Ω-ADMISSIBLE THEORY

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Introduction

Almost twenty years ago, starting with Arakelov metrics on compact Riemann surfaces, Arakelov developed an admissible theory for arithmetic surfaces [**2**].

Recall that the Ricci forms of Arakelov metrics on compact Riemann surfaces are the so-called canonical forms, which are the pull-back of the Kähler forms corresponding to the ordinary flat metrics on the associated Jacobians. Hence, the admissible theory of Arakelov and Faltings is essentially in the nature of Euclidean geometry.

But, from the moduli point of view, hyperbolic metrics are more natural. First of all, the Weil–Petersson metric on the moduli space $\mathcal{M}_q$ of compact Riemann surfaces of genus $q\,(\geqslant 2)$ comes directly from hyperbolic metrics of these surfaces; secondly, a result of Wolpert [**26**, § 2, p. 1485] says that the singularities of the Kähler form ω_{WP}, defined by the Weil–Petersson metric on $\mathcal{M}_q$, are sufficiently mild. (More precisely, in the sense of currents on the compactified moduli space $\overline{\mathcal{M}_q}$ of $\mathcal{M}_q$ constructed by Deligne and Mumford, $\omega_{\mathrm{WP}}/\pi^2$ is the curvature form of a continuous metric h_{WP} on a certain line bundle and the metric h_{WP} may be approximated by smooth positive curvature metrics.)

Thus we should develop an admissible theory with respect to hyperbolic volume forms. For this purpose, we create a more general ω-admissible theory in this paper: for any normalized volume form ω on a compact Riemann surface M, we introduce the Arakelov metric $\rho_{\mathrm{Ar}}(\omega)$ with respect to ω on the canonical line bundle K_M of M. With such an ω-Arakelov metric $\rho_{\mathrm{Ar}}(\omega)$, we can then develop an ω-admissible theory for arithmetic surfaces, which is parallel to what was done in [**2**] and [**7**]. Moreover, we show that various ω-admissible theories are closely related by the so-called Mean Value Lemma: for any arithmetic surface $\pi\colon X \to S$, the self-intersection of the canonical line bundle K_π equipped with the ω-Arakelov metric $\rho_{\mathrm{Ar}}(\omega)$ is independent of ω, and hence is exactly the self-intersection of the classical Arakelov canonical divisor. As a by-product we give the ω-insertion formula, which has some applications in string theory. All this is done in § 1.

There are two interesting examples for this ω-admissible theory. The first is obtained by taking ω as the canonical form ω_{can}. In this case, the ω-admissible theory coincides with the classical admissible theory of Arakelov. The second is obtained by taking ω as the (normalized) hyperbolic volume form ω_{hyp}. In this case, we show that the metric $\rho_{\mathrm{Ar}}(\omega_{\mathrm{hyp}})$ on the Riemann surface M induced by

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the ω_{hyp}-Arakelov metric is proportional to the original hyperbolic metric ρ_{hyp} on M. Thus the ratio of $\rho_{\text{Ar}}(\omega_{\text{hyp}})$ and ρ_{hyp} defines a natural, yet important, invariant for the compact Riemann surface M. We call this invariant the Arakelov–Poincaré volume of M and show that it actually forms a Weil function on the moduli space $\overline{\mathcal{M}_q}$. In fact we do much more: we first evaluate this invariant in terms of Quillen metrics, and then give precise degeneration behaviour on the boundary of $\overline{\mathcal{M}_q}$.

As an application of the admissible theory with respect to hyperbolic volume forms, we offer an upper bound for the self-intersection of the classical Arakelov canonical divisor in terms of Petersson norms of modular forms, by applying Soulé's arithmetic vanishing theorem. All this is done in § 2.

In § 3, we develop an ω-admissible theory for singular arithmetic surfaces, which is motivated by the work of Aitken in [**1**]. As a by-product, we introduce a metric on the determinant of cohomology, and hence give an arithmetic Riemann–Roch theorem for singular arithmetic surfaces.

We end this paper with an appendix, in which Deligne pairings are expressed in terms of determinant line bundles. Such a relation will be used in a forthcoming study on Chow–Mumford stability and Kähler–Einstein metrics.

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1. Ω-*Admissible theory for compact Riemann surfaces*

1.1. *Admissible metrics on line bundles with respect to any volume form*

In this section, we introduce an ω-admissible theory for line bundles over a compact Riemann surface with respect to any fixed volume form ω. In particular, we define the ω-Arakelov metric on the canonical line bundle, prove the Mean Value Lemma for arithmetic intersections, and give the ω-adjunction formula.

Let M be a compact Riemann surface. For any normalized volume form ω on M, that is, for any positive (1,1) form ω on M such that $\int_M \omega = 1$, there is a unique Green's function $g_\omega(P, Q)$ with respect to ω satisfying the following conditions [**15**, Chapter 1, Theorem 1.4]:

(a) if f is a rational function on an open subset U of M such that, over U, the divisor of f is P, then there exists a smooth function α on U such that for $Q \neq P$,

$$g_\omega(P, Q) = -\log|f(Q)|^2 + \alpha(Q);$$

(b) $d_P d_P^c g_\omega(P, Q) = \omega(P) - \delta_Q$;

(c) $\int_M g_\omega(P, Q)\omega(P) = 0$.

Indeed, if ω is the canonical form ω_{can} on M, the existence of $g_{\omega_{\text{can}}}$ is proved by Arakelov in [**2**] (see [**15**, Chapter 2] for more details). On the other hand, for any fixed normalized volume form ω', there exists a unique smooth real function $\beta_{\omega',\omega}$

on M such that

$$dd^c\beta_{\omega',\omega} = \omega' - \omega, \qquad \int_M \beta_{\omega',\omega}(\omega' + \omega) = 0. \tag{1}$$

Furthermore, if $P \neq Q$, we have $g_{\omega'}(P, Q) = g_\omega(P, Q) + \beta_{\omega',\omega}(P) + \beta_{\omega',\omega}(Q)$. This then gives the existence of the Green's functions with respect to any normalized volume form.

By definition, a hermitian metric ρ on a line bundle L of M is called ω-*admissible* if the first Chern form of the hermitian line bundle (L, ρ) satisfies $c_1(L, \rho) = d(L) \cdot \omega$. Here $d(L)$ denotes the degree of the line bundle L. It is easy to see that the set of ω-admissible hermitian metrics on a fixed line bundle is parametrized by $\mathbb{R}^+$. Moreover, by using the Green's functions with respect to ω, we can construct ω-admissible metrics over all line bundles on M: first, for any point $P \in M$, on $\mathcal{O}_M(P)$, we set the inner product of the defining section $\mathbf{1}_P$ (with itself) at any point Q to be $G_\omega(P, Q) := \exp(-g_\omega(P, Q))$; then we use linearity to get the ω-admissible metrics on all line bundles by assuming that the natural algebraic isomorphisms $\mathcal{O}_M(D) \otimes \mathcal{O}_M(D') \simeq \mathcal{O}_M(D + D')$ are isometries for all divisors D and D' on M. If a hermitian metric h on a line bundle L is ω-admissible, we call the corresponding hermitian line bundle (L, h) an ω-admissible hermitian line bundle, and denote it $\overline{L}$, by abuse of notation.

Therefore, on the canonical bundle K_M of M, there exist ω-admissible metrics. But such metrics are far from being unique. For our purpose, we fix a normalization as follows:

$$\|dz\|^2_{\mathrm{Ar}(\omega)}(P) := \lim_{Q \to P} \frac{|z(P) - z(Q)|^2}{G_\omega(P, Q)} \exp[-2q\beta_{\omega,\omega_{\mathrm{can}}}(P)].$$

Here z denotes a holomorphic coordinate of M at P, and q denotes the genus of M. (From now on, for simplicity, we will use $\beta_\omega(P)$ to denote $\beta_{\omega,\omega_{\mathrm{can}}}(P)$.) In the sequel, we call this metric the *Arakelov metric with respect to* ω, or the ω-*Arakelov metric*, *on* K_M.

This normalized hermitian metric on K_M is ω-admissible. Indeed, by [**15**, Chapter 4, Theorem 5.4], the metric on K_M defined by

$$\|dz\|^2_{\mathrm{Ar}(\omega_{\mathrm{can}})}(P) := \lim_{Q \to P} \frac{|z(P) - z(Q)|^2}{G_{\omega_{\mathrm{can}}}(P, Q)}$$

is an ω_{can}-admissible metric on K_M. So, for any non-zero section s of K_M,

$$\begin{aligned}
&dd^c(-\log\|s\|^2_{\mathrm{Ar}(\omega)}) \\
&\quad = dd^c(-\log\|s\|^2_{\mathrm{Ar}(\omega_{\mathrm{can}})}) - dd^c \lim_{Q \to P} \log\left(\frac{G_{\omega_{\mathrm{can}}}(P, Q)}{G_\omega(P, Q)} \exp[-2q\beta_\omega] \right) \\
&\quad = (2q - 2)\omega_{\mathrm{can}} + 2(\omega_{\mathrm{can}} - \omega) + 2q\,dd^c\beta_\omega \\
&\quad = (2q - 2)\omega_{\mathrm{can}} + 2(\omega_{\mathrm{can}} - \omega) + 2q(\omega - \omega_{\mathrm{can}}) \\
&\quad = (2q - 2)\omega.
\end{aligned}$$

Hence we have the following.

Proposition 1.1. *With the same notation as above,*

$$\|dz\|^2_{\mathrm{Ar}(\omega)}(P) := \lim_{Q\to P}\frac{|z(P)-z(Q)|^2}{G_\omega(P,Q)}\exp[-2q\beta_\omega(P)]$$

defines an admissible metric $\rho_{\mathrm{Ar}(\omega)}$ with respect to ω on K_M.

The Arakelov metric with respect to ω on K_M has another more geometric interpretation. To explain it, we first introduce the ω-Arakelov metric on the line bundle $\mathcal{O}_M(P)$ associated to a point $P\in M$: define the *ω-Arakelov metric* $\rho_{\mathrm{Ar}}(\omega;P)$ on $\mathcal{O}_M(P)$ by setting the inner product (with itself) of the defining section $\mathbf{1}_P$ of $\mathcal{O}_M(P)$ at any point $Q\in M$ equal to $\exp[-g_\omega(P,Q)+\beta_\omega(P)]$. Obviously $\rho_{\mathrm{Ar}}(\omega;P)$ on $\mathcal{O}_M(P)$ is ω-admissible. On the other hand, for two distinct points P and Q on M, if we impose the ω-Arakelov metrics $\rho_{\mathrm{Ar}}(\omega;P)$ and $\rho_{\mathrm{Ar}}(\omega;Q)$ on $\mathcal{O}_M(P)$ and $\mathcal{O}_M(Q)$, respectively, then the induced metric on $\mathcal{O}_M(P)|_Q$ is quite different from the induced metric on $\mathcal{O}_M(Q)|_P$: for the first, the norm of the section $\mathbf{1}$ is $\exp[-\frac{1}{2}\cdot g_\omega(P,Q)+\frac{1}{2}\cdot\beta_\omega(P)]$, while for the second, the norm of the section $\mathbf{1}$ is $\exp[-\frac{1}{2}\cdot g_\omega(Q,P)+\frac{1}{2}\cdot\beta_\omega(Q)]$. So the Arakelov metric $\rho_{\mathrm{Ar}}(\omega;P)$ on $\mathcal{O}_M(P)$ is not compatible with restriction. To remedy this, we make the following modification of the metric on the restriction: let $\overline{L}$ be an ω-admissible hermitian line bundle on M; then on the restriction $L|_P$, we introduce a new metric by multiplying the restriction metric from L to P by an additional factor $\exp[d(L)\cdot\beta_\omega(P)]$, and we will use the symbol $\overline{L\|_P}$ to indicate the vector space $L|_P$ together with the above modified metric. Thus, in particular, on $\overline{\mathcal{O}_M(P)\|_Q}$, the inner product of $\mathbf{1}$ (with itself) at Q is just $\exp[-g_\omega(P,Q)+\beta_\omega(P)+\beta_\omega(Q)]$. So $\overline{\mathcal{O}_M(P)\|_Q}$ is isometric to $\overline{\mathcal{O}_M(Q)\|_P}$.

Now we easily see that the Arakelov metric with respect to ω on K_M is the unique metric such that, at each point $P\in M$, the natural residue map $\mathrm{res}: K_M(P)|_P\to\mathbb{C}|_P$ induces an isometry $\mathrm{res}: \overline{K_M(P)\|_P}\to\overline{\mathbb{C}|_P}$ with respect to the metrics ρ_{Ar} on K_M, $\rho_{\mathrm{Ar}}(\omega;P)$ on $\mathcal{O}_M(P)$, and the standard flat metric on $\mathbb{C}$.

We next give some applications of the ω-Arakelov metrics to the theory of arithmetic surfaces.

Let F be a number field with $\mathcal{O}_F$ the ring of integers. Let $\pi: X\to S$ be a regular arithmetic surface over $S:=\mathrm{Spec}(\mathcal{O}_F)$. Fix a normalized volume form ω on $X(\mathbb{C})$, the fibre of π at infinity. Denote the ω-Arakelov metrics on $K_{X(\mathbb{C})}$ and on $P\in X(\mathbb{C})$ by $\rho_{\mathrm{Ar}}(\omega)$ and $\rho_{\mathrm{Ar}}(\omega;P)$, respectively. For any algebraic point P of X_F, the generic fibre of π, denote the corresponding Zariski closure of P in X by E_P. Denote by $K_\pi(\omega)$ and $E_P(\omega)$ the hermitian line bundles $(K_\pi,\rho_{\mathrm{Ar}}(\omega))$ and $(\mathcal{O}(E_P),\rho_{\mathrm{Ar}}(\omega;P))$, respectively.

