and the vanishing theorem for semi-stable bundles, tautologically, we have

Theorem 1.1 (ζ Properties [14]). *The rank n-zeta function $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ of a genus g regular projective curve X over $\mathbb{F}_q$ satisfies the following properties:*

(0) $\widehat{\zeta}_{X,\mathbb{F}_q;1}(s)$ coincides with the (completed) Artin zeta $\widehat{\zeta}_{X/\mathbb{F}_q}(s)$ of $X/\mathbb{F}_q$.

(1) $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ is a rational function in $T := (q^{-s})^n$.

(2) (Functional equation) $\widehat{\zeta}_{X,\mathbb{F}_q;n}(1-s) = \widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$.

Indeed, (0) can be deduced by expressing the Artin zeta $\widehat{\zeta}_{X/\mathbb{F}_q}(s)$ of $X/\mathbb{F}_q$ as a sum on the rationally equivalence classes of divisors, or better, the rational line bundles, of non-negative degrees on $X/\mathbb{F}_q$. Furthermore, if we introduce the non-abelian geo-arithmetic

[1] While this paper is fundamental to the field, it has never been submitted for a formal publication.

α- and β-invariants of the curve X over $\mathbb{F}_q$ associated to rank n semi-stable vector bundles by

$$\alpha_{X,\mathbb{F}_q;n}(d) = \sum_{\mathcal{E}} \frac{q^{h^0(X,\mathcal{E})}-1}{\#\mathrm{Aut}(\mathcal{E})} \quad \text{and} \quad \beta_{X,\mathbb{F}_q;n}(d) := \sum_{\mathcal{E}} \frac{1}{\#\mathrm{Aut}(\mathcal{E})} \quad (\forall d \geq 0) \tag{2}$$

where $\mathcal{E}$ rums through rank n semi-stable $\mathbb{F}_q$-rational vector bundles of degree d on X,[2] then by the vanishing theorem for semi-stable vector bundles $\mathcal{E}$ and the Riemann-Roch theorem, we conclude that

$$\alpha_{X,\mathbb{F}_q;n}(mn) = \beta_{X,\mathbb{F}_q;n}(mn) \cdot \left(q^{n(m-(g-1))}-1\right) \qquad (\forall m \geq g). \tag{3}$$

In addition, direct from the definition, we have

$$\beta_{X,\mathbb{F}_q;n}(mn) = \beta_{X,\mathbb{F}_q;n}(0) \qquad (\forall m \in \mathbb{Z}). \tag{4}$$

Therefore, by the standard ζ-technique for curves, we have the following

Theorem 1.2 ([14]). *The rank n-zeta function $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ of a genus g projective regular curve X over $\mathbb{F}_q$ is given by, with $Q = q^n$,*

$$\begin{aligned}&\widehat{Z}_{X,\mathbb{F}_q;n}(T) := \widehat{\zeta}_{X,\mathbb{F}_q;n}(s)\\ &= \sum_{m=0}^{g-2} \alpha_{X,n}(mn)\left(T^{m-(g-1)} + Q^{(g-1)-m}T^{(g-1)-m}\right) + \alpha_{X,n}(n(g-1)) + \frac{(Q-1)\beta_{X,n}(0)\cdot T}{(1-T)(1-QT)}.\end{aligned}$$

In particular, when $n = 1$, we have recovered the following standard, but less well-known formula for the Artin Zeta function of the curve $X/\mathbb{F}_q$, with $t = q^{-s}$,

$$\begin{aligned}&\widehat{Z}_{X/\mathbb{F}_q}(T) := \widehat{\zeta}_{X/\mathbb{F}_q}(s)\\ &= \sum_{m=0}^{g-2} \alpha_{X/\mathbb{F}_q}(m)\left(t^{m-(g-1)} + q^{(g-1)-m}t^{(g-1)-m}\right) + \alpha_{X/\mathbb{F}_q}(g-1) + \frac{(q-1)\beta_{X/\mathbb{F}_q}(0)\, t}{(1-t)(1-qt)}.\end{aligned}$$

where, to simplify our notation, we have set $\alpha_{X/\mathbb{F}_q}(d) := \alpha_{X,\mathbb{F}_q;1}(d)$ and $\beta_{X/\mathbb{F}_q}(d) := \beta_{X,\mathbb{F}_q;1}(d)$.

This theorem clearly implies the zeta properties on rationality and the functional equation in the previous theorem. Indeed, if we set

$$\widehat{Z}_{X,\mathbb{F}_q;n}(T) = \frac{P_{X,\mathbb{F}_q;n}(T)}{(1-T)(1-QT)\cdot T^{g-1}} \tag{5}$$

Then $P_{X,\mathbb{F}_q;n}(T)$ is a degree $2g$ polynomial in T with real coefficients whose leading coefficient and constant term are $q^{ng}\alpha_{X,\mathbb{F}_q;n}(0)$ and $\alpha_{X,\mathbb{F}_q;n}(0)$, respectively.

After examining many examples in lower ranks, we formulate the following

[2] The beta invariant was first introduced in [4].

Conjecture 1.1 (Riemann Hypothesis [14])**.** The rank n-zeta function $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ of a projective regular curve X over $\mathbb{F}_q$ satisfies the Riemann hypothesis. That is, all roots of $p_{X,\mathbb{F}_q;n}(s) := P_{X,\mathbb{F}_q;n}(q^{-s})$ lies on the line $\Re(s) = \frac{1}{2}$.

Obviously, this is equivalent to the condition that all reciprocal roots of $P_{X,\mathbb{F}_q;n}(T)$ are of norm $Q^{\frac{1}{2}}$. Still, there is an apparently weak but equivalent form is that all reciprocal roots of $P_{X,\mathbb{F}_q;n}(T)$ are not real, thanks to the functional equation of the non-abelian zeta $\widehat{\zeta}_{X,\mathbb{F}_1;n}(s)$.

The first break-through in this direction is the following result relying on basic properties of Atiyah bundles [1] and a heavy use of combinatorics:

Theorem 1.3 ([16])**.** *Let E be an elliptic curve over $\mathbb{F}_q$. Then the rank n zeta function $\widehat{\zeta}_{E,\mathbb{F}_q;n}(s)$ of E satisfies the Riemann hypothesis.*

1.2 SL_n-zeta functions of a curve over a finite field

Let X be a regular projective curve over $\mathbb{F}_q$ and let G be a split connected reductive algebraic group of rank r over $\mathbb{F}_q(X)$, the function field of $X/\mathbb{F}_q$. Let

$$\Big(V, \langle\cdot,\cdot\rangle, \Phi = \Phi^+ \cup \Phi^-, \Delta = \{\alpha_1, \dots, \alpha_r\}, \varpi := \{\varpi_1, \dots, \varpi_r\}, W\Big)$$

be the root system associated to a fixed minimal parabolic subgroup P_0 of G and its maximal split torus T. Here, as usual, V can be identified with the real vector space of $\mathbb{R}$-span of rational characters of T, and is equipped with a natural inner product $\langle\cdot,\cdot\rangle$, with which we may and hence will identify V with its dual $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. In addition, $\Phi^+ \subset V$, resp. $\Phi^- := -\Phi^+$, is the set of so-called positive roots, resp. negative roots, $\Delta \subset V$, resp. $\varpi \subset V$, is the set of simple roots, resp. of fundamental weights, and W is the Weyl group generated by the reflections σ_α ($\alpha \in \Delta$). By definition, the fundamental weights are characterized by the formula $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, 2, \dots, r$, where $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\,\alpha$ denotes the coroot corresponding to a root $\alpha \in \Phi$. We also define the Weyl vector ρ by

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \tag{6}$$

and introduce a *coordinate system* on V (with respect to the base $\{\varpi_1, \dots, \varpi_r\}$ of V and the vector ρ) by writing an element $\lambda \in V$ in the form

$$\lambda \;=\; \sum_{j=1}^{r}(1 - s_j)\varpi_j \;=\; \rho - \sum_{j=1}^{r} s_j\varpi_j\,, \tag{7}$$

which in turn induces natural identifications of V and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$, respectively. For each Weyl element $w \in W$, we set

$$\Phi_w := \Phi^+ \,\cap\, w^{-1}\Phi^-, \tag{8}$$

be the collection of positive roots whose w-images are negative. It is well-known that the cardinality of Φ_w coincides with the length ℓ_w of w, i.e. the minimal number expressing w in terms of σ_α ($\alpha \in \Delta$).

As usual, by a *standard parabolic subgroup* of G, we mean a parabolic subgroup of G that contains the fixed minimal parabolic subgroup P_0. From Lie theory (see e.g., [5]), there is an one-to-one correspondence between standard parabolic subgroups P of G and subsets Δ_P of Δ. In particular, if P is maximal, we may and will write $\Delta_P = \Delta \smallsetminus \{\alpha_p\}$ for a certain unique $p = p(P) \in \{1, \dots, r\}$. For such a standard parabolic subgroup P, denote by V_P the $\mathbb{R}$-span of rational characters of the maximal split torus T_P contained in P, by V_P^* its dual space, and by $\Phi_P \subset V_P$ the set of non-trivial characters of T_P occurring in the space V. Then, by standard theory of reductive groups (see e.g., [2]), V_P, resp. V_P^*, admits a canonical embedding in V, resp. in V^*, which is known to be orthogonal to the fundamental weight ϖ_p, and hence Φ_P can be viewed as a subset of Φ. Set

$$\Phi_P^+ = \Phi^+ \cap \Phi_P, \quad \rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P^+} \alpha \quad \text{and} \quad c_P = 2\langle \varpi_p - \rho_P, \alpha_p^\vee \rangle. \tag{9}$$

Now, for a regular projective curve X of genus g over a finite field $\mathbb{F}_q$, in [13], motivated by the study of zeta functions for number fields,[3] for a connected split reductive algebraic group G, and its maximal standard parabolic subgroup P (defined over the function field of $X/\mathbb{F}_q$), we introduce the *period of* G and the *period of* (G, P) for $X/\mathbb{F}_q$ by

$$\omega_{X,\mathbb{F}_q}^G(\lambda) := \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta}(1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle})} \prod_{\alpha \in \Phi_w} \frac{\widehat{\zeta}_{X/\mathbb{F}_q}(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_{X/\mathbb{F}_q}(\langle \lambda, \alpha^\vee \rangle + 1)}$$

and

$$\begin{aligned} \omega_{X/\mathbb{F}_q}^{G,P}(s) &:= \operatorname{Res}_{\langle \lambda - \rho, \alpha^\vee \rangle = 0,\, \alpha \in \Delta_P} \omega_{X,\mathbb{F}_q}^G(\lambda)\Big|_{s_p = s} \\ &= \operatorname{Res}_{s_r = 0} \cdots \operatorname{Res}_{s_{p+1} = 0} \operatorname{Res}_{s_{p-1} = 0} \cdots \operatorname{Res}_{s_1 = 0}\, \omega_{X/\mathbb{F}_q}^G(\lambda)\Big|_{s_p = s}, \end{aligned}$$

respectively, where s is a complex variable[4] and where for the last equality we used the facts that

$$\langle \rho, \alpha^\vee \rangle = 1 \quad (\forall \alpha \in \Delta) \quad \text{and} \quad \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \quad (\forall 1 \le i,\ j \le r). \tag{10}$$

As proved in [7, 13], the ordering of taking residues along singular hyperplanes $\langle \lambda - \rho, \alpha^\vee \rangle = 0$ for $\alpha \in \Delta_P$ does not affect the outcome, so that the definition is independent of the numbering of the simple roots used in the definition.

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[3] For number fields, the analogue of the two functions to be introduced below are special kinds of Eisenstein periods, defined as integrals of Eisenstein series over moduli spaces of semi-stable lattices. For details, see [15].

[4] We should warn the reader that in [13], [15] and [14] a different normalization is used, with the argument of $\omega_{X,\mathbb{F}_q}^{G,P}$ (and later of $\zeta_X^{G,P}$) being given by $s = c_p(s_p - 1)$ ($= n(s_p - 1)$ in the special case $(G, P) = (\mathrm{SL}_n, P_{n-1,1})$) rather than $s = s_p$ as chosen here. With the normalization used here the functional equation relates s and $1 - s$ rather than s and $-n - s$.

To get the zeta function associated to (G,P) for $X/\mathbb{F}_q$, certain normalizations should be made. For this purpose, write $\omega^G_{X,\mathbb{F}_q}(\lambda) = \sum_{w\in W} T_w(\lambda)$, where, for each $w\in W$,

$$T_w(\lambda) := \frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})}\prod_{\alpha\in\Phi_w}\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_{X/\mathbb{F}_q}(\langle\lambda,\alpha^\vee\rangle+1)}.$$

Accordingly, we need to undertand the residue

$$\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P} T_w(\lambda).$$

Clearly, we care only about those elements $w\in W$ (which we will call *special*) that give non-trivial residues, namely, those satisfying the condition that $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P} T_w(\lambda)\not\equiv 0$. This can happen only if all singular hyperplanes are of one of the following two forms:

(1) $\langle w\lambda-\rho,\alpha^\vee\rangle = 0$ for some $\alpha\in\Delta$, giving a simple pole of the rational factor $\frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})}$;

(2) $\langle\lambda,\alpha^\vee\rangle = 1$ for some $\alpha\in\Phi_w$, giving a simple pole of the zeta factor $\widehat{\zeta}_{X/\mathbb{F}_q}(\langle\lambda,\alpha^\vee\rangle)$.

For special $w\in W$, and $(k,h)\in\mathbb{Z}^2$, following [7] (see also [13]) we define

$$\begin{aligned} N_{P,w}(k,h) &:= \#\{\alpha\in w^{-1}\Phi^- : \langle\varpi_p,\alpha^\vee\rangle = k,\ \langle\rho,\alpha^\vee\rangle = h\} \\ M_P(k,h) &:= \max_{w\text{ special}}\left(N_{P,w}(k,h-1)-N_{P,w}(k,h)\right). \\ &= N_{P,w_0}(k,h-1)-N_{P,w_0}(k,h)\,, \end{aligned}\tag{11}$$

where w_0 is the longest element of the Weyl group. Indeed, the last equality is guaranteed by Corollary 8.7 of [6]. Note that $M_P(k,h)=0$ for almost all but finitely many pairs of integers (k,h), so it makes sense to introduce the product

$$D^{G,P}_{X,\mathbb{F}_q}(s) := \prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\widehat{\zeta}_{X,\mathbb{F}_q}(kn(s-1)+h)^{M_P(k,h)}.\tag{12}$$

Following [15, 13], we define the *zeta function of $X/\mathbb{F}_q$ associated to* (G,P) by

$$\widehat{\zeta}^{G,P}_{X,\mathbb{F}_q}(s) := q^{(g-1)\dim N_u(B)}\cdot D^{G,P}_{X,\mathbb{F}_q}(s)\cdot\omega^{G,P}_{X,\mathbb{F}_q}(s)\,.\tag{13}$$

Here $N_u(P_0)$ denote the nilpotent radical of the Borel subgroup P_0 of G.

Remark. For special $w\in W$, even after taking residues, there are some zeta factors $\widehat{\zeta}_{X/\mathbb{F}_q}(ks+h)$ left in the denominator of $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P} T_w(\lambda)$. The reason for introducing the factor $D^{G,P}_X(s)$ in our normalization of the zeta functions, based on formulas in [7] and [13], is to clear up all of those zeta factors appearing in the denominators associated to special Weyl elements.

In particular, we have the following

Theorem 1.4 (Functional Equation[14]). *For a regular projective curve X over $\mathbb{F}_q$,*

$$\widehat{\zeta}^{G,P}_{X,\mathbb{F}_q}(c_P - s) = \widehat{\zeta}^{G,P}_{X,\mathbb{F}_q}(s). \tag{14}$$

The proof follows closely [7], where the functional equation is established for the parallel structures on the so-called (G, P)-zeta function of number fields F.

With all these, we are now ready to introduce the SL_n-zeta function of $X/\mathbb{F}_q$ by specializing to the case when G is the special linear group SL_n and P is the maximal parabolic subgroup $P_{n-1,1}$ consisting of matrices whose final row vanishes except for its last entry, corresponding to the ordered partition $(n-1)+1$ of n. That is to say, the SL_n*-zeta function* $\widehat{\zeta}^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(s)$ *of* $X/\mathbb{F}_q$ is defined to be

$$\widehat{\zeta}^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(s) := \widehat{\zeta}^{\mathrm{SL}_n,\,P_{n-1,1}}_{X,\mathbb{F}_q}(s) := q^{\frac{n(n-1)}{2}(g-1)} \cdot D^{\mathrm{SL}_n,P_{n-1,1}}(s) \cdot \omega_X^{(\mathrm{SL}_n,P_{n-1,1})}(s)\,. \,. \tag{15}$$

As the first step to understand this zeta function, we have the following

Lemma 1.5 (Lemma 5 of [17]). *The finction $D^{\mathrm{SL}_n,P_{n-1,1}}(s)$ is given by*

$$D^{\mathrm{SL}_n,P_{n-1,1}}(s) = \prod_{k=2}^{n-1} \widehat{\zeta}_{X/\mathbb{F}_q}(k) \cdot \widehat{\zeta}_{X/\mathbb{F}_q}(ns). \tag{16}$$

Motivated by our study on the parallel structures for number fields, after verifying some concrete examples, in [14], we formulate the following

Conjecture 1.2 (Special Uniformity of Zetas). For a regular projective curve of genus g, up to some constant factor depending only on n and g, we have

$$\widehat{\zeta}_{X,\mathbb{F}_q;n}(s) = \widehat{\zeta}^{\mathrm{SL}_n}_{X/\mathbb{F}_q}(s) \tag{17}$$

For number fields F, this uniformity of zeta functions is established by using Mellin transforms to write down the rank n non-abelian zeta function of F in terms of integrations of Epstein zeta functions over the moduli space of semi-stable $\mathcal{O}_F$-lattices of rank n and degree zero. But Epstein zeta function is a special kind of Eisenstein series, which can be realized as the residue of the Siegel-Langlands Eisenstein series associated to the constant function on the Levi subgroup of the minimal parabolic subgroup $P_{1,1,\ldots,1}$ corresponding to the decomposition $n = \overbrace{1 + \ldots + 1}^{n\ times}$. Furthermore, the moduli space of of semi-stable $\mathcal{O}_F$-lattices of rank n and degree zero can be identified with the truncated domain of Arthur type within the fundamental domain of $\mathrm{SL}_n(\mathbb{Z})$. Consequently, with an use of relative trace formula yields the desired zeta uniformity. For details, please refer to Chapter 15 of [15].

To pave the same path to establish the special uniformity of zetas for function fields, the first difficulty is that the analogue construction of Mellin transform has yet to be developed (see however a work of K. Adachi at Kyushu university on "Rankin-Selberg & Zagier Methods for Function Fields over Finite Fields").

1.3 Special uniformity of zeta functions

As said, the special uniformity of zetas claims that, for a global field F, the geometrically defined rank n zeta function $\widehat{\zeta}_{F,n}(s)$ coincides with the Lie theoretically defined SL_r-zeta function $\widehat{\zeta}_F^{\mathrm{SL}_n}(s)$. When F is a number field, this conjectured in confirmed in [15] using the theories of Eisenstein series by Siegel and Langlands, Arthur's analytic truncation and geo-arithmetic truncation of stability, and relative trace formula. When F is a function field, a totally different approach has been used, thanks to an unexpected work of Mozgovoy-Reineke [9]. The uniformity of zetas for functional fields has been finally verified in the paper [17] of Zagier and myself, as a direct consequence of Theorem 7.2 of [9] and Theorem 2 of [17].

Theorem 1.6 (Special Uniformity of Zetas; Theorem 1 of [17]). *For a regular projective curve X of genus g over $\mathbb{F}_q$, we have*

$$\widehat{\zeta}_{X,\mathbb{F}_q;n} = \widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(s) = q^{\binom{n}{2}(g-1)} \sum_{a=1}^{n} \sum_{\substack{k_1,\ldots,k_p>0\\ k_1+\ldots+k_p=n-a}} \frac{\widehat{v}_{k_1}\ldots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \frac{1}{(1-q^{ns-n+a+k_p})}$$
$$\times \widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a) \sum_{\substack{l_1,\ldots,l_r>0\\ l_1+\ldots+l_r=a-1}} \frac{1}{(1-q^{-ns+n-a+1+l_1})} \frac{\widehat{v}_{l_1}\ldots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}$$

Indeed, in [9], based on the theories of Hall algebra and wall-crossing, Mozgovoy-Reineke are able to offer a close formula for $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ in terms of partitions of n and abelian zeta function $\widehat{\zeta}_{X/\mathbb{F}_q}(s)$ of $X/\mathbb{F}_q$. On the other hand, by examining the Lie structures involved in great details in [17], Zagier and myself are able to obtain the explicit formula for $\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(s)$ as stated in the theorem above. It is not difficult to verify that this formula of $\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(s)$ coincides with the one for $\widehat{\zeta}_{X,\mathbb{F}_q;n}(s)$ of [9]. Consequently, the special uniformity of zetas for curves over finite fields is established successfully.

1.4 General counting miracle

As the first application of the special uniformity of zetas of curves $X/\mathbb{F}_q$, we next give closed formulas for the non-abelian geo-arithmetic invariants $\alpha_{X,\mathbb{F}_q;n}(mn)$ and $\beta_{X,\mathbb{F}_q;n}(mn)$ of the curve X over $\mathbb{F}_q$ associated to rank n semi-stable vector bundles. Indeed, by Theorem 1.2 and Theorem 1.6, we have

$$\sum_{m=0}^{g-2} \alpha_{X,\mathbb{F}_q;n}(mn)\left(T^{m-(g-1)} + Q^{(g-1)-m}T^{(g-1)-m}\right) + \alpha_{X,\mathbb{F}_q;n}(n(g-1)) + \frac{(Q-1)\beta_{X,\mathbb{F}_q;n}(0)\cdot T}{(1-T)(1-QT)}$$
$$= q^{\binom{n}{2}(g-1)} \sum_{a=1}^{n} \sum_{\substack{k_1,\ldots,k_p>0\\ k_1+\ldots+k_p=n-a}} \frac{\widehat{v}_{k_1}\ldots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \frac{T}{(T-q^{-n+a+k_p})} \times$$

$$\times\left(\sum_{m=0}^{g-2}\alpha_{X/\mathbb{F}_q}(m)\left(q^{(n-a)(m-(g-1))}T^{m-(g-1)}+q^{(n-a+1)((g-1)-m)}T^{(g-1)-m}\right)\right.$$
$$\left.+\alpha_{X/\mathbb{F}_q}((g-1))+\frac{(q-1)\beta_{X/\mathbb{F}_q}(0)\cdot q^{n-a}T}{(1-q^{n-a}T)(1-q^{n-a+1}T)}\right)$$
$$\times\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{n-a+1+l_1}T)}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}.$$

since

$$\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)=\sum_{m=0}^{g-2}\alpha_{X/\mathbb{F}_q}(m)\left(q^{(n-a)(m-(g-1))}T^{m-(g-1)}+q^{(g-1)-m}q^{(n-a)((g-1)-m)}T^{(g-1)-m}\right)$$
$$+\alpha_{X/\mathbb{F}_q}((g-1))+\frac{(q-1)\beta_{X/\mathbb{F}_q}(0)\cdot q^{n-a}T}{(1-q^{n-a}T)(1-qq^{n-a}T)}.$$

To simplify our notation, set, for each $a=1,\dots,n$,

$$q^{-\binom{n}{2}(g-1)}\widehat{\zeta}^{[(a)]}(s):=\widehat{Z}^{[(a)]}(T)=q^{\binom{n}{2}(g-1)}\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{T}{(T-q^{-n+a+k_p})}$$
$$\times\left(\sum_{m=0}^{g-2}\alpha_{X/\mathbb{F}_q}(m)\left(q^{(n-a)(m-(g-1))}T^{m-(g-1)}+q^{(n-a+1)((g-1)-m)}T^{(g-1)-m}\right)\right.$$
$$\left.+\alpha_{X/\mathbb{F}_q}((g-1))+\frac{(q-1)\beta_{X/\mathbb{F}_q}(0)\cdot q^{n-a}T}{(1-q^{n-a}T)(1-q^{n-a+1}T)}\right)$$
$$\times\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{n-a+1+l_1}T)}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}.$$

For $m=0,1,\dots,g-1$, using the expansion $\frac{1}{1-z}=\sum_{k=0}^\infty z^k$, we have

$$q^{-\binom{n}{2}(g-1)}\alpha_{X,\mathbb{F}_q;n}(mn)=\mathrm{Res}_{T=0}T^{g-2-m}\left(\widehat{Z}_{X,\mathbb{F}_q;n}(T)\right)=\sum_{a=1}^{n}\mathrm{Res}_{T=0}\left(T^{g-2-m}\widehat{Z}^{[(a)]}(T)\right)$$
$$=\sum_{a=1}^{n}q^{(n-a)(m-(g-1))}\left(\sum_{m=k+\ell+\kappa}\sum_{k=0}^{g-2}\alpha_{X/\mathbb{F}_q}(k)\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}(-1)\sum_{\ell=1}^{\infty}(q^{-k_p})^{\ell}\right.$$
$$\left.\times\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}.\sum_{\kappa=0}^{\infty}(q^{1+l_1})^{\kappa}\right)+$$
$$+\delta_{m,g-1}\left(\alpha_{X/\mathbb{F}_q}((g-1))\cdot\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\right)$$

For example, if $m = 0$, we get

$$q^{-\binom{n}{2}(g-1)}\alpha_{X,\mathbb{F}_q;n}(0) = \begin{cases} \alpha_{X/\mathbb{F}_q}(0)\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})} & g\geq 2\\ \delta_{0,g-1}\Big(\alpha_{X/\mathbb{F}_q}((g-1))\cdot\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\Big) & g=1 \end{cases}$$

That is to say, for $g \geq 1$

$$q^{-\binom{n}{2}(g-1)}\alpha_{X,\mathbb{F}_q;n}(0) = \alpha_{X/\mathbb{F}_q}(0)\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})} = \beta_{X,\mathbb{F}_q,n-1}(0) \tag{18}$$

since

$$\alpha_{X/\mathbb{F}_q}(0) = \sum_{L\in\mathrm{Pic}^0(X)} \frac{q^{h^0(X,L)}-1}{q-1} = \frac{q^{h^0(X,\mathcal{O}_X)}-1}{q-1} = 1. \tag{19}$$