Mean Value Lemma I. *With the same notation as above, we have the following relations for arithmetic intersections:*

(i) $K_\pi(\omega)^2 = K_\pi(\omega_{\mathrm{can}})^2$;

(ii) $K_\pi(\omega)\cdot E_P(\omega) = K_\pi(\omega_{\mathrm{can}})\cdot E_P(\omega_{\mathrm{can}})$;

(iii) $E_Q(\omega)\cdot E_P(\omega) = E_Q(\omega_{\mathrm{can}})\cdot E_P(\omega_{\mathrm{can}})$, *for any two algebraic points P,Q of X_F.*

Proof. We only prove (i), as the proof of the others is similar. Denote the first arithmetic Chern class of a hermitian line bundle (L, ρ) on X by $c_{1,\mathrm{Ar}}(L, \rho)$. Then

$$c_{1,\mathrm{Ar}}(K_\pi(\omega)) = c_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + (0, (2q-2)\beta_\omega(P)).$$

Thus,

$$\begin{aligned}
c^2_{1,\mathrm{Ar}}(K_\pi(\omega)) &= c_{1,\mathrm{Ar}}(K_\pi(\omega))[c_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + (0, (2q-2)\beta_\omega(P))] \\
&= c_{1,\mathrm{Ar}}(K_\pi(\omega))c_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + c_{1,\mathrm{Ar}}(K_\pi(\omega))(0, (2q-2)\beta_\omega(P)) \\
&= c^2_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) \\
&\quad + [c_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + c_{1,\mathrm{Ar}}(K_\pi(\omega))](0, (2q-2)\beta_\omega(P)) \\
&= c^2_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + \tfrac{1}{2}\int (2q-2)\beta_\omega(P)[c_1(K_\pi(\omega_{\mathrm{can}})) + c_1(K_\pi(\omega))] \\
&= c^2_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + \tfrac{1}{2}\int (2q-2)\beta_\omega(P)[(2q-2)\omega_{\mathrm{can}} + (2q-2)\omega] \\
&= c^2_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})) + \tfrac{1}{2}\int (2q-2)^2\beta_\omega(P)[\omega_{\mathrm{can}} + \omega] \\
&= c^2_{1,\mathrm{Ar}}(K_\pi(\omega_{\mathrm{can}})),
\end{aligned}$$

by the normalization condition (1) of $\beta_\omega(P)$.

REMARK. The Mean Value Lemma says that, in terms of arithmetic intersection, the ω-admissible theory is just the classical Arakelov admissible theory. In particular, we can use different normalized volume forms to calculate the self-intersection of the classical Arakelov canonical divisor. On the other hand, the reader should notice that our ω-Arakelov metric on $\mathcal{O}_M(P)$ is not defined by using the ω-Green's function only. Thus, E_P in [**15**] is not our $E_P(\omega) = (E_P, \rho_{\mathrm{Ar}}(\omega; P))$, while, for any ω-admissible hermitian line bundle (L, ρ) on X, $(L, \rho) \cdot (E_P, \rho_{\mathrm{Ar}}(\omega; P))$ is not just the arithmetic degree of the pull-back of (L, ρ) via the section of π corresponding to P on S.

As a direct consequence, we have the following.

Ω-ADJUNCTION FORMULA. *With the same notation as above, if P is an F-rational point on X_F, then*

$$K_\pi(\omega) \cdot E_P(\omega) + E_P(\omega)^2 = 0.$$

1.2. *Admissible metrics on determinants with respect to any volume form*

In this section, we are going to show that there exists a natural ω-determinant metric on the cohomology determinant of ω-admissible hermitian line bundles. In particular, if we take ω to be the canonical form, this ω-determinant metric coincides with the Faltings metric.

Let M be a compact Riemann surface of genus q. Fix a normalized volume form ω on M. Denote the ω-Arakelov metric on the canonical line bundle K_M of M by $\rho_{\mathrm{Ar}}(\omega)$, and for any point $P \in M$, denote the ω-Arakelov metric on $\mathcal{O}_M(P)$ by $\rho_{\mathrm{Ar}}(\omega)$ also. In the sequel, we always assume that the metrics on K_M (and hence on the Riemann surface M) and on $\mathcal{O}_M(P)$ are $\rho_{\mathrm{Ar}}(\omega)$. For any ω-admissible hermitian line bundle $\overline{L}$, denote $\overline{L} \otimes (\mathcal{O}_M(P), \rho_{\mathrm{Ar}}(\omega))^{\otimes -1}$ by $\overline{L}(-\underline{P})$.

THEOREM 1.2. *Let M be a compact Riemann surface of genus q. Let ω be a normalized volume form on M. Then, for any ω-admissible hermitian line bundle $\overline{L}$ on M, there exists a unique ω-determinant metric, denoted by $h(\overline{L})$, over the cohomology determinant of L,*

$$\lambda(L) := \operatorname{Det} R\Gamma(M, L) := \bigwedge^{\max} H^0(M, L) \otimes \left(\bigwedge^{\max} H^1(M, L)\right)^*,$$

up to a universal constant multiple, such that the following conditions are satisfied:

(a) *an isometry of ω-admissible hermitian line bundles $\overline{L} \to \overline{L}'$ induces an isometry from $(\lambda(M, L),\ h(\overline{L}))$ to $(\lambda(M, L'),\ h(\overline{L}'))$;*

(b) *if the ω-admissible metric on L is changed by a factor $\alpha \in \mathbb{R}^+$, then the metric on $\lambda(M, L)$ is changed by the factor $\alpha^{\chi(M,L)}$;*

(c) (Riemann–Roch condition for closed immersions) *for any point P on M, the algebraic isomorphism*

$$\lambda(M, L) \simeq \lambda(M, L(-P)) \otimes L|_P$$

induced by the short exact sequence of coherent sheaves

$$0 \to L(-P) \to L \to L|_P \to 0$$

gives an isometry

$$(\lambda(M, L),\ h(\overline{L})) \simeq (\lambda(M, L(-P)),\ h(\overline{L}(-\underline{P}))) \otimes \overline{L\|_P}.$$

Before proving this theorem, let us make a few comments. First, if ω is the canonical volume form, this theorem is proved by Faltings [**7**] with one more condition:

(d) *the metric on $\lambda(M, \Omega_M^1) = \bigwedge^g H^0(M, \Omega_M^1)$ is the determinant of the L^2-metric on $H^0(M, \Omega_M^1)$ induced by the canonical pairing.*

Note that this condition is merely a normalization, which will fix the above-mentioned universal constant once and for all. So we drop it for our purposes, as such a normalization is not important for the arithmetic Riemann–Roch theorem. (On the other hand, a normalization is crucial for the arithmetic Noether formula.)

Secondly, in the proof of Faltings' theorem [**7**], it appears that the use of the canonical form is essential, as it plays a central role in the curvature calculation. But this is not quite true: one has Mean Value Lemma III at the end of this section, and hence can prove our theorem by using Faltings' method (see, for example, [**23**]). Nevertheless, here we use another approach.

Proof of Theorem 1.2. We will show that the Quillen metric satisfies the conditions above. Hence, up to a universal constant, we can take the Quillen metric as the ω-determinant metric.

Denote the ω-admissible hermitian metric on L in the theorem by ρ_{Ad}. Then, with respect to ρ_{Ad} on L and the ω-Arakelov metric ρ_{Ar} on M, we naturally get the Quillen metric on the cohomology determinant $\lambda(L)$, which we simply denote by h_Q. By definition, the Quillen metric h_Q satisfies conditions (a) and (b). So, in order to get a good admissible metric theory on the cohomology determinant with respect to ω, we only need to check condition (c).

To do so, without loss of generality, we suppose that there is a smooth family π: $X \to Y$ for M, together with a section s: $Y \to X$ of π corresponding to P, and that there is a family of line bundles $\mathbb{L}$ on X for L/M. That is to say, M is a special fibre of π at a certain point y_0, $s(y_0) = P \in M$, and $\mathbb{L}$ is a line bundle on X such that $\mathbb{L}|_{M=\pi^{-1}(y_0)} = L$. In this way, we get two determinant line bundles on Y: one, denoted by $\lambda(\mathbb{L})$, comes from the family $\mathbb{L}$ for L, while the other, denoted by $\lambda(\mathbb{L}(-s(Y))$, comes from the family $\mathbb{L}(-s(Y))$ for $L(-P)$.

Fix a $(1,1)$-form, denoted also by ω by abuse of notation, on X, such that, for each $y \in Y$, the restriction of ω to $X_y := \pi^{-1}(y)$ is a normalized volume form ω_y and $\omega_{y_0} = \omega$ is the original normalized volume form on M. Now, on $\mathbb{L}$, choose a hermitian metric, denoted by ρ_{Ad} too by abuse of notation, such that the restriction of ρ_{Ad} to $\mathbb{L}|_{X_y}$ is an ω_y-admissible hermitian metric and the restriction to $\mathbb{L}|_{X_{y_0}} = L$ is just the original hermitian metric ρ_{Ad} on L. Moreover, on the relative canonical line bundle K_π of π, choose a hermitian metric, denoted by ρ_{Ar} such that the induced metric on each fibre of π is the ω_y-Arakelov metric. (This then induces a hermitian metric, denote by $\rho_{\mathrm{Ar}}^\vee$ on the relative tangent bundle T_π of π.) Similarly, on $\mathcal{O}(s(Y))$, choose a hermitian metric, denoted by $\rho_{\mathrm{Ar}}(s)$, such that the restriction of $\rho_{\mathrm{Ar}}(s)$ to $\mathcal{O}(s(y))$ is the ω_y-Arakelov metric. For simplicity, denote by ρ_{Ad} also the hermitian metric induced by using a tensor product on $\mathbb{L}(-s(Y)) = \mathbb{L} \otimes \mathcal{O}_X(-s(Y))$.

Then, with respect to these hermitian metrics, on $\lambda(\mathbb{L})$ and on $\lambda(\mathbb{L}(-s(Y))$, we have the associated Quillen metrics. Denote both of them simply by h_Q.

Now, by the well-known formula of the first Chern form for the Quillen metric, we have

$$\begin{aligned}
&c_1(\lambda(\mathbb{L}), h_Q) - c_1(\lambda(\mathbb{L}(-s(Y))), h_Q)\\
&\qquad = \left(\int_\pi \mathrm{ch}(\mathbb{L}, \rho_{\mathrm{Ad}})(1 - \mathrm{ch}(\mathcal{O}(-s(Y)), \rho_{\mathrm{Ar}}))\mathrm{td}(T_\pi, \rho_{\mathrm{Ar}}^\vee)\right)^{(1)}\\
&\qquad = \left(\int_\pi c_1(\mathcal{O}(-s(Y)), \rho_{\mathrm{Ar}})[\tfrac{1}{2}c_1(\mathcal{O}(-s(Y)), \rho_{\mathrm{Ar}}) - \tfrac{1}{2}c_1(K_\pi, \rho_{\mathrm{Ar}})\right.\\
&\qquad\qquad \left. + c_1(\mathbb{L}, \rho_{\mathrm{Ad}})]\right)^{(1)}.
\end{aligned}$$

Here $(\cdot)^{(1)}$ denotes the $(1,1)$-part of the differential form $(\cdot)$. Thus, by the fact that the metric on $\mathcal{O}_M(P)$ is the Arakelov metric with respect to ω, we see that the first Chern form of $(\mathcal{O}(-P), \rho_{\mathrm{Ar}})$ is given by $c_1(\mathcal{O}(-P), \rho_{\mathrm{Ar}}) = -dd^c g_\omega - \delta_P$. Hence

$$\begin{aligned}
&c_1(\lambda(\mathbb{L}), h_Q) - c_1(\lambda(\mathbb{L}(-s(Y))), h_Q)\\
&\qquad = -dd^c \int_\pi g_\omega[\tfrac{1}{2}\, c_1(\mathcal{O}(-s(Y)), \rho_{\mathrm{Ar}}) - \tfrac{1}{2}c_1(K_\pi, \rho_{\mathrm{Ar}}) + c_1(\mathbb{L}, \rho_{\mathrm{Ad}})]\\
&\qquad\quad + s^*(c_1(\mathbb{L}, \rho_{\mathrm{Ad}})) - \tfrac{1}{2}s^*(c_1(K_\pi \otimes \mathcal{O}(s(Y)), \rho_{\mathrm{Ar}})).
\end{aligned}$$

But, by the admissible property of the metrics ρ_{Ar} and ρ_{Ad},

$$\tfrac{1}{2}c_1(\mathcal{O}(-s(Y)), \rho_{\mathrm{Ar}}) - \tfrac{1}{2}c_1(K_\pi, \rho_{\mathrm{Ar}}) + c_1(\mathbb{L}, \rho_{\mathrm{Ad}})$$

is just a multiple of ω. So, by the normalization condition (c) in § 1.1 of the

Green's function g_ω with respect to ω, we have

$$\int_\pi g_\omega[\tfrac{1}{2}c_1(\mathcal{O}(-s(Y)),\rho_{\mathrm{Ar}})-\tfrac{1}{2}c_1(K_\pi,\rho_{\mathrm{Ar}})+c_1(\mathbb{L},\rho_{\mathrm{Ad}})]=0.$$

Thus

$$\begin{aligned} &c_1(\lambda(\mathbb{L}),h_Q)-c_1(\lambda(\mathbb{L}(-s(Y))),h_Q)\\ &\qquad=s^*(c_1(\mathbb{L},\rho_{\mathrm{Ad}}))-\tfrac{1}{2}c_1(s^*(K_\pi\otimes\mathcal{O}(s),\rho_{\mathrm{Ar}})).\end{aligned}$$

Therefore, by the fact that the Arakelov metric on M with respect to ω is defined so that the isomorphism induced by the residue map gives an isometry $\overline{K_M(P)\|_P}\simeq\overline{\mathbb{C}}|_P$, we see that if we understand s^* as the ω-restriction $\|$ defined in the geometric interpretation of the Arakelov metric ρ_{Ad} in § 1.1, then

$$c_1(s^*(K_\pi\otimes\mathcal{O}(s),\rho_{\mathrm{Ar}}))=0,$$

as it corresponds to $\mathbb{C}$ together with the standard flat metric, and hence we get

$$c_1(\lambda(\mathbb{L}),h_Q)-c_1(\lambda(\mathbb{L}(-s(Y))),h_Q)=s^*(c_1(\mathbb{L},\rho_{\mathrm{Ad}})),$$

which gives condition (c) as s^* means $\|$. This completes the proof of the theorem.