This formula was first conjectured in [14] for elliptic curves, which is established in [16] after examining Atiyah bundles in details and a heavily combinatorial technique in 2016. In September 2016, using a totally different method, K. Sugahara generalized this counting miracle to all curves over $\mathbb{F}_q$, which was reproved in [9] using Hall algebra and wall crossing. Similarly,

$$\begin{aligned} &q^{-\binom{n}{2}(g-1)}\Big(\beta_{X,\mathbb{F}_q;n}(0) + Q\alpha_{X,\mathbb{F}_q;n}((g-2)n)\Big) = \mathrm{Res}_{T=0}T^{-2}\widehat{Z}_{X,\mathbb{F}_q;n}(T)\\ &= \sum_{a=1}^{n} q^{n-a}\Bigg(-\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\\ &\qquad\qquad \times \sum_{k+\ell+\kappa=g}\sum_{k=0}^{g-2}\alpha_{X/\mathbb{F}_q}(k)\sum_{\ell=1}^{g}(q^{-k_p})^{\ell}\sum_{\kappa=0}^{g}(q^{1+l_1})^{\kappa}\\ &\qquad - \alpha_{X/\mathbb{F}_q}((g-1))\cdot \sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}(q^{-k_p})\\ &\qquad\qquad \times \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\\ &\qquad + (q-1)\beta_{X/\mathbb{F}_q}(0)\cdot \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\Bigg) \end{aligned}$$

This then gives a closed formula for $\beta_{X,\mathbb{F}_q;n}(0)$ after substracting $Q\alpha_{X,\mathbb{F}_q;n}((g-2)n)$ obtained above. Thus all in all, we have proved the following

Theorem 1.7 (General Counting Miracle)**.** *For a regular projective curve X of genus g on $\mathbb{F}_q$, its non-abelian invariants $\alpha_{X,\mathbb{F}_q,n}(mn)$ $(0 \leq m \leq g-1)$ and $\beta_{X,\mathbb{F}_q;n}(0)$ for semi-stable vector bundles of rank n are given by*

$$
\begin{aligned}
&q^{-\binom{n}{2}(g-1)}\alpha_{X,\mathbb{F}_q;n}(mn)\\
&=\sum_{a=1}^{n} q^{(n-a)(m-(g-1))}\Bigg(-\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\\
&\qquad\qquad \times \sum_{m=k+\ell+\kappa}\sum_{k=0}^{g-2}\alpha_{X/\mathbb{F}_q}(k)\sum_{\ell=1}^{\infty}(q^{-k_p})^{\ell}\sum_{\kappa=0}^{\infty}(q^{1+l_1})^{\kappa}\Bigg)\\
&\quad+\delta_{m,g-1}\Bigg(\alpha_{X/\mathbb{F}_q}((g-1))\cdot\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\Bigg)\\
&q^{-\binom{n}{2}(g-1)}\Big(\beta_{X,\mathbb{F}_q;n}(0)\Big)\\
&=\sum_{a=1}^{n}\Bigg(-\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\\
&\quad\times\Bigg(q^{n-a}\sum_{k+\ell+\kappa=g}-q^{a}\sum_{k+\ell+\kappa=g-2}\Bigg)\Bigg(\sum_{k=0}^{g-2}\alpha_{X/\mathbb{F}_q}(k)\sum_{\ell=1}^{\infty}(q^{-k_p})^{\ell}\sum_{\kappa=0}^{\infty}(q^{1+l_1})^{\kappa}\Bigg)\\
&\quad-\alpha_{X/\mathbb{F}_q}((g-1))\cdot\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}(q^{n-a-k_p})\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\\
&\qquad\qquad+(q-1)q^{n-a}\beta_{X/\mathbb{F}_q}(0)\cdot\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=n-1}} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\Bigg)
\end{aligned}
$$

Recall that

$$\beta_{X,\mathbb{F}_q;n}(mn)=\beta_{X,\mathbb{F}_q;n}(0)\forall m\in\mathbb{Z} \qquad \text{and} \qquad \alpha_{X,\mathbb{F}_q;n}(mn)=q^{m(n-(g-1)}\beta_{X,\mathbb{F}_q;n}(0) \quad (m\geq g) \tag{20}$$

this theorem in fact gives all the values of $\alpha_{X,\mathbb{F}_q;n}(mn)$ and $\beta_{X,\mathbb{F}_q;n}(mn)$ for all $m\in\mathbb{Z}$, since easily

$$\alpha_{X,\mathbb{F}_q;n}(mn)=0 \qquad (m<0). \tag{21}$$

We point out in passing when n does not divide d[5], the value of $\alpha_{X,\mathbb{F}_q;n}(d)$ and $\beta_{X,\mathbb{F}_q;n}(d)$ have been obtained in [9] and [19], respectively.

We end this subsection with the following comments on $\widehat{Z}^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(T)$. By the special uniformity, this function is equal to the rank n zeta function of X, which itself is a rational

[5] Mozegovoy-Reineke call such pairs (n,d) generic in [9].

function of the form

$$\widehat{Z}_{X,\mathbb{F}_q;n}(T) = \frac{P_{X,\mathbb{F}_q;n}(T)}{T^{g-1}(1-T)(1-QT)}. \tag{22}$$

Here $P^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(T)$ is a polynomial of degree $2g$ in T with real coefficients. However, in the summand of $\widehat{\zeta}^{[(a)]}(s)$, from the first group $\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{T}{(T-q^{-n+a+k_p})}$, particularly the term $\frac{T}{(T-q^{-n+a+k_p})} = \frac{q^{(k_1+\dots+k_{p-1})}T}{q^{(k_1+\dots+k_{p-1})}T-1}$, we see that the denominators are given by

$$\frac{1}{T-1},\ \frac{1}{qT-1},\ \dots,\ \frac{1}{q^{n-a-1}T-1} \tag{23}$$

From the second consisting of the whole bracket, particularly the term $\frac{(q-1)\beta_{X/\mathbb{F}_q}(0)\cdot q^{n-a}T}{(1-q^{n-a}T)(1-q^{n-a+1}T)}$. we see that the denominators are given by

$$\frac{1}{q^{n-a}T-1},\ \frac{1}{q^{n-a+1}T-1}, \tag{24}$$

From the third group, particularly the term $\frac{1}{(1-q^{n-a+1+l_1}T)}$, we see that the denominators are given by

$$\frac{1}{q^{n-a+1+1}T-1},\ \frac{1}{(1-q^{n-a+1+2}T)},\ \dots,\ \frac{1}{(1-q^{n-a+1+a-1}T)} \tag{25}$$

Consequently,

$$\begin{aligned}
\xi^{[(a)]}(s) :=& \left(T^{g-1}\prod_{\ell=0}^{n}(q^\ell T-1)\right)\cdot\widehat{\zeta}^{[(a)]}(s)\\
=& \sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} q^{k_1+\dots+k_{p-1}}T \prod_{\substack{0\le\ell\le n-a-1\\ \ell\neq k_1+\dots+k_{p-1}}}(q^\ell T-1)\\
&\times\Bigg(\Big(\sum_{m=0}^{g-2}\alpha_{X/\mathbb{F}_q}(m)\Big(q^{(n-a)(m-(g-1))}T^m + q^{(n-a+1)((g-1)-m)}T^{2(g-1)-m}\Big)\\
&\qquad + \alpha_{X/\mathbb{F}_q}((g-1))T^{g-1}\Big)(1-q^{n-a}T)(1-q^{n-a+1}T)\\
&\qquad + (q-1)\beta_{X/\mathbb{F}_q}(0)\cdot q^{n-a}T^g\Bigg)\\
&\times\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}\prod_{\substack{n-a+2\le\ell\le n\\ \ell\neq n-a+1+l_1}}(q^\ell T-1).
\end{aligned}$$

becomes a polynomial of degree $(n-a)+2g+(a-2) = n+2(g-1)$ which is independent of a. Therefore, $\widehat{Z}^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(T)$ is a rational function of the form

$$\frac{P^{\mathrm{SL}_n}_{X,\mathbb{F}_q}(T)}{T^{g-1}\prod_{\ell=0}^{n}(q^\ell T-1)} \tag{26}$$

where $P_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(T)$ is a polynomial of degree $n+2(g-1)$ in T. By comparing this with $\frac{P_{X,\mathbb{F}_q;n}(T)}{T^{g-1}\prod_{\ell=0}^{n}(q^\ell T-1)}$, we see that in fact there are significant cancellations among the $Z^{[a]}(s)$ when taking the summation $\sum_{a=1}^n Z^{[a]}(T)$ to obtain $\widehat{Z}_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(T)$ so that among the product $\prod_{\ell=0}^{n}(q^\ell T-1)$, all the factors $(qT-1),\ (q^2T-1),\dots,(q^{n-1}T-1)$ will be finally cancelled out from the numerator (so as to leave only the factor $T^{g-1}(1-T)(1-QT)$ in the denominator). This is one of the reasons why the Riemann Hypothesis for high rank functions of curves over finite fields becomes quite complicated, even comparing with what has happened for high rank zeta functions of number fields.

2 Riemann hypothesis for rank two zeta: Yoshida's approach

Applying Theorem 1.6 to $n=2$, easily we conclude that, up to a constant fact depending only on the genus g of the curve $X/\mathbb{F}_q$,

$$\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_2}(s)=\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(2s)}{1-q^{2-2s}}-\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(2s-1)}{q^{2s}-1} \tag{27}$$

We first use the functional equation to obtain

$$\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_2}(s)=\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(1-2s)}{1-q^{2-2s}}-\frac{\widehat{\zeta}_{X,/F_q}(2s-1)}{q^{2s}-1}=\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(2\sigma)}{1-q^{1+2\sigma}}-\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(-2\sigma)}{q^{1-2\sigma}-1} \tag{28}$$

where we have set $s=\frac{1}{2}-\sigma$. Therefore,

$$\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_2}(s)=0 \qquad \text{if and only if} \qquad \frac{\widehat{\zeta}_{X/\mathbb{F}_q}(2\sigma)}{1-q^{1+2\sigma}}=\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(-2\sigma)}{q^{1-2\sigma}-1}. \tag{29}$$

Write now

$$\widehat{\zeta}_{X/\mathbb{F}_q}(s)=\frac{\prod_{i=1}^{g}(1-\omega_i q^{-s})(1-\overline{\omega}_i q^{-s})}{q^{-s(g-1)}(1-q^{-s})(1-q^{1-s})} \tag{30}$$

where $\omega_i\in\mathbb{C}\smallsetminus\mathbb{R}$ and $|\omega_i|=\sqrt{q}$, guaranteed by the Hasse-Weil Theorem, or better, the Riemann hypothesis for the Artin zeta function of $X/\mathbb{F}_q$. In particular,

$$|\omega_i+\overline{\omega}_i|\leq 2\sqrt{q}<q+1 \qquad (\forall 1\leq i\leq g). \tag{31}$$

Then (29) becomes

$$q^{4\sigma(g-1)}(q^{1-2\sigma}-1)\frac{\prod_{i=1}^{g}(1-\omega_i q^{-2\sigma})(1-\overline{\omega}_i q^{-2\sigma})}{(1-q^{-2\sigma})(1-q^{1-2\sigma})}=(1-q^{1+2\sigma})\frac{\prod_{i=1}^{g}(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})}{(1-q^{2\sigma})(1-q^{1+2\sigma})} \tag{32}$$

This is equivalent to

$$q^{4\sigma(g-1)}(1-q^{2\sigma})\prod_{i=1}^{g}(1-\omega_i q^{-2\sigma})(1-\overline{\omega}_i q^{-2\sigma}) = (1-q^{-2\sigma})\prod_{i=1}^{g}(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma}) \quad (33)$$

In particular, we should have

$$|q^{4\sigma(g-1)}|\cdot|(1-q^{2\sigma})|\cdot\prod_{i=1}^{g}|(1-\omega_i q^{-2\sigma})(1-\overline{\omega}_i q^{-2\sigma})| = |(1-q^{-2\sigma})|\cdot\prod_{i=1}^{g}|(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})| \quad (34)$$

Lemma 2.1 (Yoshida)**.** *Fix a real number $q > 1$. Let $\alpha, \beta \in \mathbb{C}$ and write $c = \alpha + \beta$. Assume that $\alpha\beta = q$ and that $c \in \mathbb{R}$ satisfies $|c| \leq q + 1$. Then for $w \in \mathbb{C}$, we have*

$$|w-\alpha|\cdot|w-\beta| \begin{cases} > |1-\alpha w|\cdot|1-\beta w| & \text{if} \quad |w| < 1 \\ < |1-\alpha w|\cdot|1-\beta w| & \text{if} \quad |w| > 1. \end{cases} \quad (35)$$

When $\alpha = 1$ and $\beta = q$, this lemma degenerates to an estimate on the fractional transformation $T_q(W) := \frac{w-q}{1-qw}$. In this sense, Yoshida's lemma is a natural degree 2 generalization from that for fractional transformations. Even an elementary proof can be given immediately, we delay the details till the proof of Lemma 4.5 below, which itself is a generalization of Yoshida's lemma.