With this theorem, we then see that the ω-admissible metric theory for the cohomology determinant is very similar to the Faltings theory. Thus all results in [**15**] or in [**4**] can be reproduced here. In particular, we find that the Riemann–Roch theorem stands without any change. But we do not list any of these results here. Instead, we show that the ω-determinant metrics do not depend on ω.

MEAN VALUE LEMMA II. *Let M be a compact Riemann surface. For any line bundle L over M, denote the cohomology determinant* Det $R\Gamma(M,L)$ *of L by $\lambda(L)$. For any normalized volume form ω on M, put the ω-Arakelov metrics on K_M and on $\mathcal{O}_M(P)$ for a point $P\in M$. Denote the associated Quillen metrics on $\lambda(K_M)$ and $\lambda(P):=\lambda(\mathcal{O}_M(P))$ simply by $h_Q(\omega)$. Then, for any two normalized volume forms ω_1 and ω_2 on M, we have the following isometries:*

(i) $(\lambda(K_M),h_Q(\omega_1))\simeq(\lambda(K_M),h_Q(\omega_2))$;

(ii) $(\lambda(P),h_Q(\omega_1))\simeq(\lambda(P),h_Q(\omega_2))$ *for any point $P\in M$.*

Proof. We only need to show that the result is valid for the case where ω_2 is the canonical form of M. By the Polyakov variation formula for Quillen metrics [**8**, Formula (3.31)], if two metrics ρ_1 and ρ_0 on the Riemann surface M satisfy $\rho_1=e^\phi\rho_0$ for some smooth function ϕ, then, for any non-zero element α of $\lambda(K_M)$, the variation of the associated Quillen metrics is given by

$$h_Q(\rho_1)(\alpha)-h_Q(\rho_0)(\alpha)=\tfrac{1}{6}\int_M\phi\,\tfrac{1}{2}(c_1(\rho_0)+c_1(\rho_1)).$$

Thus, if we take the Arakelov metric to be ρ_0 and the ω-Arakelov metric $\rho_{\mathrm{Ar}}(\omega)$ to be ρ_1, respectively, then we know that $\phi=(-q+1)\beta_\omega$. In particular,

$$\int_M\phi(c_1(\rho_0)+c_1(\rho_1))=\int_M(-q+1)(2q-2)\beta_\omega(\omega_{\mathrm{can}}+\omega)=0.$$

This completes the proof of (i). Using (i), we see that (ii) is a direct consequence of the Riemann–Roch theorem and Mean Value Lemma I in the previous subsection.

INSERTION FORMULA. *Let ω be a normalized volume form on a compact Riemann surface M. Let L be a line bundle on M. Put an ω-admissible metric on L. Put the ω-Arakelov metrics on K_M, on M, and on $\mathcal{O}_M(P)$ for any point $P \in M$. Denote simply by h_Q the associated Quillen metrics on $\lambda(L)$ and $\lambda(L - \sum_{i=1}^{\alpha} n_i P_i)$, where $n_i \in \mathbb{Z}$, and $P_i \in M$. Then we have the following isometry:*

$$(\lambda(L), h_Q) \simeq \left(\lambda\left(L - \sum_{i=1}^{\alpha} n_i P_i\right), h_Q\right) \otimes \left(\bigotimes_{i=1}^{\alpha} (\overline{L\|_{P_i}})^{\otimes n_i}\right) \otimes \left(\bigotimes_{i=1}^{\alpha} (\overline{K_M\|_{P_i}})^{\otimes n_i(n_i-1)/2}\right) \otimes \left(\bigotimes_{1\leqslant i<j\leqslant\alpha} (\overline{P_i\|_{P_j}})^{-\otimes n_i n_j}\right).$$

Proof. By using the fact that the residue map gives an isometry $\overline{K_M(P)\|_P} \simeq \mathbb{C}|_P$, we can obtain the above isometry from the following two isometries, which can be checked directly from condition (c) of Theorem 1.2:

$$(\lambda(L), h_Q) \simeq \left(\lambda\left(L - \sum_{i=1}^{\alpha} P_i\right), h_Q\right) \otimes \left(\bigotimes_{i=1}^{\alpha} \overline{L\|_{P_i}}\right) \otimes \left(\bigotimes_{1\leqslant i<j\leqslant\alpha} (\overline{P_i\|_{P_j}})^{-1}\right)$$

and

$$(\lambda(L), h_Q) \simeq (\lambda(L - nP), h_Q) \otimes ((\overline{L\|_P})^{\otimes n}) \otimes ((\overline{K_M\|_P})^{\otimes n(n-1)/2}).$$

MEAN VALUE LEMMA III. *With the same notation as above, for any two normalized volume forms ω_1 and ω_2 on M, we have the isometry*

$$\left(\lambda\left(\sum_{i=1}^{\alpha} n_i P_i\right), h_Q(\omega_1)\right) \simeq \left(\lambda\left(\sum_{i=1}^{\alpha} n_i P_i\right), h_Q(\omega_2)\right).$$

That is, for the admissible theory, the Quillen metric does not depend on the choice of the normalized volume forms on the Riemann surface.

Proof. This is a direct consequence of the Riemann–Roch theorem and the Mean Value Lemmas I and II.

2. *Admissible theory with respect to hyperbolic metrics*

In this chapter, we give an application of the ω-admissible theory developed in § 1 by taking ω to be the normalized volume form induced from the hyperbolic metric on a compact Riemann surface. As main results, we construct a Weil function on the moduli space of stable curves and give an upper bound for the self-intersection of the Arakelov canonical divisor in terms of Peterson norms of some modular forms.

2.1. *The Arakelov–Poincaré volume*

In this section, we define an invariant, the Arakelov-Poincaré volume, for any compact Riemann surface of genus at least 2.

Let M be a compact Riemann surface of genus $q \geqslant 2$. For any metric ρ on K_M, denote the corresponding normalized volume form on M by μ_ρ. Let $\rho_{\mathrm{Ar}}(\mu_\rho)$ be the Arakelov metric with respect to μ_ρ. Then there exists a function $C(\rho)$ on M such that

$$C(\rho) := \rho_{\mathrm{Ar}}(\mu_\rho)/\rho.$$

In general, $C(\rho)$ is not a constant on M: by definition, $C(\rho)$ is a constant only when ρ is proportional to the standard hyperbolic metric ρ_{hyp}. (By the standard hyperbolic metric on M, we mean the hyperbolic metric on M with the total volume $2\pi(2q-2)$.) In particular, if $\rho = \rho_{\mathrm{hyp}}$, we call the constant $C(\rho_{\mathrm{hyp}})$ the *Arakelov–Poincaré volume of M*. Indeed, if ω_{hyp} is defined to be the normalized volume form induced from the standard hyperbolic metric μ_{hyp}, that is, if $\omega_{\mathrm{hyp}} := \mu_{\mathrm{hyp}}/(2\pi(q-2))$, then

$$C(\rho_{\mathrm{hyp}}) = \frac{A_{\mathrm{Ar}}(\mathrm{hyp})}{2\pi(2q-2)},$$

where $A_{\mathrm{Ar}}(\mathrm{hyp})$ denotes the volume of the ω_{hyp}-Arakelov metric of M.

Before making an intensive study of the Arakelov–Poincaré volume, we give an application of such a quantity to the theory of arithmetic surfaces.

Let $\pi\colon X \to \mathrm{Spec}(\mathcal{O}_F)$ be a regular arithmetic surface defined over the ring of integers $\mathcal{O}_F$ of a number field F. Consider three hermitian line bundles on X:

$$(K_\pi, \rho_{\mathrm{Ar}}), \quad (K_\pi, \rho_{\mathrm{hyp}}), \quad (K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})),$$

where K_π denotes the (relative) canonical line bundle, and ρ_{Ar}, ρ_{hyp}, and $\rho_{\mathrm{Ar}}(\mathrm{hyp})$ denote the Arakelov metric, the hyperbolic metric and the Arakelov metric with respect to the normalized hyperbolic volume form on the Riemann surface $X(\mathbb{C})$, the fibre of π at infinity, respectively. For a real place σ of F, set N_σ to be 1, and for a complex place σ of F, set $N_\sigma = 2$. By the Mean Value Lemma I in § 1, we know that

$$c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}) = c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})).$$

Therefore, by $\rho_{\mathrm{Ar}}(\mathrm{hyp}) = C(\rho_{\mathrm{hyp}}) \cdot \rho_{\mathrm{hyp}}$, we get the following.

PROPOSITION 2.1. *With the same notation as above,*

$$c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}) = c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{hyp}}) - (2q-2)\sum_\sigma N_\sigma \log C(\rho_{\sigma,\mathrm{hyp}}).$$

In other words, the difference between the self-intersection of the Arakelov canonical divisor and the self-intersection of the canonical line bundle equipped with the standard hyperbolic metric is measured by $C(\rho_{\mathrm{hyp}})$.

Thus, in order to study the self-intersection of the classical Arakelov canonical divisor, which is the key part to all interesting applications, we only need to study $C(\rho_{\mathrm{hyp}})$ and the self-intersection of the canonical line bundle equipped with the standard hyperbolic metric.

Next we go back to the study of $C(\rho_{\mathrm{hyp}})$. First we give another evaluation of $C(\rho_{\mathrm{hyp}})$ in terms of Quillen metrics.

For any hermitian metric ρ on K_M, we naturally get a metric on M. With respect to these metrics, denote the associated Quillen metric on $\lambda(K_M)$ by $h_Q(\rho)$. By the Polyakov variation formula [**8**, Formula (3.31)], for any constant c and any non-zero section α of $\lambda(K_M)$,

$$h_Q(e^c\rho)(\alpha) - h_Q(\rho)(\alpha) = \tfrac{1}{6}\int_M c\cdot c_1(\rho) = \tfrac{1}{6}c\int_M c_1(\rho) = \tfrac{1}{6}c(2q-2).$$

LEMMA 2.2. *With the same notation as above, we have*

$$h_Q(\rho_{\text{hyp}})(\alpha) - h_Q(\rho_{\text{Ar}})(\alpha) = -\tfrac{1}{6}(2q-2)c(\rho_{\text{hyp}}).$$

Here $c(\rho_{\text{hyp}})$ is defined by $C(\rho_{\text{hyp}}) =: \exp[c(\rho_{\text{hyp}})]$.

Proof. By Mean Value Lemma II(i) in § 1, we get

$$h_Q(\rho_{\text{Ar}})(\alpha) - h_Q(\rho_{\text{Ar}}(\omega_{\text{hyp}}))(\alpha) = 0.$$

Thus, by the relation stated before this lemma, we get

$$\begin{aligned} h_Q(\rho_{\text{hyp}})(\alpha) - h_Q(\rho_{\text{Ar}}(\omega_{\text{hyp}}))(\alpha) &= -\tfrac{1}{6}c(\rho_{\text{hyp}})(2q-2) \\ &= -\tfrac{1}{6}(2q-2)c(\rho_{\text{hyp}}). \end{aligned}$$

This completes the proof of the lemma.

On the other hand, by definition,

$$\lambda(K_M) = \det H^0(K_M) \otimes \det H^1(K_M)^{-1} \simeq \det H^0(K_M) \otimes \mathbb{C}^{-1},$$

so if we choose α to be $\omega_1 \wedge \ldots \wedge \omega_q \wedge \mathbf{1}^{-1}$ with $\omega_1, \ldots, \omega_q$ an orthonormal basis of $H^0(K_M)$ with respect to the natural pairing, we see that for any hermitian metric ρ on K_M,

$$h_Q(\rho)(\omega_1 \wedge \ldots \wedge \omega_q \wedge 1^{-1}) = -\zeta'_\rho(0) - \log A_\rho(M),$$

where $\zeta_\rho(s)$ is defined as a formal sum $\sum_j \lambda_j^{-s}$ with λ_j the non-zero eigenvalues of the Laplacian associated with ρ on K_M and hence on M, and $A_\rho(M)$ denotes the volume of M associated to ρ. Hence, if we define the regularized determinant of the corresponding Laplacian by $\det^* \Delta_\rho := \exp(-\zeta'_\rho(0))$, we have the following.