Therefore, the left hand side of (34) is simply

$$\begin{aligned} & |q^{4\sigma(g-1)}|\cdot|(1-q^{2\sigma})|\cdot\prod_{i=1}^{g}|(1-\omega_i q^{-2\sigma})(1-\overline{\omega}_i q^{-2\sigma})| \\ =& |(1-q^{2\sigma})||q^{-4\sigma}|\cdot\prod_{i=1}^{g}|(q^{2\sigma}-\omega_i)(q^{2\sigma}-\overline{\omega}_i)| \\ & \begin{cases} > \prod_{i=1}^{g}|(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})|\cdot|(q^{-2\sigma}-1)||q^{(1-2g)2\sigma}| & \text{if} \quad \Re(\sigma) < 0 \\ < \prod_{i=1}^{g}|(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})|\cdot|(q^{-2\sigma}-1)||q^{(1-2g)2\sigma}| & \text{if} \quad \Re(\sigma) > 0. \end{cases} \\ & \begin{cases} > \prod_{i=1}^{g}|(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})|\cdot|(q^{-2\sigma}-1)| & \text{if} \quad \Re(\sigma) < 0 \\ < \prod_{i=1}^{g}|(1-\omega_i q^{2\sigma})(1-\overline{\omega}_i q^{2\sigma})|\cdot|(q^{-2\sigma}-1)| & \text{if} \quad \Re(\sigma) > 0. \end{cases} \end{aligned}$$

which is nothing but the right hand side of (34), provided that $g \geq 1$. This implies that unless $\Re(\sigma) = 0$, i.e., σ is a pure imaginary complex number, $\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_2}(s)$ cannot be zero. This then proves the following

Theorem 2.2. *(Yoshida) Let X be a regular projective curve of genus g, then the* SL_2 *zeta function $\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_2}(s)$ and hence the rank two zeta function $\widehat{\zeta}_{X/\mathbb{F}_q,2}(s)$ satisfy the Riemann hypothesis.*

This result is due to Yoshida [18], during our intensive lectures on non-abelian zeta functions for number fields in Kyoto. Yoshida, motivated by our works on rank two zeta functions of number fields [**?**, 10, **?**], actually proves the Riemann hypothesis for a slight more general zeta function

$$\widehat{\zeta}_{X,\mathbb{F}_q}(s;C_1) := C_1(s)\frac{\widehat{\zeta}_{X/\mathbb{F}_q}(2s)}{1-q^{1-s}} - C_2(s)\frac{q^{-s}\widehat{\zeta}_{X/\mathbb{F}_q}(2s-1)}{1-q^{2s-1}}, \tag{36}$$

where $C_1(s)$ takes the form

$$C_1(s) = q^{as}(1+q^{-s})q^{-hs}\prod_{j=1}^{h}(1-\gamma_i q^{s-1/2})(1-\delta_i q^{s-1/2}) \tag{37}$$

with constants γ_i and δ_i satisfying the conditions that $\gamma_i+\delta_i \in \mathbb{R}$, $|\gamma_i+\delta_i| \leq q+1$ for a non-negative real number a and a natural number h, and $C_2(s)$ determined by

$$C_2(s) = C_1(1-s) \tag{38}$$

so that

$$\widehat{\zeta}^{\mathrm{SL}_2}_{X,\mathbb{F}_q}(1-s) = \widehat{\zeta}^{\mathrm{SL}_2}_{X,\mathbb{F}_q}(s). \tag{39}$$

Our SL_2-zeta function $\widehat{\zeta}^{\mathrm{SL}_2}_{X,\mathbb{F}_q}(s)$ is certainly a special form of Yoshida's type.

3 What can we get from the RH for the rank n zetas?

Before going further, we here deduce the natural upper and lower bounds for the non-abelian geo-arithmetic invariants $\alpha_{X,\mathbb{F}_q;n}(mn)$ $(m=0,\dots,g-1)$ and $\beta_{X,\mathbb{F}_q;n}(0)$ of a curve $X/\mathbb{F}_q$ by assuming the Riemann hypothesis for rank n zeta function of $X/\mathbb{F}_q$. Indeed as we will see later, in turn, when proving the Riemann Hypothesis for rank three zeta functions, these bounds in case of $n=2$ plays a very important role.

Set now

$$\alpha'_{X,\mathbb{F}_q;n}(mn) := \frac{\alpha_{X,\mathbb{F}_q;n}(2m)}{\alpha_{X,\mathbb{F}_q;n}(0)} \qquad \text{and} \qquad \beta_{X,\mathbb{F}_q;n}(0) = \frac{\beta_{X,\mathbb{F}_q;n}(0)}{\alpha_{X,\mathbb{F}_q;n}(mn)}.^{6} \tag{40}$$

Assume the Riemann hypothesis for the rank n non-abelian zeta function of $X/\mathbb{F}_q$. By (5), we have

$$\frac{1}{\alpha_{X,\mathbb{F}_q;n}(0)}P_{X,\mathbb{F}_q;n}(s) = \frac{T^{g-1}(1-T)(1-QT)}{\alpha_{X,\mathbb{F}_q;n}(0)}\cdot\widehat{Z}_{X,\mathbb{F}_q;n}(T) =$$

[6] This definition make sense, since $\alpha_{X,\mathbb{F}_q;n}(0) \geq \frac{q^{h^0(X,\mathcal{O}_X^{\oplus n})}-1}{\mathrm{Aut}\,\mathcal{O}_X^{\oplus n}} > 0$. In particular, $\alpha_{X/\mathbb{F}_q}(0) = \frac{q^{h^0(X,\mathcal{O}_X)}-1}{\#\mathrm{Aut}\,\mathcal{O}_X} = 1$.

$$=\left(\left(\sum_{m=0}^{g-2}\alpha'_{X,\mathbb{F}_q;n}(mn)\left(T^m+(QT)^{2(g-1)-m}\right)+\alpha'_{X,\mathbb{F}_q;n}(n(g-1))T^{g-1}\right)(1-(Q+1)T+QT^2)\right.$$
$$\left.+(Q-1)T^g\beta'_{X,\mathbb{F}_q;n}(0)\right)$$
$$=\prod_{i=1}^{g}(1-\omega_{X/\mathbb{F}_q,n;i}T)(1-\overline{\omega}_{X/\mathbb{F}_q,n;i}T).$$

where $\omega_{X/\mathbb{F}_q,n;i}$ denotes the reciprocal roots of $P_{X,\mathbb{F}_q;n}(s)$. Indeed, $\frac{1}{\alpha_{X,\mathbb{F}_q;n}(0)}P_{X,\mathbb{F}_q;n}(s)\in\mathbb{R}[T]$ is a degree $2g$ polynomial in T of real coefficients with leading coefficient Q^g and constant term 1. By the functional equation of $\widehat{Z}_{X,\mathbb{F}_q;n}(T)$, we may regroup all $2g$ reciprocal roots of $P_{X,\mathbb{F}_q;n}(s)$ into g pairs, within each pair of which the product of two elements is always equals to Q. Consequently, the Riemann hypothesis for rank n zeta function of $X/\mathbb{F}_q$ is equivalent to the condition that each such a pair is of the form $\left\{\omega_{X/\mathbb{F}_q,n;i},\ \overline{\omega}_{X/\mathbb{F}_q,n;i}\right\}$. This is certainly equivalent to the condition that

$$\left|\omega_{X/\mathbb{F}_q,n;i}\right|=Q^{\frac{1}{2}}\qquad(i=1,\dots g)\tag{41}$$

since $\omega_{X/\mathbb{F}_q,n;i}\cdot\overline{\omega}_{X/\mathbb{F}_q,n;i}=Q$.

Set now $a_{X,\mathbb{F}_q,n;i}=\omega_{X/\mathbb{F}_q,n;i}+\overline{\omega}_{X/\mathbb{F}_q,n;i}$. From Vieta's theorem between the reciprocal roots and coefficients of polynomials, by comparing the coefficients of T^i for $i=1,\dots,g$ in both sides of the above identity, under the Riemann hypothesis (41), we conclude that the follows hold:

$$\begin{cases}\left|\alpha'_{X,\mathbb{F}_q;n}(n)-(Q+1)\alpha'_{X,\mathbb{F}_q;n}(0)\right|=\left|\sum_{i=1}^g a_{n,i}\right|\leq 2g\sqrt{Q},\\ \left|\alpha'_{X,\mathbb{F}_q;n}(mn)-(Q+1)\alpha'_{X,\mathbb{F}_q;n}((m-1)n)+Q\alpha'_{X,\mathbb{F}_q;n}((m-2)n)\right| \\ \qquad\leq\sum_{1\leq i_1<i_2<\dots<i_m\leq 2g}Q^{m/2}=\binom{2g}{m}Q^{m/2} & (2\leq m\leq g-1)\\ \left|(Q-1)\beta'_{X,\mathbb{F}_q;n}(0)-(Q+1)\alpha'_{X,\mathbb{F}_q;n}((g-1)n)+Q\alpha'_{X,\mathbb{F}_q;n}((g-2)n)\right| \\ \qquad\leq\sum_{1\leq i_1<i_2<\dots<i_g\leq 2g}Q^{g/2}=\binom{2g}{g}Q^{g/2}\end{cases}$$

Now expand each absolute value inequality, say $|\kappa|<c$ as $-c<\kappa<c$, first, and then add three consecutive lower and upper bounds, we obtain the follows

$$\begin{cases}-2g\sqrt{Q}\leq\alpha'_{X,\mathbb{F}_q;n}(n)-(Q+1)\alpha'_{X,\mathbb{F}_q;n}(0)\leq 2g\sqrt{Q},\\ -\sum_{k=3}^{m+1}Q^{k-3}\sum_{i=1}^k\binom{2g}{i-1}Q^{(i-1)/2}-2gQ^{m-1}\sqrt{Q}\\ \qquad\leq\alpha'_{X,\mathbb{F}_q;n}(mn)-\left(Q^m+\dots+Q^2+Q+1\right)\alpha'_{X,\mathbb{F}_q;n}(0) & (2\leq m\leq g-1)\\ \qquad\qquad\leq\sum_{k=3}^{m+1}Q^{k-3}\sum_{i=1}^k\binom{2g}{i-1}Q^{(i-1)/2}+2gQ^{m-1}\sqrt{Q},\\ -\sum_{k=3}^{g+1}Q^{k-3}\sum_{i=1}^k\binom{2g}{i-1}Q^{(i-1)/2}-2gQ^{g-1}\sqrt{Q}\\ \qquad\leq(Q-1)\beta'_{X,\mathbb{F}_q;n}(0)-\left(Q^g+\dots+Q^2+Q+1\right)\alpha'_{X,\mathbb{F}_q;n}(0)\\ \qquad\qquad\leq\sum_{k=3}^{g+1}Q^{k-3}\sum_{i=1}^k\binom{2g}{i-1}Q^{(i-1)/2}+2gQ^{g-1}\sqrt{Q}.\end{cases}$$

But, by definition, $\alpha'_{X,\mathbb{F}_q;n}(0)=1$. Therefore, we have proved the following:

Theorem 3.1 (Bounds of non-abelian invariants). *Assume the Riemann Hypothesis for the rank two zeta functions of a projective regular curve X over $\mathbb{F}_q$ of genus g, we have for the invariants $\beta_{X,\mathbb{F}_q;n}(0)$ and $\alpha'_{X,\mathbb{F}_q;n}(2m)$ $(m = 1, \ldots, g-1)$*

$$\begin{cases} -2g\sqrt{Q} \leq \alpha'_{X,\mathbb{F}_q;n}(n) - (Q+1) \leq 2g\sqrt{Q} \\ \\ -\sum_{k=3}^{m+1} Q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} Q^{(i-1)/2} - 2gQ^{m-1}\sqrt{Q} \\ \qquad\qquad \leq \alpha'_{X,\mathbb{F}_q;n}(mn) - \Big(Q^m + \ldots + Q^2 + Q + 1\Big) \qquad\qquad (2 \leq m \leq g) \\ \qquad\qquad\qquad\qquad \leq \sum_{k=3}^{m+1} Q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} Q^{(i-1)/2} + 2gQ^{m-1}\sqrt{Q} \\ \\ -\sum_{k=3}^{g+1} Q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} Q^{(i-1)/2} - 2gQ^{g-1}\sqrt{Q} \\ \qquad\qquad \leq (Q-1)\beta'_{X,\mathbb{F}_q;n}(0) - \Big(Q^g + \ldots + Q^2 + Q + 1\Big) \\ \qquad\qquad\qquad\qquad \leq \sum_{k=3}^{g+1} Q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} Q^{(i-1)/2} + 2gQ^{g-1}\sqrt{Q} \end{cases}$$

Example 3.1 (Brill-Noether Loci). Even when $n = 1$, the above result exposes some intrinsic geo-arithmetic properties of the curve $X/\mathbb{F}_q$. Indeed, we may introduce the Brill-Noether loci within the degree Picard group $\mathrm{Pic}^d(X)$ of X by setting

$$W^{\geq i}_{X/\mathbb{F}_q}(d) := \{L \in \mathrm{Pic}^d(X) : h^0(X,L) \geq i\} \quad \text{and} \quad W^{=i}_{X/\mathbb{F}_q}(d) := \{L \in \mathrm{Pic}^d(X) : h^0(X,L) = i\}. \tag{42}$$