THEOREM 2.3. *With the same notation as above,*

$$c(\rho_{\text{hyp}}) = 12\frac{1}{2q-2}\left(\log\frac{\det^* \Delta_{\rho_{\text{Ar}}}}{A_{\rho_{\text{Ar}}}(M)} - \log\frac{\det^* \Delta_{\rho_{\text{hyp}}}}{A_{\rho_{\text{hyp}}}(M)}\right).$$

2.2. *Degeneration of $c(\rho_{\text{hyp}})$*

In this section, we are going to discuss the degeneration of $c(\rho_{\text{hyp}})$ along with a degenerating family of compact Riemann surfaces. Here, by a degenerating family of Riemann surfaces, we mean a holomorphic map of the unit disc $\mathbb{D}$ into the compactified moduli space $\overline{\mathscr{M}_q}$ of moduli space $\mathscr{M}_q$ of compact Riemann surfaces of genus q (in the sense of Deligne and Mumford) such that the restriction of this map to the punctured disc $\mathbb{D} - \{0\}$ is a holomorphic map into $\mathscr{M}_q$. Hence, the fibre over the origin, the so-called central fibre, in $\mathbb{D}$ is a nodal curve. As the general discussion is the same, we assume in the sequel that the central fibre contains only one single node.

We first study the quantity $\log((\det^* \Delta_{\rho_{\rm Ar}})/A_{\rho_{\rm Ar}}(M))$. By the works of Faltings [**7**], Deligne [**5**], Smit, Moret-Bailly [**16**], and Gillet and Soulé [**10**], among others, we see that $\log((\det^* \Delta_{\rho_{\rm Ar}})/A_{\rho_{\rm Ar}}(M))$ is essentially the Faltings delta function. Indeed, if we let $\delta_F(M)$ be the Faltings delta function of M, then, by (52) of [**20**], we get

$$6\log\frac{\det^* \Delta_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M)} = -\delta_F(M) - 2q\log\pi + 4q\log 2 + a(q).$$

Here, $a(q)$ is the Deligne constant. Thus by using the arithmetic Riemann–Roch formula [**10**], we know that $a(q) = -(1-q)(-24\zeta'(-1)+1)$. Here, $\zeta(s)$ denotes the Riemann zeta function. (The reader may also find the value of $a(g)$ by using the degeneration discussion in [**13**].) So we have the following.

LEMMA 2.4. *With the same notation as above,*

$$\log\frac{\det^* \Delta_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M)} = -\tfrac{1}{6}\big(\delta_F(M) + 2q\log\pi - 4q\log 2 + (1-q)(-24\zeta'(-1)+1)\big).$$

Thus the degeneration of $\log((\det^* \Delta_{\rho_{\rm Ar}})/A_{\rho_{\rm Ar}}(M))$ is simply the degeneration of Faltings' delta function, which was discussed in detail in [**12**] and [**24**]. Applying their results, we get the following.

PROPOSITION 2.5. (i) *For a family of compact Riemann surfaces M_t of genus q, degenerating as $t \to 0$ to surfaces $M_{0,1}$ and $M_{0,2}$ of genera $q_1, q_2 > 0$ joined at a separating node, we get*

$$\lim_{t\to 0}\left(\log\frac{\det^* \Delta^t_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M_t)} - \frac{2}{3}\frac{q_1q_2}{q}\log|t|\right)$$
$$= \log\frac{\det^* \Delta^1_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M_{0,1})} + \log\frac{\det^* \Delta^2_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M_{0,2})} + \tfrac{1}{6}(-24\zeta'(-1)+1).$$

Here $\Delta^t_{\rho_{\rm Ar}}$, $\Delta^1_{\rho_{\rm Ar}}$ and $\Delta^2_{\rho_{\rm Ar}}$ denote the Laplacians on M_t, $M_{0,1}$ and $M_{0,2}$, respectively.

(ii) *For a family of compact Riemann surfaces M_t of genus q degenerating as $t \to 0$ to a surface M_0 of genus $q-1$ with two punctures R and S identified at a non-separating node, denote the Arakelov–Green function on the normalization of the central fibre M_0 by $g_0(\cdot,\cdot)$. We have*

$$\lim_{t\to 0}\left(\log\frac{\det^* \Delta^t_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M)} - \frac{4q-1}{18q}\log|t| - \log(-\log|t|)\right)$$
$$= \log\frac{\det^* \Delta^0_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}(M_0)} + \frac{2q-5}{9q}g_0(R,S)$$
$$+ \tfrac{1}{3}\log 2\pi + \tfrac{1}{6}(-2\log\pi + 4\log 2 + (-24\zeta'(-1)+1)).$$

Here $\Delta^t_{\rho_{\rm Ar}}$ denotes the Laplacian on M_t.

Next, we consider the degeneration of $\log((\det^* \Delta_{\rho_{\rm hyp}})/A_{\rho_{\rm hyp}(M)})$. Related problems for such a degeneration have been studied by many authors, notably

Hejhal [**11**], Wolpert [**25**], and Jorgenson and Lundelius [**14**]. We will follow [**13**]. The starting point is the following result of Wolpert.

LEMMA 2.6 [**25**]. *Let* $[\gamma]$ *denote a homology class that generates* $H_1(M_t, \mathbb{Z})$ *and let* $l(\gamma)$ *denote the length of the geodesic path* γ *in* $[\gamma]$*. Then as* $|t|$ *approaches zero, we get*

$$l(\gamma) = \frac{2\pi^2}{-\log|t|} + o((\log|t|)^{-2}).$$

Thus we may use the length of the geodesic path to parametrize our family of degenerating hyperbolic Riemann surfaces.

To go further, we now recall the definition of the Selberg zeta function for Riemann surfaces. For a connected hyperbolic Riemann surface X with a cusp, let $\{\gamma\}$ denote the set of primitive closed hyperbolic geodesics on X, that is, those γ such that $[\gamma]$ generates its centralizer in the fundamental group of X. The Selberg zeta function $Z(s; X)$ of X is defined, for $\mathrm{Re}(s) > 1$, by $Z(s; X) := \prod_{k=0}^{\infty}\prod_{\{\gamma\}}(1 - e^{-(s+k)l(\gamma)})$. In general, if X is a finite-volume hyperbolic Riemann surface with connected components $X_1, \ldots, X_{r+1}$, we define the Selberg zeta function of X by $Z(s; X) := \prod_{j=1}^{r+1} Z(s; X_j)$. For convenience, we define the delta function of X with respect to the hyperbolic metric by

$$\delta_{\mathrm{hyp}}(X) := \tfrac{1}{2}\chi(X)(-4\zeta'(-1) + \tfrac{1}{2} - \log(2\pi)) + \log\left(\frac{Z^{(r+1)}(1; X)}{(r+1)!}\,\frac{1}{A_{\mathrm{hyp}}(X)}\right),$$

where $\chi(X)$ denotes the Euler characteristic of X, and $A_{\mathrm{hyp}}(X)$ denotes the volume of M with respect to the usual hyperbolic metric. This definition makes sense, as by the Selberg trace formula for weight-zero forms, we know that $Z(s; X)$ has a meromorphic continuation to the whole complex s-plane with a zero at $s = 1$ of order $r + 1$. With this, we may state the following fundamental result of D'Hoker and Phong.

LEMMA 2.7 [**6**]. *If X is a connected compact Riemann surface of genus* $q \geqslant 2$*, then*

$$\log\frac{\det^*\Delta_{\rho_{\mathrm{hyp}}}}{A_{\rho_{\mathrm{hyp}}(X)}} = (-4\zeta'(-1) + \tfrac{1}{2} - \log(2\pi))(1-q) + \log\frac{Z'(1; X)}{A_{\rho_{\mathrm{hyp}}}(X)} = \delta_{\mathrm{hyp}}(X).$$

So the degeneration behaviour of $\log((\det^*\Delta_{\rho_{\mathrm{hyp}}})/A_{\rho_{\mathrm{hyp}}(M)})$ is essentially the degeneration of the Selberg zeta function.

PROPOSITION 2.8. (i) *For a family of compact Riemann surfaces* M_t *of genus* q*, degenerating as* $t \to 0$ *to surfaces* $M_{0,1}$ *and* $M_{0,2}$ *of genera* $q_1, q_2 > 0$*, joined at a separating node,*

$$\lim_{t\to 0}\left(\log\frac{\det^*\Delta^t_{\rho_{\mathrm{hyp}}}}{A_{\rho_{\mathrm{hyp}}}(M_t)} - \tfrac{1}{6}\log|t|\right) = \delta_{\mathrm{hyp}}(M_0) - \log\pi + \log\left(\frac{1}{2q_1-1} + \frac{1}{2q_2-1}\right).$$

(ii) *For a family of compact Riemann surfaces* M_t *of genus* q *degenerating as* $t \to 0$ *to a surface* M_0 *of genus* $q-1$ *with two punctures identified at a* *non-separating node,*

$$\lim_{t\to 0}\left(\log\frac{\det^*\Delta^t_{\rho_{\mathrm{hyp}}}}{A_{\rho_{\mathrm{hyp}}}(M_t)}-\tfrac{1}{6}\log|t|-\log(-\log|t|)\right)=\delta_{\mathrm{hyp}}(M_0)-\log\pi.$$

Proof. Note that, in general, for a degenerating family, some eigenvalues of the corresponding Laplacians will approach zero. So when we measure the change for $(\det^*\Delta^t_{\rho_{\mathrm{hyp}}})/A_{\rho_{\mathrm{hyp}}}(M_t)$, we should study the behaviour of these eigenvalues. To regularize it, we set $\prod_{\mathrm{sev}}(M_t)$ to be the product of small eigenvalues of the Laplacian on functions of M_t, that is, the eigenvalues of the Laplacian which are less than $\frac{1}{4}$. (If there are no small eigenvalues, we set $\prod_{\mathrm{sev}}(M_t):=1$.) The first result we use here is the following degeneration behaviour of the difference

$$\log\frac{\det^*\Delta^t_{\rho_{\mathrm{hyp}}}}{A_{\rho_{\mathrm{hyp}}}(M_t)}-\log\left(\prod_{\mathrm{sev}}(M_t)\right),$$

which is well known to experts.

LEMMA 2.9 [**14**]. *Let M_t denote a degenerating family of hyperbolic Riemann surfaces. Let $[\gamma]$ denote a homology class that generates $H_1(M_t,\mathbb{Z})$, and let $l(\gamma)$ denote the length of the geodesic path γ in $[\gamma]$. Then*

$$\log\frac{\det^*\Delta^t_{\rho_{\mathrm{hyp}}}}{A_{\rho_{\mathrm{hyp}}}(M_t)}-\log\left(\prod_{\mathrm{sev}}(M_t)\right)$$
$$=-\frac{(2\pi)^2}{12l}+\log\frac{2\pi}{l}+\delta_{\mathrm{hyp}}(M_0)-\log\left(\prod_{\mathrm{sev}}(M_0)\right)+o(1).$$

Note that the degeneration behaviour of $-(2\pi)^2/(12l)+\log(2\pi/l)$ is given by Lemma 2.6, so we still need to consider the degeneration behaviour of $\prod_{\mathrm{sev}}(M_t)$. For this, we have the following.

LEMMA 2.10. (i) *If there is only one component for M_0, then*

$$\lim_{t\to 0}\log\frac{\prod_{\mathrm{sev}}(M_t)}{\prod_{\mathrm{sev}}(M_0)}=0.$$

(ii) (Burger [**3**]) *If there are two components for M_0, then*

$$\lim_{t\to 0}\left(\log\frac{\prod_{\mathrm{sev}}(M_t)}{\prod_{\mathrm{sev}}(M_0)}-\log\frac{(2\pi)^2}{-\log|t|^2}\right)=\log\left(\frac{1}{2\pi^2}\left(\frac{1}{2q_1-1}+\frac{1}{2q_2-1}\right)\right).$$

Proof. The first statement comes from the fact that the first eigenvalues of M_t will not go to zero as t goes to zero, while the second statement comes from the result of Burger [**3**] on small eigenvalues of Riemann surfaces together with Lemma 2.6, since it is known that the small eigenvalues vary continuously over the stable compactified moduli space [**11**]. This completes the proof of Lemma 2.10 and hence Proposition 2.5.

With Propositions 2.5 and 2.8, we then easily have the following.

THEOREM 2.11. (i) *For M_t a family of compact Riemann surfaces of genus q, degenerating as $t \to 0$ to surface $M_{0,1}$ and $M_{0,2}$ of genera $q_1, q_2 > 0$ joined at a node,*

$$\lim_{t\to 0}\left[c(\rho_{\mathrm{hyp}}^t) - \frac{6}{2q-2}\left(\tfrac{2}{3}\frac{q_1q_2}{q} - \tfrac{1}{6}\right)\log|t|\right]$$
$$= \frac{6}{2q-2}\left[\left(\log\frac{\det^*\Delta^1_{\rho_{\mathrm{Ar}}}}{A_{\rho_{\mathrm{Ar}}}(M_{0,1})} - \log\frac{Z_1'(1)}{A_{\mathrm{hyp}}(M^0_{0,1})}\right)\right.$$
$$\left. + \left(\log\frac{\det^*\Delta^2_{\rho_{\mathrm{Ar}}}}{A_{\rho_{\mathrm{Ar}}}(M_{0,2})} - \log\frac{Z_2'(1)}{A_{\mathrm{hyp}}(M^0_{0,2})}\right)\right] + \sigma_1(q_1, q_2).$$

Here $\Delta^t_{\rho_{\mathrm{Ar}}}$, $\Delta^1_{\rho_{\mathrm{Ar}}}$ and $\Delta^2_{\rho_{\mathrm{Ar}}}$ denote the Laplacians on M_t, $M_{0,1}$ and $M_{0,2}$, respectively, with respect to the Arakelov metric, Z_1 and Z_2 denote the Selberg zeta functions of $M_{0,1}$ and $M_{0,2}$ respectively, and $\sigma_1(q_1, q_2)$ is a function of q_1 and q_2.