The $W^{\geq i}_{X/\mathbb{F}_q}(d)$'s induce a natural stratification structure on $\mathrm{Pic}^d(X)$ since

$$W^{\geq i}_{X/\mathbb{F}_q}(d) = \bigsqcup_{j \geq i} W^{=j}_{X/\mathbb{F}_q}(d) \qquad \text{and} \qquad W^{\geq 0}_{X/\mathbb{F}_q}(d) = \mathrm{Pic}^d(X). \tag{43}$$

It is natural to ask what are the topological or better motivic properties of these refined structures. Set accordingly

$$w^{\geq i}_{X/\mathbb{F}_q}(d) = \# W^{\geq i}_{X/\mathbb{F}_q}(d) \qquad \text{and} \qquad w^{=i}_{X/\mathbb{F}_q}(d) = \# W^{=i}_{X/\mathbb{F}_q}(d). \tag{44}$$

and

$$\alpha^{\geq i}_{X/\mathbb{F}_q}(d) = \sum_{L \in W^{\geq i}_{X/\mathbb{F}_q}(d)} \frac{q^{h^0(X,L)-1}}{q-1} \quad \text{and} \quad \alpha^{=i}_{X/\mathbb{F}_q}(d) = \sum_{L \in W^{=i}_{X/\mathbb{F}_q}(d)} \frac{q^{h^0(X,L)-1}}{q-1} = \frac{q^i-1}{q-1} w^{=i}_{X/\mathbb{F}_q}(d). \tag{45}$$

Then

$$w^{\geq 0}_{X/\mathbb{F}_q}(d) = \sum_{i \geq 0} w^{=i}_{X/\mathbb{F}_q}(d) = \#\mathrm{Pic}^0(X) \quad \text{and} \quad \alpha_{X/\mathbb{F}_q}(d) = \alpha^{\geq 0}_{X/\mathbb{F}_q}(d) = \sum_{i \geq 0} \frac{q^i-1)}{q-1} w^{=i}_{X/\mathbb{F}_q}(d), \tag{46}$$

which by Theorem 3.1, is controlled by

$$\begin{cases} -2g\sqrt{q} \leq \alpha_{X/\mathbb{F}_q}(1) - (q+1) \leq 2g\sqrt{q} \\ -\sum_{k=3}^{m+1} q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} q^{(i-1)/2} - 2gq^{m-1}\sqrt{q} \\ \qquad \leq \alpha'_{X/\mathbb{F}_q}(m) - \left(q^m + \ldots + q^2 + q + 1\right) \qquad (2 \leq m \leq g) \\ \qquad\qquad \leq \sum_{k=3}^{m+1} q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} q^{(i-1)/2} + 2gq^{m-1}\sqrt{q} \\ -\sum_{k=3}^{g+1} q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} q^{(i-1)/2} - 2gq^{g-1}\sqrt{q} \\ \qquad \leq (q-1)\beta_{X/\mathbb{F}_q}(0) - \left(q^g + \ldots + q^2 + q + 1\right) \\ \qquad\qquad \leq \sum_{k=3}^{g+1} q^{k-3} \sum_{i=1}^{k} \binom{2g}{i-1} q^{(i-1)/2} + 2gq^{g-1}\sqrt{q} \end{cases} \tag{47}$$

since $\alpha_{X/\mathbb{F}_q}(0) = 1$.

Recall that, by definition,

$$\beta_{X/\mathbb{F}_q}(0) = \frac{1}{q-1}\#\mathrm{Pic}^0(X) = \widehat{v}_1 = \frac{1}{q-1}\prod_{i=1}^{g}(1-\omega_{X/F_q,i})(1-\overline{\omega}_{X/F_q,i}). \tag{48}$$

Hence it admits a natural bound

$$0 < \beta_{X/\mathbb{F}_q}(0) \leq \frac{1}{q-1}\prod_{i=1}^{g}(1+\sqrt{q})(1+\sqrt{q}) \tag{49}$$

So, at least for the β-invariant, (47) may not be the sharpest bounds.

Furthermore, we may introduce natural associated invariants

$$\alpha_{X/\mathbb{F}_q}^{\geq i}(d) := \sum_{V \in W_{X/\mathbb{F}_q}^{\geq i}(d)} \frac{q^{h^0(X,L)}-1}{q-1} = \sum_{j \geq i} w_{X/\mathbb{F}_q}^{=j}(d)\frac{q^j-1}{q-1} \tag{50}$$

and their associated generating function, for fixed i and d,

$$A_{X/\mathbb{F}_q}(u,v) := \sum_{i,d \geq 0} a_{X/\mathbb{F}_q}^{=i}(d) \cdot u^i v^d. \tag{51}$$

Recall that, by the vanishing theorem, we have

$$\alpha_{X/\mathbb{F}_q}^{=i}(d) = \delta_{i,d-(g-1)}\frac{q^{d-(g-1)}-1}{q-1}\#\mathrm{Pic}^0(X) \qquad d \geq g \tag{52}$$

So, $A_{X/\mathbb{F}_q}(u,v)$ is indeed a rational function of u and v satisfying standard functional equation, thanks to the duality theorem.

Obviously, there is a natural generalization of this discussion for rank n semi-stable bundles. We expect that the non-abelian motivic structures of $X/\mathbb{F}_q$ may be understood via the rank n generating function

$$B(X,\mathbb{F}_q;u,v;z) := \sum_{n \geq 0} A(X,\mathbb{F}_q;n;u,v)z^n = \sum_{n,i,m \geq 0} \alpha_{X,\mathbb{F}_q;n}^{=i}(mn)u^i v^{mn} z^n. \tag{53}$$

Similarly, by the vanishing theorem, we have

$$\alpha_{X/\mathbb{F}_q}^{=i}(mn)=\delta_{i,mn-n(g-1)}\frac{q^{d-n(g-1)}-1}{q-1}\#M_{X,\mathbb{F}_q;n}(0)\qquad d\geq ng \tag{54}$$

where $M_{X,\mathbb{F}_q;n}(0)$ denotes the space of isomorphism classes, or better by including the geo-arithmetic structures, the moduli stack of semi-stable vector bundles of rank n and degree mn on X rationally over $\mathbb{F}_q$. So, $B_{X/\mathbb{F}_q}(u,v;z)$ is indeed a rational function of u and v satisfying standard functional equation, thanks to the duality theorem. For details, please refer the final section on motivic structures of curves over general base fields.

We end this discussion with the following application of the Riemann hypothesis.

Lemma 3.2. *For a regular projective curve X of genus g over $\mathbb{F}_q$, we have*

$$\left(\frac{\beta_{X,\mathbb{F}_q;2}(0)}{\widehat{\nu}_1^2}-\frac{1}{q+1}\right)>0.\qquad \text{In particular,}\qquad \frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}>\frac{q}{q^2-1} \tag{55}$$

Slight differently,

$$\frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}>\frac{3}{2}\frac{1}{q-1}. \tag{56}$$

Proof. Note that

$$\left(\frac{\beta_{X,\mathbb{F}_q;2}(0)}{\widehat{\nu}_1^2}-\frac{1}{q+1}\right)=\frac{\widehat{\nu}_2+\frac{1}{1-q^2}\widehat{\nu}_1^2}{\widehat{\nu}_1^2}-\frac{1}{q+1}=\frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}-\frac{q}{q^2-1}$$

Now

$$\begin{aligned}
\frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}=&\left(\prod_{i=1}^g\frac{(1-\omega_{X/\mathbb{F}_q;i}q^{-2})(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-2})}{(1-\omega_{X/\mathbb{F}_q;i}q^{-1})^2(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-1})^2}\right)\frac{q^{-2(g-1)}(q-1)^2}{q^{-2(g-1)}(1-q^-2)(1-qq^{-2})}\\
=&\frac{1}{q^g}\left(\prod_{i=1}^g\frac{(\overline{\omega}_{X/\mathbb{F}_q;i}-q^{-1})(\omega_{X/\mathbb{F}_q;i}-q^{-1})}{(1-\omega_{X/\mathbb{F}_q;i}q^{-1})(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-1})}\frac{1}{(1-\omega_{X/\mathbb{F}_q;i}q^{-1})(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-1})}\right)\frac{q^3}{q+1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\text{since }\omega_{X/\mathbb{F}_q;i}\overline{\omega}_{X/\mathbb{F}_q;i}=q)\\
=&\left(\prod_{i=1}^g\frac{(\overline{\omega}_{X/\mathbb{F}_q;i}-q^{-1})(\omega_{X/\mathbb{F}_q;i}-q^{-1})}{(1-\omega_{X/\mathbb{F}_q;i}q^{-1})(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-1})}\right)\frac{q^2}{q^2-1}\cdot\widehat{\nu}_1\geq\left(\prod_{i=1}^g 1\right)\frac{q^2}{q^2-1}\cdot\widehat{\nu}_1\\
&\qquad\qquad\text{by Yoshida's lemma, since }|q^{-1}|<1\ (\text{and }\widehat{\nu}_1>0\text{ and }\frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}>0)\\
=&\frac{q^2}{q^2-1}\cdot\widehat{\nu}_1=\frac{q^2}{(q^2-1)(q-1)}\cdot\#\mathrm{Pic}^0(X)>\frac{q}{q^2-1}
\end{aligned}$$

where in the last step, we have used the fact that there exists at least one $\mathbb{F}_q$ rational point in $\mathrm{Pic}^0(X)$. This verifies (55).

In addition, since $\#\mathrm{Pic}^0(X) \geq 2$, we have $2q^2\#\mathrm{Pic}^0(X) + 3 > 3q^2$. This implies that $\frac{q^2}{q^2-1}\#\mathrm{Pic}^0(X) > \frac{3}{2}$. Therefore,

$$\frac{\widehat{v}_2}{\widehat{v}_1^2} > \frac{q^2}{(q^2-1)(q-1)}\#\mathrm{Pic}^0(X) > \frac{3}{2}\frac{1}{q-1}, \tag{57}$$

as wanted. □

This result will be used in the next section to prove the rank three Riemann hypothesis.

4 Riemann hypothesis for rank three zeta of a curve over a finite field

4.1 Decompose rank three zeta

Let X be a regular projective curve of genus g over a finite field $\mathbb{F}_q$. By the special uniformity of zetas, we have, (up to a constant factor depending only on n and g),

$$\begin{aligned}
\widehat{\zeta}_{X,\mathbb{F}_q;n} =&\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_n}(s)\\
=&\sum_{a=1}^{n}\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{ns-n+a+k_p})}\\
&\times\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{-ns+n-a+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{r-1}(1-q^{l_j+l_{j+1}})}
\end{aligned}$$

In particular, when $n=3$, we have

$$\begin{aligned}
\widehat{\zeta}_{X,\mathbb{F}_q;3}(s) =&\widehat{\zeta}_{X,\mathbb{F}_q}^{\mathrm{SL}_3}(s)\\
=&\left(\frac{\widehat{v}_1^2}{(1-q^2)(1-q^{3s-1})}+\frac{\widehat{v}_2}{(1-q^{3s})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s-2)\\
&+\frac{\widehat{v}_1^2}{(1-q^{3s})(1-q^{-3s+3})}\widehat{\zeta}_{X,\mathbb{F}_q}(3s-1)\\
&+\left(\frac{\widehat{v}_2}{(1-q^{-3s+3})}+\frac{\widehat{v}_1^2}{(1-q^2)(1-q^{-3s+2})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s)\\
=&\sum_{a=1}^{3}\widehat{\zeta}_{X/\mathbb{F}_q}^{[a]}(s) =: \widehat{\zeta}_{X/\mathbb{F}_q;3}^{\leq 2}+\widehat{\zeta}_{X/\mathbb{F}_q;3}^{\geq 2}\\
=&\widehat{\zeta}_{X/\mathbb{F}_q;3}^{\geq 2}(s)\left(1+\frac{\widehat{\zeta}_{X/\mathbb{F}_q;3}^{\leq 2}(s)}{\widehat{\zeta}_{X/\mathbb{F}_q;3}^{\geq 2}(s)}\right)
\end{aligned}$$

Here, as before,

$$\begin{aligned}
\widehat{\zeta}^{[1]}_{X/\mathbb{F}_q}(s) &:= \left(\frac{\widehat{v}_1^2}{(1-q^2)(1-q^{3s-1})} + \frac{\widehat{v}_2}{(1-q^{3s})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s-2)\\
\widehat{\zeta}^{[2]}_{X/\mathbb{F}_q}(s) &:= \frac{\widehat{v}_1^2}{(1-q^{3s})(1-q^{-3s+3})}\widehat{\zeta}_{X,\mathbb{F}_q}(3s-1)\\
\widehat{\zeta}^{[3]}_{X/\mathbb{F}_q}(s) &:= \left(\frac{\widehat{v}_2}{(1-q^{-3s+3})} + \frac{\widehat{v}_1^2}{(1-q^2)(1-q^{-3s+2})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s)
\end{aligned}$$

and we have used the following definition

$$\begin{cases}
\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s) := \frac{1}{2}\widehat{\zeta}^{[2]}_{X/\mathbb{F}_q}(s) + \widehat{\zeta}^{[3]}_{X/\mathbb{F}_q}(s)\\
\qquad = \frac{1}{2}\frac{\widehat{v}_1^2}{(1-q^{3s})(1-q^{-3s+3})}\widehat{\zeta}_{X,\mathbb{F}_q}(3s-1) + \left(\frac{\widehat{v}_2}{(1-q^{-3s+3})} + \frac{\widehat{v}_1^2}{(1-q^2)(1-q^{-3s+2})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s)\\
\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(s) := \widehat{\zeta}^{[1]}_{X/\mathbb{F}_q}(s) + \frac{1}{2}\widehat{\zeta}^{[2]}_{X/\mathbb{F}_q}(s)\\
\qquad = \left(\frac{\widehat{v}_1^2}{(1-q^2)(1-q^{3s-1})} + \frac{1}{2}\frac{\widehat{v}_2}{(1-q^{3s})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s-2) + \frac{\widehat{v}_1^2}{(1-q^{3s})(1-q^{-3s+3})}\widehat{\zeta}_{X,\mathbb{F}_q}(3s-1)
\end{cases} \tag{58}$$

Our strategy, motivated by [10] where a similar result is proved for rank three zeta of the field of rational, to prove the Riemann Hypothesis for $\widehat{\zeta}_{X/\mathbb{F}_q;3}(s)$ is first show the following

Proposition 4.1 (Riemann Hypothesis for $\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)$)**.** *For a regular projective curve X over $\mathbb{F}_q$, all zeros of $\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)$ lie on the line $\Re(s) = \frac{1}{3}$.*

Then based on this proposition, we prove the following

Theorem 4.2. *For a regular projective curve X over $\mathbb{F}_q$, there is no zero of $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$ lies in the half plane $\Re(s) < \frac{1}{2}$.*

Assuming this theorem, easily we have the following

Theorem 4.3 (Riemann Hypothesis in Rank Three)**.** *For a regular projective curve X over $\mathbb{F}_q$, all zeros of the rank three non-abelian zeta $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$ and SL_3-zeta $\widehat{\zeta}^{\mathrm{SL}_3}_{X,\mathbb{F}_q}(s)$ of $X/\mathbb{F}_q$ lie on the line $\Re(s) = \frac{1}{2}$.*

Proof. This is a direct consequence of Theorem 4.2. Indeed, the functional equation claims that

$$\widehat{\zeta}_{X,\mathbb{F}_q;3}(1-s) = \widehat{\zeta}_{X,\mathbb{F}_q;3}(s). \tag{59}$$

Hence, there is no zero of $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$ lies in the half plane $\Re(s) > \frac{1}{2}$ as well. Therefore, all zeros of rank three zeta $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$ lies on the line $\Re(s) = \frac{1}{2}$. □

4.2 Estimation on the ratio $\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}$ when $a+b=n+1$

In this subsection, we will establish the following remarkable

Proposition 4.4. *Let X be a regular projective curve over $\mathbb{F}_q$. Then for any integers a and b satisfying the condition that $a+b=n+1$, we have*

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}\right| \begin{cases} >1 & \text{if} \quad |q^{n\sigma}|<1 \\ <1 & \text{if} \quad |q^{n\sigma}|>1. \end{cases} \tag{60}$$

where σ is defined by $s=\frac{1}{2}+\sigma$.

Proof. The point is to apply the functional equation for the abelian zeta function

$$\widehat{\zeta}_{X/\mathbb{F}_q}(1-s)=\widehat{\zeta}_{X/\mathbb{F}_q}(s)=\frac{\prod_{i=1}^g(1-\omega_{X/\mathbb{F}_q;i}q^{-s})(1-\overline{\omega}_{X/\mathbb{F}_q;i}q^{-s})}{q^{-s(g-1)}(1-q^{-s})(1-q^{1-s})} \tag{61}$$

to the abelian zeta factor $\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+b)$ appeared in the denominator of $\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}$ when $a+b=n+1$. Accordingly, for $s=\frac{1}{2}+\sigma$, we have

$$\begin{aligned}
&\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)\cdot q^{(1-ns+n-b)(g-1)}}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)\cdot q^{(ns-n+a)(g-1)}}=\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-n\sigma+\frac{n}{2}-b)}\cdot q^{(1-2n\sigma+n-(a+b))(g-1)}\\
=&\left(\prod_{i=1}^g\frac{(1-\omega_{X/\mathbb{F}_q,i}q^{-n\sigma+\frac{n}{2}-a})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-n\sigma+\frac{n}{2}-a})}{(1-\omega_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})}\right)\frac{(1-q^{-1+n\sigma-\frac{n}{2}+b})(1-q^{n\sigma-\frac{n}{2}+b})}{(1-q^{-n\sigma+\frac{n}{2}-a)})(1-q^{1-n\sigma+\frac{n}{2}-a})}\\
=&q^{-2n\sigma(g-1)}\left(\prod_{i=1}^g\frac{(q^{n\sigma}-\omega_{X/\mathbb{F}_q,i}q^{\frac{n}{2}-a})(q^{n\sigma}-\overline{\omega}_{X/\mathbb{F}_q,i}q^{\frac{n}{2}-a})}{(1-\omega_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})}\right)\\
&\qquad\times\frac{(1-q^{-1+n\sigma-\frac{n}{2}+b})(1-q^{n\sigma-\frac{n}{2}+b})}{(q^{n\sigma}-q^{\frac{n}{2}-a)})(q^{n\sigma}-q^{1+\frac{n}{2}-a})}
\end{aligned}$$

To estimate this latest expression, we next give a the following generalization of Lemma 2.1 of Yoshida.

Lemma 4.5. *Let q and κ be real numbers satisfying $q>1$ and $\kappa\geq 0$. For any complex number α,β satisfying $\alpha\beta=q$ and $|\alpha+\beta|\leq q+1$, we have,*

$$|w-\alpha q^\kappa|\cdot|w-\beta q^\kappa|=|1-\alpha q^\kappa w|\cdot|1-\beta q^\kappa w|\cdot\begin{cases} >1 & \text{if } |w|<1 \\ <1 & \text{if } |w|>1 \end{cases} \tag{62}$$

In particular, when $\kappa=0$, we recover Yoshida's inequality:

$$|w-\alpha|\cdot|w-\beta|=|1-\alpha w|\cdot|1-\beta w|\cdot\begin{cases} >1 & \text{if} \quad |w|<1 \\ <1 & \text{if} \quad |w|>1. \end{cases} \tag{63}$$

Proof. We start with the following

Sublemma 4.6. *For a fixed real number $q > 1$, as a function of x in the region $x \geq 0$,*

$$f_q(x) := q^{2x+1} + 1 - q^x(q+1) \geq 0 \tag{64}$$

Proof. Clearly, $f(0) = 0$. Hence it suffices to verify that

$$f'(x) = \Big(2q^{2x+1} - q^x(q+1)\Big)\log q = q^x \log q \cdot \Big(2q^{x+1} - (q+1)\Big) > 0. \tag{65}$$

But this is a direct consequence of the fact that for $g(x) = 2q^{x+1} - (q+1)$,

$$g'(x) = 2q^{x+1}\log q > 0 \tag{66}$$

since $g(0) = q - 1 > 0$. ◻

Back to the proof of the lemma. We follow Yoshida's approach closely. By a direct calculation,

$$|w - \alpha q^\kappa|^2 |w - \beta q^\kappa|^2 = \Big(w^2 - (\alpha+\beta)q^\kappa w + \alpha\beta q^{2\kappa}\Big)\Big(\overline{w}^2 - (\alpha+\beta)q^\kappa \overline{w} + \alpha\beta q^{2\kappa}\Big)$$
$$= |w|^4 - (\alpha+\beta)q^\kappa |w|^2 (w + \overline{w}) + (\alpha\beta)q^{2\kappa}(w^2 + \overline{w}^2) \tag{a}$$
$$+ (\alpha+\beta)^2 q^{2\kappa}|w|^2 - (\alpha+\beta)(\alpha\beta)q^{3\kappa}(w+\overline{w}) + (\alpha\beta)^2 q^{4\kappa}$$
$$|1 - \alpha q^\kappa w|^2 |1 - \beta q^\kappa w|^2 = \Big(1 - (\alpha+\beta)q^\kappa w + \alpha\beta q^{2\kappa}w^2\Big)\Big(1 - (\alpha+\beta)q^\kappa \overline{w} + \alpha\beta q^{2\kappa}\overline{w}^2\Big)$$
$$= (\alpha\beta)^2 q^{4\kappa}|w|^4 - (\alpha+\beta)(\alpha\beta)q^{3\kappa}|w|^2(w+\overline{w}) + (\alpha\beta)q^{2\kappa}(w^2+\overline{w}^2) \tag{b}$$
$$+ (\alpha+\beta)^2 q^{2\kappa}|w|^2 - (\alpha+\beta)q^\kappa(w+\overline{w}) + 1$$

Subtracting (b) from (a), we get

$$|w - \alpha q^\kappa|^2 |w - \beta q^\kappa|^2 - |1 - \alpha q^\kappa w|^2 |1 - \beta q^\kappa w|^2$$
$$= \Big(1 - (\alpha\beta)q^{2\kappa}\Big)\Big(|w|^2 - 1\Big)\Big(\Big(1 + (\alpha\beta)q^{2\kappa}\Big)\Big(|w|^2 + 1\Big) - (\alpha+\beta)q^\kappa(w+\overline{w})\Big)$$

Set now $|w| = r$. We claim that

$$\Big(1+(\alpha\beta)q^{2\kappa}\Big)\Big(r^2+1\Big) - 2r(\alpha+\beta)q^\kappa = \Big(1+q^{2\kappa+1}\Big)\Big(r^2+1\Big) - 2r(\alpha+\beta)q^\kappa > 0 \qquad (r \neq 1) \tag{67}$$

since the discriminant of this degree two polynomial with real coefficients in r is given by

$$\Delta := 4(\alpha+\beta)^2 q^{2\kappa} - 4(1+q^{2\kappa+1})^2 \leq 4\Big((q+1)^2 q^{2\kappa} - \Big(1 + 2q^{2\kappa+1} + q^{4\kappa+2}\Big)\Big)$$
$$= 4\Big((q^2+2q+1)q^{2\kappa} - \Big(1 + 2q^{2\kappa+1} + q^{4\kappa+2}\Big)\Big) = 4\Big((q^2+1)q^{2\kappa} - \Big(1 + q^{4\kappa+2}\Big)\Big)$$
$$= 4 f_{q^2}(2\kappa) \leq 0$$

by Sublemma 4.6. Indeed, when $D < 0$, the claim is trivial since the leading coefficient $(1 + q^{2\kappa+1})$ is strictly positive. In addition, even if this discriminant is zero, which is equivalent to $\kappa = 0$ and $\alpha + \beta = q + 1$, (67) holds as well, since the degree two polynomial becomes $(q+1)(r-1)^2$, which is strictly positive when $r \neq 1$. ◻

Back to the proof of Proposition 4.4. By the calculation above Lemma **??**, we have

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}\right| = \left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-n\sigma+\frac{n}{2}-b)}\right|$$
$$= \left|q^{-2n\sigma(g-2)}\cdot q^{-(1+n-(a+b))(g-1)}\right|$$
$$\times\left(\prod_{i=1}^{g}\left|\frac{(q^{n\sigma}-\omega_{X/\mathbb{F}_q,i}q^{\frac{n}{2}-a})(q^{n\sigma}-\overline{\omega}_{X/\mathbb{F}_q,i}q^{\frac{n}{2}-a})}{(1-\omega_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})}\right|\right)\cdot\left|\frac{(1-q^{-1+n\sigma-\frac{n}{2}+b})(1-q^{n\sigma-\frac{n}{2}+b)})}{(q^{n\sigma}-q^{\frac{n}{2}-a)})(q^{n\sigma}-q^{1+\frac{n}{2}-a)})}\right|$$

Now apply Lemma 4.5 to each pair of factors of the numerator within the product $\prod_{i=1}^{g}$, namely, with the parameters $\alpha=\omega_{X/\mathbb{F}_q,i}$, $\beta=\overline{\omega}_{X/\mathbb{F}_q,i}$ $(i=1,\dots,g)$, $\kappa=\frac{n}{2}-a$ and $w=q^{n\sigma}$, which is applicable thanks to the Riemann hypothesis, or better, the corresponding theorem of Weil, for the Artin zeta function $\widehat{\zeta}_{X/\mathbb{F}_q}(s)$ of the curve $X/\mathbb{F}_q$, we have, for $g\geq 2$,

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}\right| = \left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-n\sigma+\frac{n}{2}-b)}\right|$$
$$= \left|q^{-(1+n-(a+b))(g-1)}\right|$$
$$\times\left(\prod_{i=1}^{g}\left|\frac{(1-\omega_{X/\mathbb{F}_q,i}q^{n\sigma+\frac{n}{2}-a})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{n\sigma+\frac{n}{2}-a})}{(1-\omega_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-1+n\sigma-\frac{n}{2}+b})}\right|\right)\cdot\left|\frac{(1-q^{-1+n\sigma-\frac{n}{2}+b})(1-q^{n\sigma-\frac{n}{2}+b)})}{(q^{n\sigma}-q^{\frac{n}{2}-a)})(q^{n\sigma}-q^{1+\frac{n}{2}-a)})}\right|$$
$$\times\begin{cases}>1 & \text{if} \quad |q^{n\sigma}|<1\\ <1 & \text{if} \quad |q^{n\sigma}|>1.\end{cases}$$