(ii) *For M_t a family of compact Riemann surfaces of genus q degenerating as $t \to 0$ to a surface M_0 of genus $q-1$ with two punctures R and S identified at a non-separating node,*

$$\lim_{t\to 0}\left[c(\rho_{\mathrm{hyp}}^t) - \frac{6}{2q-2}\left(\frac{4q-1}{18q} - \tfrac{1}{6}\right)\log|t|\right]$$
$$= \frac{6}{2q-2}\left(\log\frac{\det^*\Delta^0_{\rho_{\mathrm{Ar}}}}{A_{\rho_{\mathrm{Ar}}}(M_0)} - \log\frac{Z_0'(1)}{A_{\mathrm{hyp}}(M^0_0)}\right)$$
$$+ \frac{2(2q-5)}{3q(2q-2)} g_0(R, S) + \sigma_2(q).$$

Here $\Delta^t_{\rho_{\mathrm{Ar}}}$ denotes the Laplacian on M_t with respect to the Arakelov metric, Z_0 denotes the Selberg zeta function of M_0, and $\sigma_2(q)$ is a function of q.

We remark first, that the functions σ_1 and σ_2 can be precisely evaluated, and second, that the singularity of $c(\rho^t_{\mathrm{hyp}})$ has order approximately $\log|t|$. In particular, there is no $\log(-\log|t|)$-term.

2.3. *An upper bound for the self-intersection of the Arakelov canonical divisor*

In this section, we use the Arakelov–Poincaré volume to give an upper bound for the self-intersection of the Arakelov canonical divisor in terms of Weil–Petersson norms of certain modular forms.

Let $\pi\colon X \to S = \operatorname{Spec}\mathcal{O}_F$ be a regular arithmetic surface defined over a number field F with a ring of integers $\mathcal{O}_F$. Let ω be a normalized volume form on $X(\mathbb{C})$, the fibre of X at infinity. For any ω-admissible hermitian line bundle (L, ρ), let

$$s(L, \rho) := \inf_{P \in X(\overline{F})} h_{L,\rho}(P)$$

and

$$s'(L,\rho) := \liminf_{P\in X(\overline{F})} h_{L,\rho}(P).$$

Here $h_{L,\rho}$ denotes the height corresponding to (L,ρ).

THEOREM 2.12. *With the same notation as above, if the degree of the restriction of L to any component of any fibre of X on S is non-negative, and $d := \deg_F L$ is strictly positive, then, for any non-torsion element $e \in H^1(X, L^{-1})$, we have the following two relations:*

(1) [**21**, Theorem 2] $c_{1,\mathrm{Ar}}^2(L,\rho) \leqslant -d(d-2)s(L,\rho) + [F:\mathbb{Q}]d^2(\log\|e\|_{L^2(\rho,\omega)}+1)$;

(2) [**27**, Theorem 6.3] $2s'(L,\rho) \geqslant c_{1,\mathrm{Ar}}^2(L,\rho)/d \geqslant s'(L,\rho)+s(L,\rho)$.

Moreover, if $\pi: X \to S$ is semi-stable, then

(3) [**7**, Theorem 5] $c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}) \geqslant 0$. *Here K_π denotes the Arakelov canonical line bundle of X equipped with the Arakelov metric ρ_{Ar}.*

From now on, we always assume that $\pi: X \to S$ is semi-stable. Apply the ω-admissible theory to the normalized hyperbolic volume form ω_{hyp}. Denote the Arakelov metric with respect to ω_{hyp} by $\rho_{\mathrm{Ar}}(\mathrm{hyp})$. Applying Theorem 2.12(1) to $(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp}))$, we get

$$\begin{aligned} c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})) \leqslant & -(2q-2)(2q-4)s(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})) \\ & + [F:\mathbb{Q}](2q-2)^2(\log\|e\|_{L^2(\rho_{\mathrm{Ar}}(\mathrm{hyp}),\omega_{\mathrm{hyp}})}+1). \end{aligned}$$

Next, we discuss the quantities on the right-hand side term by term.

First, let us consider the s-term. By definition,

$$c_{1,\mathrm{Ar}}(E_P, \rho_{\mathrm{Ar}}(\omega)) = c_{1,\mathrm{Ar}}(E_P, \rho_{\mathrm{Ar}}) + (0, \beta_\omega(P)+\beta_\omega(Q));$$

thus we see that

$$s(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})) \geqslant s(K_\pi, \rho_{\mathrm{Ar}}) + \tfrac{1}{2}(2q-2)[F:\mathbb{Q}] \inf_{P\in X(\overline{F})} \beta_{\mathrm{hyp}}(P).$$

Here β_{hyp} denotes $\beta_{\omega_{\mathrm{hyp}}}$. On the other hand, essentially, by (2), (3) and the Hodge index theorem, we know that $s(K_\pi, \rho_{\mathrm{Ar}}) \geqslant 0$ [**7**, Theorem 5]. Thus we get

$$s(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})) \geqslant \tfrac{1}{2}(2q-2)[F:\mathbb{Q}] \inf_{P\in X(\overline{K})} \beta_{\mathrm{hyp}}(P).$$

Hence

$$\begin{aligned} c_{1,\mathrm{Ar}}^2(K_\pi, \rho_{\mathrm{Ar}}(\mathrm{hyp})) \leqslant & -\tfrac{1}{2}(2q-2)^2(2q-4)[F:\mathbb{Q}] \inf_{P\in X(\overline{F})} \beta_{\mathrm{hyp}}(P) \\ & + [F:\mathbb{Q}](2q-2)^2(\log\|e\|_{L^2(\rho_{\mathrm{Ar}}(\mathrm{hyp}),\omega_{\mathrm{hyp}})}+1). \end{aligned}$$

Secondly, we consider the e-term. It follows from the definition that, for any $e \in H^1(X, K_\pi^{-1})$, $\|e\|^2_{L^2(\rho_{\mathrm{Ar}}(\mathrm{hyp}),\omega_{\mathrm{hyp}})}$ is given by

$$\sup_\sigma \int_{X_\sigma} \langle e_\sigma, e_\sigma\rangle_{\rho_{\mathrm{Ar}}(\mathrm{hyp})} \omega_{\mathrm{hyp},\sigma}.$$

Since $\rho_{\rm Ar}(\rm hyp) = C(\rm hyp)\cdot\rho(\rm hyp)$, we obtain

$$\|e\|_{L^2(\rho_{\rm Ar}(\rm hyp),\,\omega_{\rm hyp})} = \sup_\sigma \left[C_\sigma(\rm hyp)\int_{X_\sigma}\langle e_\sigma, e_\sigma\rangle_{\rho(\rm hyp)}\omega_{\rm hyp,\,\sigma}\right].$$

But

$$H^1(X_\sigma, K_{X_\sigma}^{-1}) \simeq H^0(X_\sigma, K_{X_\sigma}^{\otimes 2}),$$

so $\int_{X_\sigma}\langle e_\sigma, e_\sigma\rangle_{\rho(\rm hyp)}\omega_{\rm hyp,\,\sigma}$ is nothing but the Weil–Petersson norm of the quadratic differential corresponding to e at X_σ. Denote this quantity by $\|e\|_{\rm WP}^2$. We then have

$$\begin{aligned} c_{1,\rm Ar}^2(K_\pi, \rho_{\rm Ar}(\rm hyp)) \leqslant & -\tfrac{1}{2}(2q-2)^2(2q-4)[F:\mathbb{Q}]\inf_{P\in X(\bar F)}\beta_{\rm hyp}(P) \\ & + [F:\mathbb{Q}](2q-2)^2(\log\|e\|_{\rm WP}+1) \\ & + [F:\mathbb{Q}](2q-2)^2 c(\rm hyp). \end{aligned}$$

Here $\exp[c(\rm hyp)] = C(\rm hyp)$ denotes the maximum of the various $C_\sigma(\rm hyp)$.

Thus, using Theorem 2.3 from § 2.1, we have the following.

THEOREM 2.13. *With the same notation as above, the self-intersection of the Arakelov canonical divisor of X is bounded from above as follows:*

$$\begin{aligned} c_{1,\rm Ar}^2(K_\pi, \rho_{\rm Ar}) \leqslant & \frac{6}{2\pi}[F:\mathbb{Q}]\left(\log\frac{\det^*\Delta_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}} - \log\frac{\det^*\Delta_{\rho_{\rm hyp}}}{A_{\rho_{\rm hyp}}}\right) \\ & -\tfrac{1}{2}(2q-2)^2(2q-4)[F:\mathbb{Q}]\inf_{P\in X(\mathbb{C})}\beta_{\rm hyp}(P) \\ & + [F:\mathbb{Q}](2q-2)^2(\log\|e\|_{\rm WP}+1). \end{aligned}$$

We end this section with the following observations. First, by a result of Wolpert [**26**], if we have an *analytic family* of stable curves, the term $\|e\|_{\rm WP}$ can be totally controlled. Unfortunately, the same idea cannot be applied to the arithmetic situation: we do not know how to choose an arithmetic extension, as there is no good deformation theory in arithmetic at this moment.

Secondly, to apply our result to Diophantine Geometry, one should use the construction of Kodaira and Parshin [**17**, proof of Theorem in § 3]. In this case, one finds that the limited family of Vojta is quite useful [**22**, pp. 165–166]: for a limited family,

$$\log\frac{\det^*\Delta_{\rho_{\rm Ar}}}{A_{\rho_{\rm Ar}}} - \log\frac{\det^*\Delta_{\rho_{\rm hyp}}}{A_{\rho_{\rm hyp}}}$$

can be uniformly bounded, while β can also be uniformly bounded, as a standard Moser iteration [**9**, Theorem 8.24] shows that $\sup\beta$ is bounded by $C\lambda_1^{-1}$, where C is a constant which does not depend on β, and λ_1 denotes the first (non-zero) eigenvalue of the Laplacian associated to $\omega_{\rm can}+\omega_{\rm hyp}$. Thus the only term we need to take care of is $\|e\|_{\rm WP}$, the term which involves Petersson norms of some modular forms.

Finally, observe that, for modular curves, by Rankin's work on Petersson norms of modular forms, $\|e\|_{\rm WP}$ can be calculated in terms of the special value of the Rankin L-function associated to e [**18**, Theorem 3]. So it is essential to study the

Rankin L-function in order to obtain the arithmetic Miyaoka–Yau inequality. (For this purpose, one may find that the paper [**19**] is quite useful.) Independently, such ideals have been used already with success by Abbes Ullmo and Michel Ullmer. I thank the referee for informing me of this.

3. Ω-*Admissible theory for singular arithmetic surfaces*

In this chapter, we give an ω-admissible theory for singular arithmetic surfaces associated to stable curves. When ω corresponds to a canonical volume form, such a theory was (first) given by Aitken [**1**].

3.1. *Generalized Néron family*

The ω-admissible theory for line bundles on compact Riemann surfaces consists of two parts: one concerns the ω-Green's functions, while the other deals with the ω-Arakelov metric over the canonical line bundle. In this section, we give a parallel theory for nodal curves. We begin by recalling the theory of Néron families for smooth curves C, which is equivalent to ω-admissible theory for line bundles.

Following [**15**, Chapter 1], by a Néron family on a smooth curve C of genus $q(C)$ defined over $\mathbb{C}$, we mean a map $D \mapsto \lambda_D$ from Cartier divisors on C to Weil functions, satisfying the following conditions

NF 1. The map $D \mapsto \lambda_D$ is a homomorphism.

NF 2. If $D = (f)$ is the divisor of a rational function f, and ν the valuation associated to D, then $\lambda_D = \nu \circ f -$ constant.

NF 3. If (U, f) represents D, then there exists a C^∞-function α on U such that

$$\lambda_D = \nu \circ f + \alpha.$$

NF 4. For any two points $P \neq Q$, $\lambda_P(Q) = \lambda_Q(P)$.

A Néron family always exists. (Usually, we also write $\lambda_D(E)$ as $\lambda(D, E)$.) Moreover, for a fixed volume form ω on C, if the Chern form of the Néron family λ, defined to be $d_P d_P^c \lambda_D(P)$, is proportional to ω, then λ is essentially unique; if λ' is another Néron family such that its Chern form is proportional to ω, then there exists an absolute constant c such that, for any two divisors D and E on C,

$$\lambda_D(E) - \lambda'_D(E) = c \cdot \deg D \cdot \deg E.$$

(See below, or [**15**, Chapter 1], for more details.) We call such a Néron family an ω-*admissible Néron family*, and denote it by λ_ω.

There is another way to express the admissibility condition on a Néron family λ.