In particular, when $a+b=n+1$, we have $\frac{n}{2}-a=-1-\frac{n}{2}+b$ and hence the denominator and numerators are identical for each pair of factors within the product $\prod_{i=1}^{g}$. Consequently,

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}\right| = \left|\frac{(1-q^{n\sigma+\frac{n}{2}-a})(1-q^{n\sigma+\frac{n}{2}-a+1)})}{(q^{n\sigma}-q^{\frac{n}{2}-a)})(q^{n\sigma}-q^{1+\frac{n}{2}-a)})}\right|\cdot\begin{cases}>1 & \text{if} \quad |q^{n\sigma}|<1\\ <1 & \text{if} \quad |q^{n\sigma}|>1.\end{cases}$$

Now, by applying Lemma 4.5 again to the numerators involved, with parameters $\alpha=1$, $b=q$, $\kappa=\frac{n}{2}-a$ and $w=q^{n\sigma}$, we get

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(1-ns+n-b)}\right|\begin{cases}>1 & \text{if} \quad |q^{n\sigma}|<1\\ <1 & \text{if} \quad |q^{n\sigma}|>1.\end{cases} \qquad (a+b=n+1,\ s=:\frac{1}{2}+\sigma) \quad (68)$$

as wanted. □

4.3 Estimation on the ratio $\frac{\widehat{\zeta}^{[a]}_{X,\mathbb{F}_q;n}(s)}{\widehat{\zeta}^{[b]}_{X,\mathbb{F}_q;n}(s)}$ when $b = a+1$

By definition,

$$\frac{\widehat{\zeta}^{[a]}_{X,\mathbb{F}_q;n}(s)}{\widehat{\zeta}^{[b]}_{X,\mathbb{F}_q;n}(s)} = \frac{\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-a}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \frac{1}{(1-q^{-n+a+k_p}/T)} \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=a-1}} \frac{1}{(1-Tq^{n-a+1+l_1})} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}{\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=n-b}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})} \frac{1}{(1-q^{-n+b+k_p}/T)} \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=b-1}} \frac{1}{(1-Tq^{n-b+1+l_1})} \frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}$$
$$\times \frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n-b)}$$

Write $s = \frac{1}{2}+\sigma$, then the zeta factor becomes

$$\begin{aligned}
&\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(ns-n+b)} = \frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+b)}\\
=&\frac{q^{-(n\sigma-\frac{n}{2}+b)(g-1)}}{q^{-(n\sigma-\frac{n}{2}+a)(g-1)}}\\
&\times\left(\prod_{i=1}^{g}\frac{(1-\omega_{X/\mathbb{F}_q,i}q^{-(n\sigma-\frac{n}{2}+a)})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-(n\sigma-\frac{n}{2}+a)})}{(1-\omega_{X/\mathbb{F}_q,i}q^{-(n\sigma-\frac{n}{2}+b)})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-(n\sigma-\frac{n}{2}+b)})}\right)\cdot\frac{(1-q^{-(n\sigma-\frac{n}{2}+b)})(1-q^{1-(n\sigma-\frac{n}{2}+b)})}{(-q^{-(n\sigma-\frac{n}{2}+a)})(1-q^{1-(n\sigma-\frac{n}{2}+a)})}\\
=&\left(\prod_{i=1}^{g}\frac{(1-\omega_{X/\mathbb{F}_q,i}q^{-n\sigma+\frac{n}{2}-a})(1-\overline{\omega}_{X/\mathbb{F}_q,i}q^{-n\sigma+\frac{n}{2}-a})}{(\overline{\omega}_{X/\mathbb{F}_q,i}-q^{-n\sigma+\frac{n}{2}-a})(\omega_{X/\mathbb{F}_q,i}-q^{-n\sigma+\frac{n}{2}-a})}\right)\cdot\frac{(q-q^{-n\sigma+\frac{n}{2}-a})}{(1-q^{1-n\sigma+\frac{n}{2}-a)})}\\
&\qquad(\text{since } b=a+1 \quad\text{and}\quad \omega_{X/\mathbb{F}_q,i}\overline{\omega}_{X,\mathbb{F}_q}=q\quad\forall i=1,\dots,g)
\end{aligned}$$

Therefore, by applying Yoshida's lemma to the factors in the denominator with the parameter $\alpha=\omega_{X/\mathbb{F}_q,i}$, $\beta=\overline{\omega}_{X/\mathbb{F}_q,i}$ and $w=q^{-n\sigma+\frac{n}{2}-a}$, we have

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+b)}\right| = \left|\frac{(q-q^{-n\sigma+\frac{n}{2}-a})}{(1-q^{1-n\sigma+\frac{n}{2}-a)})}\right|\cdot\begin{cases}>1 & \left|q^{-n\sigma+\frac{n}{2}-a}\right|>1\\ <1 & \left|q^{-n\sigma+\frac{n}{2}-a}\right|<1\end{cases}\quad(b=a+1)\quad(69)$$

That is to say, we have proved the following

Proposition 4.7. *For a regular projective curve X over $\mathbb{F}_q$, we have*

$$\left|\frac{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+a)}{\widehat{\zeta}_{X,\mathbb{F}_q}(n\sigma-\frac{n}{2}+b)}\right| = \left|\frac{(q-q^{-n\sigma+\frac{n}{2}-a})}{(1-q^{1-n\sigma+\frac{n}{2}-a)})}\right|\cdot\begin{cases}>1 & |q^{n\sigma}|<\left|q^{\frac{n}{2}-a}\right|\\ <1 & |q^{n\sigma}|>\left|q^{\frac{n}{2}-a}\right|\end{cases}\quad(b=a+1)\quad(70)$$

This implies that, when $b = a + 1$,

$$\left|\frac{\widehat{\zeta}^{[a]}_{X,\mathbb{F}_q;n}(s)}{\widehat{\zeta}^{[b]}_{X,\mathbb{F}_q;n}(s)}\right|$$

$$=\left|\frac{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{-n+a+k_p+\frac{n}{2}+n\sigma})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{-\frac{n}{2}-n\sigma+n-a+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a-1}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{-n+a+1+k_p+\frac{n}{2}+n\sigma})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a}}\frac{1}{(1-q^{-\frac{n}{2}-n\sigma+n-a+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}\right|$$

$$\times\left|\frac{(q-q^{-n\sigma+\frac{n}{2}-a})}{(1-q^{1-n\sigma+\frac{n}{2}-a)})}\right|\cdot\begin{cases}>1 & |q^{n\sigma}|<\left|q^{\frac{n}{2}-a}\right|\\ <1 & |q^{n\sigma}|>\left|q^{\frac{n}{2}-a}\right|\end{cases}$$

$$=\left|\frac{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(n\sigma-\frac{n}{2}+a)+k_p})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{-(n\sigma-\frac{n}{2}+a)+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a-1}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(n\sigma-\frac{n}{2}+a)+1+k_p})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a}}\frac{1}{(1-q^{-(n\sigma-\frac{n}{2}+a)+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}\right|$$

$$\times\left|\frac{(q-q^{-(n\sigma-\frac{n}{2}+a)})}{(1-q^{1-(n\sigma-\frac{n}{2}+a)})}\right|\cdot\begin{cases}>1 & |q^{n\sigma}|<\left|q^{\frac{n}{2}-a}\right|\\ <1 & |q^{n\sigma}|>\left|q^{\frac{n}{2}-a}\right|\end{cases}$$

So we are lead to consider the norm of the following rational function in the first big factor

$$r_{X/\mathbb{F}_q;n,a}(\sigma)$$

$$:=\frac{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(n\sigma-\frac{n}{2}+a)+k_p})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{-(n\sigma-\frac{n}{2}+a)+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}{\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a-1}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(n\sigma-\frac{n}{2}+a)+1+k_p})}\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a}}\frac{1}{(1-q^{-(n\sigma-\frac{n}{2}+a)+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}}$$

$$=\frac{f_{X/\mathbb{F}_q;n,a}(\sigma)\cdot g_{X/\mathbb{F}_q;n,a}(\sigma)}{f_{X/\mathbb{F}_q;n,a+1}(\sigma)\cdot g_{X/\mathbb{F}_q;n,a+1}(\sigma)}$$

Here in the last step, we have set accordingly

$$f_{X/\mathbb{F}_q;n,a}(\sigma):=\sum_{\substack{k_1,\dots,k_p>0\\k_1+\dots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(n\sigma-\frac{n}{2}+a)+k_p})}\tag{71}$$

and

$$g_{X/\mathbb{F}_q;n,a}(\sigma):=\sum_{\substack{l_1,\dots,l_r>0\\l_1+\dots+l_r=a-1}}\frac{1}{(1-q^{-(n\sigma-\frac{n}{2}+a)+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}\tag{72}$$

Therefore,

$$\left|\frac{\widehat{\zeta}^{[a]}_{X,\mathbb{F}_q;n}(s)}{\widehat{\zeta}^{[b]}_{X,\mathbb{F}_q;n}(s)}\right|=\frac{f_{X/\mathbb{F}_q;n,a}(\sigma)\cdot g_{X/\mathbb{F}_q;n,a}(\sigma)}{f_{X/\mathbb{F}_q;n,a+1}(\sigma)\cdot g_{X/\mathbb{F}_q;n,a+1}(\sigma)}\left|\frac{(q-q^{-(n\sigma-\frac{n}{2}+a)})}{(1-q^{1-(n\sigma-\frac{n}{2}+a)})}\right|\cdot\begin{cases}>1 & |q^{n\sigma}|<\left|q^{\frac{n}{2}-a}\right|\\ <1 & |q^{n\sigma}|>\left|q^{\frac{n}{2}-a}\right|\end{cases}\tag{73}$$

We end this subsection with the following comments. It appears to be very tempting to apply Yoshida's lemma to the factor in the middle with parameters $\alpha = 1$, $\beta = q$ and $w = q^{-(n\sigma - \frac{n}{2}+a)}$. Unfortunately, this would only result inequalities in opposite directions. Nevertheless, as to be seen in the subsection below, there will be a nice total cancelation on this middle factor from the factors in the first group on the ratios of f and g's.

4.4 The Riemann hypothesis for $\widehat{\zeta}_{X,\mathbb{F}_q;3}^{\geq[2]}(s)$

In this subsection, we prove the following

Proposition 4.8 (Riemann Hypothesis for $\widehat{\zeta}_{X,\mathbb{F}_q;3}^{\geq[2]}(s)$)**.** *Let X be a regular projective curve over $\mathbb{F}_q$. Then all zeros of $\widehat{\zeta}_{X,\mathbb{F}_q;3}^{\geq[2]}(s)$ lie on the line $\Re(s) = \frac{1}{3}$.*

Proof. Indeed, from the definitions in the previous subsection, particularly, in (71) and (72)taking parameters $n = 3$, $a = 2$ and $a = 3$, we have

$$f_{X/\mathbb{F}_q;3,2}(\sigma) := \sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=1}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(3\sigma-\frac{3}{2}+2)+k_p})}$$

$$= \sum_{p=1,\,k_p=1} \widehat{v}_{k_p}\frac{1}{(1-q^{(3\sigma-\frac{3}{2}+2)+k_p})} = \widehat{v}_1\frac{1}{(1-q^{(3\sigma-\frac{3}{2}+2)+1})} = \frac{\widehat{v}_1}{1-q^{3\sigma+\frac{3}{2}}}$$

and

$$f_{X/\mathbb{F}_q;3,2+1}(\sigma) := \sum_{\substack{k_1,\dots,k_p>0\\ k_1+\dots+k_p=3-3}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{\prod_{j=1}^{p-1}(1-q^{k_j+k_{j+1}})}\frac{1}{(1-q^{(3\sigma-\frac{3}{2}+3)+k_p})} = 1$$

Similarly,

$$g_{X/\mathbb{F}_q;3,2}(\sigma) := \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=2-1}} \frac{1}{(1-q^{-(3\sigma-\frac{3}{2}+2)+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}$$

$$= \sum_{r=1,l_r=1} \frac{1}{(1-q^{-(3\sigma-\frac{3}{2}+2)+1+l_1})}\widehat{v}_{l_1} = \frac{\widehat{v}_1}{1-q^{-3\sigma+\frac{3}{2}}}$$

and

$$g_{X/\mathbb{F}_q;3,2+1}(\sigma) := \sum_{\substack{l_1,\dots,l_r>0\\ l_1+\dots+l_r=3-1}} \frac{1}{(1-q^{-(3\sigma-\frac{3}{2}+3)+1+l_1})}\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{\prod_{j=1}^{p-1}(1-q^{l_j+l_{j+1}})}$$

$$= \sum_{r=1,l_r=2} \frac{1}{(1-q^{-(3\sigma-\frac{3}{2}+3)+1+l_1})}\widehat{v}_{l_1} + \sum_{\substack{r=2\\ l_1=l_r=1}} \frac{1}{(1-q^{-(3\sigma-\frac{3}{2}+3)+1+l_1})}\frac{\widehat{v}_{l_1}\widehat{v}_{l_r}}{\prod_{j=1}^{2-1}(1-q^{l_j+l_{j+1}})}$$

$$= \frac{\widehat{v}_2}{(1-q^{-3\sigma+\frac{3}{2}})} + \frac{1}{(1-q^{-3\sigma+\frac{3}{2}-1})}\frac{\widehat{v}_1^2}{1-q^2}$$

Therefore,

$$\begin{aligned} r_{X/\mathbb{F}_q;3,2}(\sigma) =& \frac{f_{X/\mathbb{F}_q;3,2}(\sigma)\cdot g_{X/\mathbb{F}_q;3,2}(\sigma)}{f_{X/\mathbb{F}_q;3,2+1}(\sigma)\cdot g_{X/\mathbb{F}_q;3,2+1}(\sigma)} \\ =& \frac{1}{\left(\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\sigma+\frac{3}{2}-1}\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\right)}\cdot\frac{1-q^{-3\sigma+\frac{3}{2}-1}}{1-q^{3\sigma+\frac{3}{2}}} \end{aligned}$$

Consequently, by the result in the previous subsection, particularly, by (73), we have

$$\begin{aligned} &\left|\frac{\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)}\right| = \left|r_{X/\mathbb{F}_q;3,2}(\sigma)\right|\cdot\left|\frac{(q-q^{-(3\sigma-\frac{3}{2}+2)})}{(1-q^{1-(3\sigma-\frac{3}{2}+2)})}\right|\cdot\begin{cases} >1 & \left|q^{3\sigma}\right|<\left|q^{\frac{3}{2}-2}\right| \\ <1 & \left|q^{3\sigma}\right|>\left|q^{\frac{3}{2}-2}\right| \end{cases} \\ =& \frac{\left|q^{-3\sigma-\frac{1}{2}}\right|}{\left|\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\sigma+\frac{3}{2}-1}\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\right|}\cdot\begin{cases} >1 & \left|q^{3\sigma}\right|<\left|q^{-\frac{1}{2}}\right| \\ <1 & \left|q^{3\sigma}\right|>\left|q^{-\frac{1}{2}}\right| \end{cases} \\ =& \frac{1}{\left|\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\sigma+\frac{3}{2}-1}\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\right|}\cdot\begin{cases} >1 & \left|q^{3\sigma}\right|<\left|q^{-\frac{1}{2}}\right| \\ <1 & \left|q^{3\sigma}\right|>\left|q^{-\frac{1}{2}}\right| \end{cases} \end{aligned}$$

Lemma 4.9. *we have*

$$\frac{1/2}{\left|\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\sigma+\frac{3}{2}-1}\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\right|}\begin{cases} >1 & \left|q^{3\sigma}\right|<\left|q^{-\frac{1}{2}}\right| \\ <1 & \left|q^{3\sigma}\right|>\left|q^{-\frac{1}{2}}\right| \end{cases} \tag{74}$$

Proof. To see it clearly, set $\sigma=-\frac{1}{6}+\tau$ so that $3\sigma=-\frac{1}{2}+3\tau$, and let

$$\begin{aligned} D_{X/\mathbb{F}_q,3,2}(\tau) :=& 2\left|\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\sigma+\frac{3}{2}-1}\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\right| \\ =& 2\left|\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)-q^{-3\tau}\left(q\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q^2}{1-q^2}\right)\right| \\ =& 2q\cdot\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}\right)\cdot\left|\frac{\left(\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{1}{1-q^2}\right)}{\left(q\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q^2}{1-q^2}\right)}-q^{-3\tau}\right| \end{aligned}$$

where in the last step, we have used Corollary 3.2 that

$$\frac{\widehat{v_2}}{\widehat{v_1^2}}+\frac{q}{1-q^2}>0. \tag{75}$$

Note that the condition $\left|q^{3\sigma}\right|<\left|q^{-\frac{1}{2}}\right|$ is equivalent to $\left|q^{3\tau}\right|<1$, and similarly for the opposite direction. Hence it suffices to verify that

$$2q \cdot \Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big) \cdot \left| \frac{\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}\Big)}{\Big(q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}\Big)} - q^{-3\tau} \right| = D_{X/\mathbb{F}_q,3,2}(\tau) \begin{cases} < 1 & \left|q^{3\tau}\right| < 1 \\ > 1 & \left|q^{3\tau}\right| > 1 \end{cases} \tag{76}$$

In other words, for $w = q^{3\tau}$, we have to show that

(1) When $|w| < 1$, then w should be contained inside the disc of radius $\frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)}$ centered at $\frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}}$; and

(2) When $|w| > 1$, then w should be totally located outside the disc of radius $\frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)}$ centered at $\frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}}$

An elementary discussion implies that this would happen if and only if the disc of radius $\frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)}$ centered at $\frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}}$ should be totally contained in the unit disc $|w| < 1$.

Since $\frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}} \in \mathbb{R}$ is a real number, this means that

$$\begin{cases} -\frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)} + \frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}} > -1 \\ \frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)} + \frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}} < 1 \end{cases} \tag{77}$$

That is to say,

$$-1 + \frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)} < \frac{\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{1}{1-q^2}}{q\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q^2}{1-q^2}} < 1 - \frac{1}{2q\Big(\frac{\widehat{v_2}}{\widehat{v_1^2}} + \frac{q}{1-q^2}\Big)} \tag{78}$$

or the same

$$\frac{\widehat{v_2}}{\widehat{v_1^2}} > \max\left\{\frac{1/2+1}{q-1}, \frac{1/2 + \frac{q^2+1}{q^2-1}}{q+1}\right\} = \frac{3}{2}\frac{1}{q-1} \tag{79}$$

which is guaranteed by (56) in Corollary 3.2. □

We are now ready to complete our proof of the proposition. Indeed,

$$\widehat{\zeta}^{\geq[2]}_{X,\mathbb{F}_q;3}(s) = \frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s) + \widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s) = \widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)\left(\frac{\frac{1}{2}\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)} + 1\right) \tag{80}$$

By Lemma 4.9 just proved, all the zeros of the second factor lie on the line $\Re(s) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. Accordingly, it suffices to show that the zeros of $\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)$ cannot be the zeros of $\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s)$. Recall that

$$\widehat{\zeta}^{[2]}_{X/\mathbb{F}_q}(s) := \frac{\widehat{v}_1^2}{(1-q^{3s})(1-q^{-3s+3})}\,\widehat{\zeta}_{X,\mathbb{F}_q}(3s-1)$$
$$\widehat{\zeta}^{[3]}_{X/\mathbb{F}_q}(s) := \left(\frac{\widehat{v}_2}{(1-q^{-3s+3})} + \frac{\widehat{v}_1^2}{(1-q^2)(1-q^{-3s+2})}\right)\widehat{\zeta}_{X,\mathbb{F}_q}(3s)$$

Obviously, the zeta zeros from the zeta factor $\widehat{\zeta}_{X/\mathbb{F}_q}(3s)$, which are on the line of $\Re(s) = \frac{1}{6}$, cannot be the zeros of $\widehat{\zeta}_{X/\mathbb{F}_q}(3s-1)$, which are on the line $\Re(s) = \frac{1}{2}$, by the Riemann hypothesis for the Artin zeta function $\zeta_{F/\mathbb{F}_q}(s)$. This then leaves the zeros of the rational function factor in $\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)$, which is clearly not on the line of $\Re(s) = \frac{1}{2}$. Therefore, all zeros of $\frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s) + \widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)$ are coming from the second factor $\frac{\frac{1}{2}\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)} + 1$ in (80) and hence lie on the line $\Re(s) = \frac{1}{3}$, as wanted. □

4.5 Rank three Riemann hypothesis

Now we are finally ready to complete our proof of Theorem 4.2 and hence Theorem 4.3. We start with the function $R_{X/\mathbb{F}_q;3}(s) := \left(1 + \frac{\widehat{\zeta}^{\leq 2}_{X,\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{\geq 2}_{X,\mathbb{F}_q;3}(s)}\right)$. Recall that

$$\widehat{\zeta}^{\geq 2}_{X,\mathbb{F}_q;3}(s) = \frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s) + \widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s) = \frac{\prod_{i=1}^{n+2(g-1)}(1-T\gamma_i)}{T^{g-1}\prod_{\ell=0}^{n}(1-q^\ell T)} \tag{81}$$

Here, by the Riemann hypothesis for $\widehat{\zeta}^{\geq 2}_{X,\mathbb{F}_q;3}(s)$ established in the previous subsection, we have

$$|\gamma_i| = Q^{1/3} \qquad \forall 1 \leq i \leq n + 2(g-1) \tag{82}$$

Then

$$\begin{aligned}\widehat{\zeta}^{\leq 2}_{X,\mathbb{F}_q;3}(s) =& \widehat{\zeta}^{[1]}_{X,\mathbb{F}_q;3}(s) + \frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s) = \frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(1-s) + \widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(1-s)\\ =& \frac{\prod_{i=1}^{n+2(g-1)}(1-\frac{\gamma_i}{QT})}{(QT)^{-g+1}\prod_{\ell=0}^{n}(1-\frac{q^\ell}{QT})} = \frac{1}{Q^{\frac{n-1}{2}+g-1}T^{g-2}}\frac{\prod_{i=1}^{n+2(g-1)}(QT-\gamma_i)}{\prod_{\ell=0}^{n}(q^\ell T-1)}\end{aligned}$$

Therefore,

$$\frac{\widehat{\zeta}^{\leq[2]}_{X,\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{\geq[2]}_{X,\mathbb{F}_q;3}(s)}=\frac{\widehat{\zeta}^{[1]}_{X,\mathbb{F}_q;3}(s)+\frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s)}{\frac{1}{2}\widehat{\zeta}^{[2]}_{X,\mathbb{F}_q;3}(s)+\widehat{\zeta}^{[3]}_{X,\mathbb{F}_q;3}(s)}=\frac{\frac{\prod_{i=1}^{n+2(g-1)}(1-T\gamma_i)}{T^{g-1}\prod_{\ell=0}^{n}(1-q^\ell T)}}{\frac{1}{Q^{\frac{n-1}{2}+g-1}T^{g-2}}\frac{\prod_{i=1}^{n+2(g-1)}(QT-\gamma_i)}{\prod_{\ell=0}^{n}(q^\ell T-1)}}=\frac{T}{\sqrt{Q}}\prod_{i=1}^{n+2(g-1)}\frac{1-T\gamma_i}{\sqrt{Q}T-\frac{\gamma_i}{\sqrt{Q}}}$$

We examine the factor $\dfrac{1-T\gamma_i}{\sqrt{Q}T-\frac{\gamma_i}{\sqrt{Q}}}$ under the condition that $|\gamma_i|=Q^{1/3}$ for all $1\leq i\leq n+2(g-1)$. Write then $\gamma_i=Q^{\frac{1}{3}}e^{i\theta_i}$. Then

$$\left|\frac{1-T\gamma_i}{\sqrt{Q}T-\frac{\gamma_i}{\sqrt{Q}}}\right|=\left|\frac{1-TQ^{1/3}e^{i\theta_i}}{\sqrt{Q}T-\frac{Q^{1/3}e^{-i\theta_i}}{\sqrt{Q}}}\right|=\left|\frac{1-\sqrt{Q}TQ^{-1/6}e^{i\theta_i}}{\sqrt{Q}T-Q^{-1/6}e^{-i\theta_i}}\right|$$

Note that

$$\left|\frac{1-\sqrt{Q}TQ^{-1/6}e^{i\theta}}{\sqrt{Q}T-Q^{-1/6}e^{-i\theta}}\right|^2=1-\frac{(Q^{1/3}-1)(1+Q^{2/3})|T|^2}{Q|T|^2-Q^{1/3}(e^{-i\theta}\overline{T}+e^{i\theta}T)+Q^{-\frac{1}{3}}}<1$$

Obviously,

$$\left|\frac{T}{\sqrt{Q}}\right|\begin{cases}<1 & |T|<\sqrt{Q}\\ >1 & |T|>\sqrt{Q}\end{cases}\tag{83}$$

This implies that $R_{X/\mathbb{F}_q;3}(s)$ has no zero in the region $\Re(s)<\frac{1}{2}$. Now note that, by §4.1, we have

$$\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)=\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(s)+\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)=\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)\left(1+\frac{\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)}\right)=\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)\cdot R_{X/\mathbb{F}_q;3}(s)\tag{84}$$

Thus, to prove Theorem 4.2, what is left to to show that the zeros of $\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)$ cannot be the zeros of $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$. But this is clear, since all zeros of $\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(s)$ lie on the line $\Re(s)=1-\frac{1}{3}$ by the functional equation.

$$\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(1-s)=\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s).\tag{85}$$

In particular, $\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(1-s)$ and $\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)$ cannot have any common zero. Therefore, from (84), the zeros of $\widehat{\zeta}_{X,\mathbb{F}_q;3}(s)$ cannot come from the first factor $\widehat{\zeta}^{\geq[2]}_{X,\mathbb{F}_q;3}(s)$, but all come from the second factor $\left(1+\frac{\widehat{\zeta}^{\leq 2}_{X/\mathbb{F}_q;3}(s)}{\widehat{\zeta}^{\geq 2}_{X/\mathbb{F}_q;3}(s)}\right)$. This proves Theorem 4.2 and hence also Theorem 4.3.

We mention in passing that an analogue of this result holds for the rank 3 zeta function $\widehat{\zeta}_{\mathbb{Q};3}(s)$ and hence the SL_n zeta function $\widehat{\zeta}^{\mathrm{SL}_3}_{\mathbb{Q}}(s)$ the field Q of rationals has been proved by Suzuki [11] (See also [15] for general discussions).

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