NF 5. For all points P in C,

$$-\lambda((\alpha), P) = \left(\log |h(P)| + \lim_{Q \to P} (\log |z(Q) - z(P)| + \lambda(P, Q)) \right) - q(C)\beta_\omega(P) + \text{constant}.$$

Here $\alpha = h(z)\, dz$ is a section of the canonical bundle, and β_ω is the unique function such that $dd^c \beta_\omega = \omega - \omega_{\text{can}}$, $\int \beta_\omega(\omega + \omega_{\text{can}}) = 0$ with ω_{can} the canonical volume form of C.

Obviously, if we let λ_ω be $\frac{1}{2}g_\omega$ with g_ω the ω-Green's function, and impose the ω-Arakelov metric on K_C, then λ_ω satisfies the above conditions.

With this in mind, we make the following definition of a generalized admissible Néron family for a nodal curve C defined over $\mathbb{C}$.

Let $C = \bigcup C_i$ be the decomposition of C into its irreducible components. Let S denote the set of singular points of C and denote the arithmetic genus of C by $p_a(C)$. Denote the normalization of C_i by $\tilde{C}_i$. Put normalized volume forms $\omega = \{\omega_i\}$ on $\{\tilde{C}_i\}$. Let $\beta_\omega = \{\beta_{\omega_i}\}$ be the functions on $\bigcup \tilde{C}_i$ defined as above. (Here, if the geometric genus of C_i is zero, then to define β_ω we take the normalized volume form associated to the standard Fubini–Study metric on $\mathbb{P}^1$ as the canonical volume form.) With this, by a *generalized admissible Néron family with respect to* ω, we mean a map $D \mapsto \lambda_D$ from Cartier divisors on C *with supports disjoint from the singular set* S to functions, satisfying the following conditions.

GNF 1. The map $D \mapsto \lambda_D$ is a homomorphism.

GNF 2. If $D = (f)$ is the divisor of a rational function f on C, then there exists a constant $\gamma_\lambda(f)$, depending only on f, such that, for any $P \in C \backslash S$,

$$\lambda_D(P) = \nu \circ f(P) - \gamma_\lambda(f).$$

GNF 3. If (U, f) represents D, then there exists a continuous function α on U such that

$$\lambda_D = \nu \circ f + \alpha.$$

GNF 4. For any two points $P \neq Q$, disjoint from S, $\lambda_P(Q) = \lambda_Q(P)$.

GNF 5. Let α be a meromorphic section of the dualizing sheaf K_C whose divisor κ has support disjoint from S. Let U be an open neighbourhood in C, disjoint from S and the support of κ, which is parametrized by a complex coordinate z. On U, write $\alpha = h(z)\,dz$ for some nowhere-zero holomorphic function h. Then there exists a constant $\gamma_\lambda(\alpha)$, depending only on α, such that for any point $P \in C \backslash S$,

$$\lambda(\kappa, P) = -\log|h(P)| - \lim_{Q \to P}(\log|z(Q) - z(P)| + \lambda(P, Q)) + p_a(C)\beta_\omega(P) - \gamma_\lambda(\alpha).$$

If ω is the collection of canonical volume forms on $\{\tilde{C}_i\}$, then in [**1**] Aitken shows that there exists a generalized admissible Néron family. We will construct later a generalized admissible Néron family with respect to any ω. But now, to understand how we end up with our construction, let us study a toy model.

For the time being, let C be a regular curve. Denote by g_ω the Green's functions on C with respect to any fixed normalized volume form ω. Let R and S be two distinct points on C and let C' be the new nodal curve of arithmetic genus $p_a(C')$ resulting from identifying R and S into an ordinary double point.

For any two regular points P and Q in C', set

$$\lambda'_\omega(P, Q) := \lambda_\omega(P, Q) - \frac{1}{2p_a(C')}\lambda_\omega(P + Q, R + S) + \frac{1}{2}\frac{1}{p_a(C')}(\beta_\omega(P) + \beta_\omega(Q) + \beta_\omega(R) + \beta_\omega(S)).$$

Extend the definition of λ'_ω by linearity. Then, it is easy to see that GNF 1, GNF 3 and GNF 4 are satisfied. So we need to check GNF 2 and GNF 5.

Let f be a rational function of C' whose divisor (f) is away from the double point. Then

$$\begin{aligned}\gamma'_\omega(f) &:= -\log|f(P)| - \lambda'_\omega((f), P)\\ &= -\log|f(P)| - \lambda_\omega((f), P) + \frac{1}{2p_a(C')}\lambda_\omega((f), R+S) - \frac{1}{2p_a(C')}\beta_\omega((f))\\ &= \gamma_\omega(f) + \frac{1}{2p_a(C')}\lambda_\omega((f), R+S) - \tfrac{1}{2}\frac{1}{p_a(C')}\beta_\omega((f)),\end{aligned}$$

which is independent of P. Here, if $D = \sum n_k P_k$ is a divisor, $\beta_\omega(D)$ denotes $\sum n_k \beta_\omega(P_k)$. This shows that GNF 2 is satisfied.

Now let α be a meromorphic section of the dualizing sheaf $K_{C'}$ of C' whose divisor κ has support away from the double point. Let U be an open neighbourhood in C' away from the double point and κ, which is parametrized by a complex coordinate z. On U, we write $\omega = h(z)\,dz$ for some holomorphic function h on U. Then, for each regular point P in C', we get

$$\begin{aligned}\gamma'_\omega(\alpha) &= -\log|h(P)| - \lambda'_\omega(\kappa, P) - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda'_\omega(P, Q)) + p_a(C')\beta_\omega(P)\\ &= -\log|h(P)| - \lambda_\omega(\kappa, P) + \frac{2p_a(C')-2}{2p_a(C')}\lambda_\omega(P, R+S) + \frac{1}{2p_a(C')}\lambda_\omega(\kappa, R+S)\\ &\quad - \frac{1}{2p_a(C')}\beta_\omega(\kappa) - \frac{2p_a(C')-2}{2p_a(C')}\beta_\omega(P) - \frac{2p_a(C')-2}{2p_a(C')}(\beta_\omega(R)+\beta_\omega(S))\\ &\quad - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda_\omega(P, Q)) + \frac{1}{p_a(C')}\lambda_\omega(P, R+S) - \frac{2}{2p_a(C')}\beta_\omega(P)\\ &\quad - \frac{1}{4p_a(C')}(\beta_\omega(R)+\beta_\omega(S)) + p_a(C')\beta_\omega(P)\\ &= -\log|h(P)| - \lambda_\omega(\kappa - R - S, P)\\ &\quad - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda_\omega(P, Q)) + p_a(C)\beta_\omega(P)\\ &\quad - \frac{1}{2p_a(C')}\beta_\omega(\kappa) + \frac{1}{2p_a(C')}\lambda_\omega(\kappa, R+S) - \frac{2p_a(C')-1}{2p_a(C')}(\beta_\omega(R)+\beta_\omega(S)).\end{aligned}$$

Thus, if we also consider α as a meromorphic section of the dualizing line bundle K_C of C, then it has divisor $\kappa - R - S$. Thus,

$$\begin{aligned}\gamma'_\omega(\alpha) = \gamma_\omega(\alpha) - \frac{1}{2p_a(C')}\beta_\omega(\kappa) + \frac{1}{2p_a(C')}\lambda_\omega(\kappa, R+S)\\ - \frac{2p_a(C')-1}{2p_a(C')}(\beta_\omega(R)+\beta_\omega(S)),\end{aligned}$$

which is independent of P. So GNF 5 is also satisfied. This completes the proof of the existence of a generalized admissible Néron family for the model.

It seems to be the case that for this toy model, the construction of λ'_ω is quite artificial. It is not so. Indeed, we have the following observation.

With respect to the canonical volume form on C, the construction of Aitken [**1**] gives the generalized admissible Néron family $\lambda'(P,Q)$ on C' on setting

$$\lambda'(P,Q) := \lambda(P,Q) - \frac{1}{2p_a(C')}\lambda(P+Q,R+S).$$

Writing the right-hand side in terms of λ_ω, we get

$$\begin{aligned}\lambda'(P,Q) &= \lambda_\omega(P,Q) - \frac{1}{2p_a(C')}\lambda_\omega(P+Q,R+S)\\ &\quad + \frac{1-p_a(C')}{2p_a(C')}(\beta_\omega(P)+\beta_\omega(Q)) + \frac{1}{2p_a(C')}(\beta_\omega(R)+\beta_\omega(S))\\ &= \lambda_\omega(P,Q) - \frac{1}{2p_a(C')}\lambda_\omega(P+Q,R+S)\\ &\quad + \frac{1}{2p_a(C')}(\beta_\omega(P)+\beta_\omega(Q)+\beta_\omega(R)+\beta_\omega(S)) - \tfrac{1}{2}(\beta_\omega(P)+\beta_\omega(Q)).\end{aligned}$$

Thus by setting $\lambda'_\omega(P,Q)$ equal to

$$\lambda_\omega(P,Q) - \frac{1}{2p_a(C')}\lambda_\omega(P+Q,R+S) + \frac{1}{2p_a(C')}(\beta_\omega(P)+\beta_\omega(Q)+\beta_\omega(R)+\beta_\omega(S)),$$

we obtain the relation

$$2\lambda'_\omega(P,Q) = 2\lambda'(P,Q) + \beta_\omega(P) + \beta_\omega(Q),$$

which is exactly the same as the relation in § 1.1 for Green's functions over smooth curves. So, by using the process in § 1, we should get a suitable theory for the generalized admissible Néron family with respect to ω.

With this in mind, we are ready now to give the following theorem.

EXISTENCE THEOREM. *Let $C = \bigcup C_i$ be a nodal curve; then there exists a generalized admissible Néron family λ_ω with respect to any normalized volume forms $\omega = \{\omega_i\}$ on $C = \bigcup C_i$.*

Proof. Denote by λ the generalized Néron family on C with respect to the canonical volume forms on C introduced by Aitken [**1**]. Define $\beta_\omega = \{\beta_{\omega_i}\}$ on C by the condition that

$$dd^c\beta_{\omega_i} = \omega_i - \omega_{\mathrm{can},\,i}, \qquad \int_{\tilde{C}_i}\beta_{\omega_i}(\omega_i + \omega_{\mathrm{can},\,i}) = 0.$$

Here $\omega_{\mathrm{can},\,i}$ denotes the canonical volume form on the normalization $\tilde{C}_i$ of the irreducible component C_i of C. (If the genus of $\tilde{C}_i$ is zero, then the canonical volume form is supposed to be the normalized volume form associated to the Fubini–Study metric on $\mathbb{P}^1$.)

Then for any two regular points P and Q of C, set

$$\lambda_\omega(P,Q) := \lambda(P,Q) + \tfrac{1}{2}\beta_\omega(P) + \tfrac{1}{2}\beta_\omega(Q).$$

Extend the definition of λ_ω by linearity. Then it is easy to see that GNF 1, GNF 3, and GNF 4 are satisfied. Next, we check GNF 2 and GNF 5.

Let f be a rational function of C whose divisor (f) is away from double points. Then

$$\begin{aligned}\gamma_\omega(f) &:= -\log|f(P)| - \lambda_\omega((f), P)\\ &= -\log|f(P)| - \lambda((f), P) - \tfrac{1}{2}\beta_\omega((f))\\ &= \gamma(f) - \tfrac{1}{2}\beta_\omega((f)),\end{aligned}$$

which is independent of P. Here $\gamma(f)$ denotes the corresponding constant in GNF 2 for Aitken's generalized admissible Néron family (with respect to the canonical volume forms). This shows that GNF 2 is satisfied.

Now let α be a meromorphic section of the dualizing sheaf K_C of C whose divisor κ has support away from double points. Let U be an open neighbourhood in C, away from double points and κ, which is parametrized by a complex coordinate z. On U, we write $\omega = h(z)\,dz$ for some holomorphic function h on U. For each regular point P in C, we get

$$\begin{aligned}\gamma_\omega(\alpha, P) &= -\log|h(P)| - \lambda_\omega(\kappa, P) - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda_\omega(P, Q)) + p_a(C)\beta_\omega(P)\\ &= -\log|h(P)| - \lambda(\kappa, P) - \tfrac{1}{2}\beta_\omega(\kappa) - (p_a(C)-1)\,\beta_\omega(P)\\ &\quad - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda(P, Q) + \tfrac{1}{2}\beta_\omega(P) + \tfrac{1}{2}\beta_\omega(Q)) + p_a(C)\,\beta_\omega(P)\\ &= -\log|h(P)| - \lambda(\kappa, P) - \lim_{Q\to P}(\log|z(Q)-z(P)| + \lambda(P, Q)) - \tfrac{1}{2}\beta_\omega(\kappa)\\ &= \gamma(\alpha) - \tfrac{1}{2}\,\beta_\omega(\kappa),\end{aligned}$$

which is independent of P. Here $\gamma(\alpha)$ denotes the corresponding constant in GNF 5 for Aitken's generalized admissible Néron family (with respect to the canonical volume forms). So GNF 5 also is satisfied. This completes the proof of the existence theorem.

3.2. λ_ω-*admissible metrics on line bundles*

Having established the existence of a generalized admissible Néron family with respect to ω in § 3.1, we now give a metric theory for line bundles over a nodal curve.

With the same notation as in § 3.1, assume that λ_ω is a generalized admissible Néron family with respect to ω. We introduce λ_ω-admissible metrics on all line bundles as follows: first, for any regular point $Q \neq P$, on $\mathcal{O}_C(P)$, define a metric by setting

$$\|1_P\|(Q) := e^{-\lambda_\omega(P,Q)},$$

where 1_P denotes the defining section of $\mathcal{O}_C(P)$; then we use linearity to define metrics on all line bundles by assuming that the natural algebraic isomorphisms $\mathcal{O}_C(D)\otimes\mathcal{O}_C(D') \simeq \mathcal{O}_C(D+D')$ are isometries for all divisors D and D' on C, whose supports are away from the double points (and call the metrics on $\mathcal{O}_C(D)$ obtained in this way restricted λ_ω-admissible metrics); finally, by a λ_ω-admissible metric on $\mathcal{O}_C(D)$, we mean a metric on $\mathcal{O}_C(D)$ which is a constant multiple of the restricted λ_ω-admissible metric on $\mathcal{O}_C(D)$.

The λ_ω-admissible metrics on line bundles are well defined: first, any line bundle L on C can be expressed as $\mathcal{O}_C(D)$ for some divisor D on C with support away from the double points; secondly, the expression of L in the form $\mathcal{O}_C(D)$ is not unique, but by GNF 2, λ_ω-admissible metrics on L are well defined, as we allow admissible metrics to differ from each other by a constant factor. Usually we will denote the line bundle $\mathcal{O}_C(D)$ together with the restricted λ_ω-admissible metric by $\underline{\mathcal{O}_C(D)}$, while a line bundle L together with a λ_ω-admissible metric is denoted by $\overline{L}$, by abuse of notation. We will call $\overline{L}$ a λ_ω-admissible hermitian line bundle.

3.3. *Deligne metric and arithmetic intersection*

With the same notation as above, let $\overline{L}$ and $\overline{M}$ be two λ_ω-admissible hermitian line bundles on C. Choose sections l and m of L and M respectively, such that the divisors (l) and (m) are away from the double points of C and have disjoint supports. Then there exist constants $c(l)$ and $c(m)$ such that

$$\overline{L} = \underline{\mathcal{O}_C((l))} \otimes \mathcal{O}_C(e^{c(l)}), \qquad \overline{M} = \underline{\mathcal{O}_C((m))} \otimes \mathcal{O}_C(e^{c(m)}).$$

Here $\mathcal{O}_C(e^c)$ denotes the trivial line bundle together with a metric such that the square of the norm of 1 is e^{-c}.

Now, we are ready to introduce a norm on the Deligne pairing $\langle L, M\rangle$ associated to $\overline{L}$ and $\overline{M}$, which we call the *Deligne norm* h_D. First choose two sections l and m and L and M respectively, such that the divisors (l) and (m) are away from the double points of C and have disjoint supports; then define a norm on a generator $\langle l, m\rangle$ of $\langle L, M\rangle$ by

$$-\log|\langle l, m\rangle|^2_{h_D} := 2\lambda((l), (m)) + \deg M \cdot c(l) + \deg L \cdot c(m).$$

Usually, we also write $(\langle L, M\rangle, h_D)$ as $\langle \overline{L}, \overline{M}\rangle$. Obviously we have the following.

PROPOSITION 3.1. *With the same notation as above, for line bundles with λ_ω-admissible metrics on C, we have the following natural isometries:*

(a) $\langle \overline{L}, \overline{M}\rangle \simeq \langle \overline{M}, \overline{L}\rangle$;

(b) $\langle \overline{L}_1 \otimes \overline{L}_2, \overline{M}\rangle \simeq \langle \overline{L}_1, \overline{M}\rangle \otimes \langle \overline{L}_2, \overline{M}\rangle$.

3.4. *λ_ω-Arakelov metrics*

In arithmetic intersection theory for regular curves, the central result is the so-called adjunction formula, which claims that the residue isomorphism $K_C(P)|_P \simeq \mathbb{C}$ naturally becomes an isometry if we put suitable metrics on $\mathcal{O}_C(P)$ and K_C, respectively. Here K_C is the dualizing line bundle of C. In this section, we study the same problem for nodal curves.

With the same notation as above, for any regular point P of C, define the λ_ω-*Arakelov metric* on $\mathcal{O}_C(P)$, denoted by $\rho_{\mathrm{Ar},\lambda_\omega,P}$, as follows:

$$\|1_P\|^2_{\rho_{\mathrm{Ar}},\lambda_\omega,P}(Q) := e^{-2\lambda_\omega(P,Q)+\beta_\omega(P)}.$$

Obviously, $\rho_{\mathrm{Ar},\lambda_\omega,P}$ is a λ_ω-admissible metric on $\mathcal{O}_C(P)$.

Next, we define the λ_ω-*Arakelov metric* $\rho_{\mathrm{Ar},\lambda_\omega}$ on K_C: for any section $\alpha = h(z)\,dz$ of K_C, whose divisor is away from the double points of C,

$$\|h(z)\,dz\|^2_{\rho_{\mathrm{Ar},\lambda_\omega}}(P) := |h(P)|^2 \cdot \lim_{Q\to P} \frac{|z(Q)-z(P)|^2}{e^{-2\lambda(P,Q)}} \cdot e^{-2p_a(C)\beta_\omega(P)}.$$

By GNF 5, we see that $\rho_{\mathrm{Ar},\lambda_\omega}$ is a λ_ω-admissible metric on K_C.

With this, if we start with $(K_C, \rho_{\mathrm{Ar},\lambda_\omega})$ and $(\mathcal{O}_C(P), \rho_{\mathrm{Ar},\lambda_\omega,P})$, then the restriction of the tensor metric on $K_C(P)$, denoted by ρ_{Ar} for simplicity, to the point P can be calculated as follows:

$$\begin{aligned}\left\|h(z)\frac{dz}{z}\otimes 1_P\right\|^2_{\rho_{\mathrm{Ar}}}(P) &= \|h(z)\ dz\|^2_{\rho_{\mathrm{Ar},\lambda_\omega}}\cdot\left\|\frac{1_P}{z}\right\|^2_{\rho_{\mathrm{Ar},\lambda_\omega,P}}(P)\\ &= |h(P)|^2\cdot\lim_{Q\to P}\frac{|z(Q)-z(P)|^2}{e^{-2\lambda(P,Q)}}\cdot e^{-2p_a(C)\beta_\omega(P)}\\ &\quad\cdot\lim_{Q\to P}\frac{1}{|z(Q)-z(P)|^2}\cdot e^{-2\lambda(P,Q)}\cdot e^{\beta_\omega(P)}\\ &= |h(P)|^2\cdot e^{(-2p_a(C)+1)\beta_\omega(P)}\\ &= |\mathrm{res}_P(\alpha)|^2\cdot e^{-d(K_C(P))\cdot\beta_\omega(P)}.\end{aligned}$$

Here $d(K_C(P))$ denotes the degree of $K_C(P)$. Therefore, for $\overline{L}$, if we denote by $\overline{L\|_P}$ the space obtained by scaling the metric of $\overline{L}|_P$ by a constant factor $e^{d(L)\beta_\omega(P)}$, then we have the following.

ADJUNCTION FORMULA. *With the same notation as above, for any regular point P of C, the natural residue map induces as isometry*

$$\overline{K_C(P)\|_P}\simeq\overline{\mathbb{C}}.$$

Here K_C and $\mathcal{O}_C(P)$ have the λ_ω-Arakelov metrics, and $\mathbb{C}$ has the standard flat metric.

REMARK. All the λ_ω-admissible metrics are only defined over regular points of C. In general, at double points, such metrics have logarithmic singularities.

From the above we see that arithmetic intersection theory for nodal curves is exactly the same as for regular curves. So we can use the same methods as for regular curves to prove the results for nodal curves. In particular, we also have the so-called Mean Value Lemmas as in § 1. We leave this to the reader.

3.5. *λ_ω-admissible metrics on cohomology determinants*

In this section, we define λ_ω-admissible metrics on cohomology determinants.

For this, let us first recall that for any two line bundles L and M over a nodal curve C, the Deligne pairing $\langle L, M\rangle$ has a natural decomposition:

$$\langle L, M\rangle = \mathrm{Det}\,R\Gamma(L\otimes M)\otimes\mathrm{Det}\,R\Gamma(L)^{-1}\otimes\mathrm{Det}\,R\Gamma(M)^{-1}\otimes\mathrm{Det}\,R\Gamma(\mathcal{O}_C).$$

Here $\mathrm{Det}\,R\Gamma(L)$ denotes the cohomology determinant associated to L. Moreover, we have Serre duality

$$\mathrm{Det}\,R\Gamma(L)\simeq\mathrm{Det}\,R\Gamma(K_C\otimes L^{-1}).$$

It is then well known that these two relations for Deligne pairings are equivalent to the Riemann–Roch isomorphism:

$$\mathrm{Det}\,R\Gamma(L)^{\otimes 2}\otimes\mathrm{Det}\,R\Gamma(\mathcal{O}_C)^{\otimes -2}\simeq\langle L, L\otimes K_C^{-1}\rangle.$$

(For more discussion about Deligne pairings, please see the appendix.)

To define admissible metrics on cohomology determinants for nodal curves, we will use the above isomorphisms of the Deligne pairings. With the same notation as in the previous sections, let λ_ω be a generalized admissible Néron family on a nodal curve C. Write $\rho_{\mathrm{Ar},\omega}$ for the λ_ω-Arakelov metric on the dualizing sheaf K_C of C. Fix a metric h_0 on Det $R\Gamma(\mathcal{O}_C)$. Then, for any line bundle L with a λ_ω-admissible metric ρ_L, we define the λ_ω-admissible metric $h_{\mathrm{Ad}}(L)$ on $\operatorname{Det} R\Gamma(L)$ by requiring that the following map is an isometry:

$$(\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(L))^{\otimes 2} \otimes (\operatorname{Det} R\Gamma(\mathcal{O}_C), h_0)^{\otimes -2} \simeq \langle (L, \rho_L), (L, \rho_L) \otimes (K_C, \rho_{\mathrm{Ar},\omega})^{-1} \rangle.$$

We call this metric $h_{\mathrm{Ad}}(L)$ the λ_ω-admissible metric on $\operatorname{Det} R\Gamma(L)$ with respect to ρ_L (and with respect to the generalized admissible Néron family λ_ω).

Obviously, with this definition of the λ_ω-admissible metric on $\operatorname{Det} R\Gamma(L)$, the Serre duality and the Deligne decomposition become isometries if we put λ_ω-admissible metrics on the corresponding data. So we have the following.

THEOREM 3.2. *With the same notation as above, for a nodal curve C, with respect to a fixed metric h_0 on $\operatorname{Det} R\Gamma(\mathcal{O}_C)$ and a fixed generalized Néron family λ_ω, for any λ_ω-admissible metrized line bundle $\overline{L}$ there exists a unique metric $h_{\mathrm{Ad}}(\overline{L})$ on $\operatorname{Det} R\Gamma(L)$, such that we have the following isometries:*

(i) (Deligne decomposition)

$$\langle \overline{L}, \overline{M} \rangle \simeq (\operatorname{Det} R\Gamma(L \otimes M), h_{\mathrm{Ad}}(\overline{L} \otimes \overline{M})) \otimes (\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(\overline{L}))^{-1} \otimes (\operatorname{Det} R\Gamma(M), h_{\mathrm{Ad}}(\overline{M}))^{-1} \otimes (\operatorname{Det} R\Gamma(\mathcal{O}_C), h_0);$$

(ii) (Serre duality)

$$(\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(\overline{L})) \simeq (\operatorname{Det} R\Gamma(K_C \otimes L^{-1}), h_{\mathrm{Ad}}((K_C, \rho_{\mathrm{Ar},\omega}) \otimes \overline{L}^{-1}));$$

(iii) (Riemann–Roch theorem)

$$(\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(\overline{L}))^{\otimes 2} \otimes (\operatorname{Det} R\Gamma(\mathcal{O}_C), h_0)^{\otimes -2} \simeq \langle \overline{L}, \overline{L} \otimes (K_C, \rho_{\mathrm{Ar},\omega})^{-1} \rangle.$$

There is another way to define the λ_ω-admissible metric $h_{\mathrm{Ad}}(\overline{L})$ associated to $\overline{L}$. Indeed, we have the following.

PROPOSITION 3.3. *With the same notation as above, for a nodal curve C, with respect to a fixed metric h_0 on $\operatorname{Det} R\Gamma(\mathcal{O}_C)$ and a fixed generalized Néron family λ_ω, for all λ_ω-admissible metrized line bundles $\overline{L}$, the metric $h_{\mathrm{Ad}}(\overline{L})$ on $\operatorname{Det} R\Gamma(L)$ defined in the previous theorem satisfies the following conditions:*

(i) *an isometry of λ_ω-admissible hermitian line bundles $\overline{L} \to \overline{L}'$ induces an isometry from $(\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(\overline{L}))$ to $(\operatorname{Det} R\Gamma(L'), h_{\mathrm{Ad}}(\overline{L}'))$;*

(ii) *if the λ_ω-admissible metric on $\overline{L}$ is changed by a multiplicative factor $\alpha \in \mathbb{R}^+$, then the metric on $(\operatorname{Det} R\Gamma(L), h_{\mathrm{Ad}}(\overline{L}))$ is changed by the factor $\alpha^{\chi(L)}$;*

(iii) (Riemann–Roch condition for closed immersions) *For any regular point P on C, take the λ_ω-Arakelov metric on $\mathcal{O}_M(P)$ and the tensor metric on $L(-P)$; then the algebraic isomorphism*

$$\operatorname{Det} R\Gamma(L) \simeq \operatorname{Det} R\Gamma(L(-P)) \otimes L|_P,$$

induced by the short exact sequence of coherent sheaves

$$0 \to L(-P) \to L \to L|_P \to 0,$$

is an isometry

$$(\mathrm{Det}\, R\Gamma(L), h_{\mathrm{Ad}}(\overline{L})) \simeq (\mathrm{Det}\, R\Gamma(L(-P)), h_{\mathrm{Ad}}(\overline{L}(-\underline{P}))) \otimes \overline{L\|_P}.$$

REMARK. One may use this proposition first to define the λ_ω-admissible metric on $\mathrm{Det}\, R\Gamma(L)$ for any λ_ω-admissible line bundle $\overline{L}$; then to prove the (local and global) arithmetic Riemann–Roch theorems by using the adjunction formula as was done in [**15**]. (Here local means the result on archimedean places, while global refers to the results on arithmetic surfaces.) We leave this to the reader.

Proof of Proposition 3.3. This is an easy consequence of the previous theorem and the adjunction formula.

Appendix. Algebraic and analytic structures of Deligne's pairing

A.1. *Algebraic structure*

Let G and H be abelian groups with H uniquely 2-divisible. The map $D\colon G \to H$ is said to be *quadratic* (following Bourbaki) if it satisfies the identity

$$D(x+y+z) - D(x+y) - D(x+z) - D(y+z) + D(x) + D(y) + D(z) - D(0) = 0,$$

or equivalently, if

$$\langle x, y\rangle := D(x+y) - D(x) - D(y) + D(0)$$

is a bilinear pairing. Define $L(x) := D(x) - \frac{1}{2}\langle x, y\rangle - D(0)$. Then an equivalent condition is that L is a homomorphism. We can regard the decomposition

$$D(x) = \tfrac{1}{2}\langle x, y\rangle + L(x) + D(0)$$

as decomposing the quadratic map D into homogeneous components of degree 2, 1 and 0. (Please compare this with the Deligne decomposition for Deligne pairing in terms of cohomology determinants.)

For $\alpha \in G$ it is equivalent to require that $L(x) = -\frac{1}{2}\langle x, \alpha\rangle$ for all $x \in G$ or that $D(\alpha - x) = D(x)$ for all $x \in G$. Such an element α is called *canonical* with respect to D. (Please compare this with Serre duality.) When a canonical element exists, the above decomposition takes the form

$$D(x) = \tfrac{1}{2}\langle x, x\rangle - \tfrac{1}{2}\langle x, \alpha\rangle + D(0). \tag{2}$$

(Please compare this with the Riemann–Roch theorem for surfaces.)

Just as in geometry, (2) does not determine D uniquely. In fact, it depends on the choice of $D(0)$. For this reason, we usually call the geometric version of (2) the weak version of the Riemann–Roch theorem.

We always have $D(\alpha) = D(0)$ and $D(2\alpha) = \langle \alpha, \alpha\rangle + D(0)$. If there is an integer a with $D(2\alpha) = aD(\alpha)$, then $\langle \alpha, \alpha\rangle = (a-1)D(0)$, so for any $n \in \mathbb{Z}$ we have

$$(a-1)D(n\alpha) = (\tfrac{1}{2}(a-1)(n^2 - n) + 1)\langle \alpha, \alpha\rangle.$$

(Please compare this with the Mumford relation on determinant line bundles over moduli spaces of stable curves. I thank Professor C. T. C. Wall for providing me with this alternative presentation of this section.)

A.2. *Analytic structure*

Let $\pi: X \to S$ be a flat family of relative dimension n. Let $L_0, \ldots, L_n$ be invertible hermitian sheaves over X. Then, in [**5**], Deligne introduces the Deligne pairing $\langle L_0, \ldots, L_n\rangle(X/S)$ together with the Deligne metric h_D, which is multilinear and symmetric in the L_i.

On the other hand, for any line bundle L over X, there exists a determinant line bundle $\lambda(L) = \operatorname{Det} R\Gamma(L)$ over S. A natural question is what should be the relation between the Deligne pairing and the cohomology determinant.

To understand this precisely, consider the case of arithmetic surfaces. Deligne shows that there exists a natural algebraic isomorphism

$$\langle L, M\rangle \simeq \lambda(L \otimes M) \otimes \lambda(L)^{-1} \otimes \lambda(M)^{-1} \otimes \lambda(\mathcal{O}_X).$$

Indeed, one may equally use this isomorphism to define the Deligne pairing.

For this algebraic isomorphism, we have an hermitian metric h_D on the Deligne pairing $\langle L, M\rangle$ as soon as we are given metrics ρ and τ on L and M respectively.

PROPOSITION A.1. *With respect to any metric on $X_{\mathbb{C}}$, if we put the Deligne metric on the Deligne pairing and the Quillen metrics on the associated determinant line bundles, then*

$$\langle L, M\rangle \simeq \lambda(L \otimes M) \otimes \lambda(L)^{-1} \otimes \lambda(M)^{-1} \otimes \lambda(\mathcal{O}_X)$$

is an isometry. In particular, the metric for the manifold plays no role in the combination on the right-hand side.

From this, one may use the cohomology determinant together with Quillen metrics to give arithmetic intersections. (Please compare this statement with the arithmetic Riemann–Roch formula, where the arithmetic intersection was used to give information about the determinant with the Quillen metric.)

Next we use determinant line bundles to express the Deligne pairing for any smooth family of regular varieties. Without loss of generality, in the sequel we only consider the global version of the Deligne pairing over arithmetic varieties.

In that case, we have the following arithmetic Riemann–Roch formula of Gillet and Soulé [**10**]. Denote by $\det R^*\pi_* L$ the cohomology determinant associated to L, by h_Q the Quillen metric, and by $\mathrm{Td}^{\mathrm{Ar}}$ the arithmetic Todd genus. Then

$$c_{1,\mathrm{Ar}}(\operatorname{Det} R^*\pi_* L, \rho_Q) = \pi_*(\mathrm{ch}_{\mathrm{Ar}}(L, \rho) \cdot \mathrm{Td}^{\mathrm{Ar}}(T_\pi, \tau_\pi))^{(1)}.$$

To express the Deligne pairing in terms of determinant line bundles, we consider $n+1$ line bundles $L_0, \ldots, L_n$ together with hermitian metrics ρ_i. Let $\lambda(x_0 + \ldots + x_n)$ be $c_{1,\mathrm{Ar}}(\operatorname{Det} R^*\pi_*(L_0 \otimes \ldots \otimes L_n), \rho_Q)$. Then we should find $a_{i_1,\ldots,i_m}^{j_1,\ldots,j_m} \in \mathbb{Q}$ such that

$$\begin{aligned}\sum a_{i_1,\ldots,i_m}^{j_1,\ldots,j_m} \lambda(i_1 x_{j_1} + \ldots + i_m x_{j_m}) &= \pi_*(c_{1,\mathrm{Ar}}(L_0, \rho_0) \ldots c_{1,\mathrm{Ar}}(L_n, \rho_n)) \\ &= c_{1,\mathrm{Ar}}(\langle L_0, \ldots, L_n\rangle(X/S), \rho_D).\end{aligned}$$

We claim that this can always be done. As a special case, let us first express $\langle L, \ldots, L\rangle$ in terms of determinant line bundles. Let $A(n) = (a_{ij})$ be the matrix defined by $a_{00} = 1$, $a_{ij} = j^i$ for $i, j = 0, \ldots, n+1$ and $(i, j) \neq (0, 0)$. Then, $A(n)$ is a Vandermonde determinant. In particular, $A(n)$ is non-degenerate. Consider

$$A(n)[x] := A(n) - \begin{pmatrix} 0_{n+1\times n+1} & 0 \\ 0 & x \end{pmatrix}.$$

There exists a unique $x \in \mathbb{Q}$ such that $A(n)[x]$ is degenerate. Moreover, the rank of $A(n)[x]$ is its size minus 1, that is, $n+1$. Thus the homogeneous linear equation $A(n)[x]X = 0$ has solution space of dimension 1, say $\mathbb{R}(x_0, \ldots, x_{n+1})^t$. Hence

$$A(n)X = \begin{pmatrix} 0_{n+1\times n+1} & 0 \\ 0 & x \end{pmatrix} X = (0, \ldots, 0, xx_{n+1})^t.$$

In particular, we see that $x_{n+1} \neq 0$ and also

$$x_{n+1} \cdot c_{1,\mathrm{Ar}}(\langle L, \ldots, L\rangle(X/S), \rho_D) = \sum_{i=0}^{n+1} x_i c_{1,\mathrm{Ar}}(\mathrm{Det}\, R^* \pi_*(L^i), \rho_Q)$$

by using the arithmetic Riemann–Roch formula. Indeed,

$$\begin{aligned}
&\sum_{i=0}^{n+1} x_i c_{1,\mathrm{Ar}}(\mathrm{Det}\, R^* \pi_*(L^i), \rho_Q) \\
&\quad = \left(\sum_{i=0}^{n+1} x_i \pi_* \left(\sum_{k=0}^{n+1} \frac{c_{1,\mathrm{Ar}}(L^{\otimes i}, \rho^{\otimes i})^k}{k!} \mathrm{Td}^{\mathrm{Ar}}_{n+1-k}(T_\pi, \tau_\pi) \right) \right)^{(1)} \\
&\quad = \sum_{k=0}^{n+1} \sum_{i=0}^{n+1} x_i \pi_* \left(i^k \frac{c_{1,\mathrm{Ar}}(L, \rho)^k}{k!} \mathrm{Td}^{\mathrm{Ar}}_{n+1-k}(T_\pi, \tau_\pi) \right)^{(1)} \\
&\quad = 0 + \ldots + 0 + x_{n+1} \pi_* (c_{1,\mathrm{Ar}}(L, \rho))^{n+1} \\
&\quad = x_{n+1} c_{1,\mathrm{Ar}}(\langle L, \ldots, L\rangle(X/S), \rho_D).
\end{aligned}$$

On the other hand, by elementary row operations followed by dividing the ith row by $i!$ for each i, we see that $A(n)[x]$ takes the form

$$\begin{pmatrix}
\binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \ldots & \binom{n}{0} & \binom{n+1}{0} \\
0 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \ldots & \binom{n}{1} & \binom{n+1}{1} \\
0 & 0 & \binom{2}{2} & \binom{3}{2} & \ldots & \binom{n}{2} & \binom{n+1}{2} \\
0 & 0 & 0 & \binom{3}{3} & \ldots & \binom{n}{3} & \binom{n+1}{3} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \binom{n}{n} & \binom{n+1}{n} \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 - \frac{x}{(n+1)!}
\end{pmatrix}.$$

So the value of x making the matrix singular is $x = (n+1)!$. This then implies that $x_i = (-1)^i \binom{n+1}{i}$. Therefore we have the following.

PROPOSITION A.2. *With the same notation as above, there exists an isometry*

$$\begin{aligned}(\langle L,\ldots,L\rangle(X/S), h_D) \simeq\ & (\operatorname{Det} R^*\pi_*(L^{n+1}), h_Q)\\ &\otimes (\operatorname{Det} R^*\pi_*(L^n), h_Q)^{\otimes -\binom{n+1}{1}} \otimes \ldots\\ &\otimes (\operatorname{Det} R^*\pi_*(L^{n+1-i}), h_Q)^{\otimes (-1)^i\binom{n+1}{i}}\\ &\otimes \ldots \otimes (\operatorname{Det} R^*\pi_*(\mathcal{O}_X), h_Q)^{\otimes (-1)^{n+1}}.\end{aligned}$$

In general, one may show the following.

PROPOSITION A.3. *With the same notation as above, there exists an isometry*

$$\begin{aligned}&(\langle L_1,\ldots,L_{n+1}\rangle(X/S), h_D)\\ &\quad\simeq (\operatorname{Det} R^*\pi_*(L_1\otimes L_2\otimes\ldots\otimes L_{n+1}), h_Q)\\ &\quad\quad\otimes ((\operatorname{Det} R^*\pi_*(L_1\otimes L_2\otimes\ldots\otimes L_n), h_Q)\\ &\quad\quad\quad\otimes\ldots\otimes(\operatorname{Det} R^*\pi_*(L_2\otimes L_3\otimes\ldots\otimes L_{n+1}), h_Q))^{\otimes -1}\\ &\quad\quad\otimes ((\operatorname{Det} R^*\pi_*(L_1\otimes L_2\otimes\ldots\otimes L_{n-1}), h_Q)\\ &\quad\quad\quad\otimes\ldots\otimes(\operatorname{Det} R^*\pi_*(L_3\otimes L_4\otimes\ldots\otimes L_{n+1}), h_Q))^{\otimes (-1)^2}\\ &\quad\quad\otimes\ldots\otimes(\operatorname{Det} R^*\pi_*(\mathcal{O}_X), h_q)^{\otimes (-1)^{n+1}.}\end{aligned}$$

In fact, the proof of Proposition A.3 comes from the arithmetic Riemann–Roch formula, together with the equality

$$\sum_{k=i+1}^{n+1}(-1)^k\binom{k}{i}\binom{n+1}{k} = (-1)^{i+1}\binom{n+1}{i}.$$

Remark. We will use the above propositions in a forthcoming study of the existence of Einstein–Kähler metrics and Chow–Mumford stability.

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