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# CURVATURE OF THE $L^2$-METRIC ON THE DIRECT IMAGE OF A FAMILY OF HERMITIAN-EINSTEIN VECTOR BUNDLES

By WING-KEUNG TO and LIN WENG

*Abstract.* For a holomorphic family of simple Hermitian-Einstein holomorphic vector bundles over a compact Kähler manifold, the locally free part of the associated direct image sheaf over the parameter space forms a holomorphic vector bundle, and it is endowed with a Hermitian metric given by the $L^2$ pairing using the Hermitian-Einstein metrics. Our main result in this paper is to compute the curvature of the $L^2$-metric. In the case of a family of Hermitian holomorphic line bundles with fixed positive first Chern form and under certain curvature conditions, we show that the $L^2$-metric is conformally equivalent to a Hermitian-Einstein metric. As applications, this proves the semi-stability of certain Picard bundles, and it leads to an alternative proof of a theorem of Kempf.

**0. Introduction.** Given a holomorphic family of simple Hermitian-Einstein holomorphic vector bundles over a fixed compact Kähler manifold and parametrized by a complex manifold, one obtains a coherent analytic sheaf over the parameter space by taking direct image. The locally free part of the direct image sheaf forms a holomorphic vector bundle over a Zariski open subset of the parameter space and is endowed with a Hermitian metric given by the $L^2$-pairings on global holomorphic sections of the Hermitian-Einstein bundles with respect to the Hermitian-Einstein metrics and the Kähler metric on the Kähler manifold. Our objective in this paper is to compute the curvature of this $L^2$-metric.

This work is partly motivated by earlier works of various authors on the analogous problem of computing the curvature of the Weil-Petersson metric on the moduli space of Hermitian-Einstein vector bundles over a compact Kahler manifold ([C], [ST], [ZT]), where the Weil-Petersson metric is defined similarly by the $L^2$-pairing on the harmonic representatives of the Kodaira-Spencer classes associated to tangent vectors of the moduli space. In particular, the approach of Schumacher-Toma [ST] is especially inspiring to us, and their observation that the curvature form of the total space contains the harmonic representatives of the Kodaira-Spencer classes also plays an important role in our computations.

In the case of a family of ample line bundles over a compact Kähler manifold of semi-positive Ricci cuvature, the curvature of the $L^2$-metric takes a simple form (see Theorem 2). When the parameter space is a compact Kähler manifold, the $L^2$-metric is conformally equivalent to a Hermitian-Einstein metric, which

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leads to the semi-stability of the direct image (see Theorem 3). Our result also leads to an alternative proof of a theorem of Kempf [Ke2] on the Picard bundle associated to an ample line bundle over an abelian variety (see (1.4)). A problem of Narasimhan asks whether for compact Riemann surfaces of genus $g$ and a line bundle of degree $d > 2g - 2$, the Hermitian-Einstein metric on the associated Picard bundle over the Jacobian variety with respect to the canonical polarization is conformally equivalent to the $L^2$-metric (see [Ke2]). Our result may be regarded as a partial solution to the higher dimensional analogue of Narasimhan's problem.

This paper is organised as follows. In §1, we introduce some notations and state our main results. In §2, we carry out the computation of the curvature of the $L^2$-metric. In §3, we treat the case of a family of Hermitian holomorphic line bundles with fixed positive first Chern form. Finally, in §4, we give a short alternative proof of Kempf's theorem.

## 1. Notation and statements of results.

(1.1) Let $X$ be a compact Kähler manifold of dimension $n$ endowed with a Kähler metric $g$. Denote the Kähler form of $g$ by $\omega$. Let $(E_0, h_0)$ be a Hermitian holomorphic vector bundle of rank $r$ over $X$, and denote the curvature form associated to the Hermitian connection of $h_0$ by $\Omega_0 \in \mathcal{A}^{1,1}(X, \mathrm{End}(E_0))$. Here $\mathcal{A}^{1,1}(X, \mathrm{End}(E_0))$ denotes the space of $\mathrm{End}(E_0)$-valued (1,1)-forms on $X$. $(E_0, h_0)$ is said to be Hermitian-Einstein if

$$\sqrt{-1}\Lambda\Omega_0 = c \cdot \mathrm{Id}_{E_0}$$

for some real constant $c$, where $\Lambda$ denotes the adjoint of the operator of exterior multiplication by $\omega$, and $\mathrm{Id}_{E_0}$ denotes the identity endomorphism on $E_0$. $E_0$ is said to be simple if any global holomorphic endomorphism of $E_0$ is a constant multiple of the identity. It is well-known that $E_0$ is a simple Hermitian-Einstein vector bundle if and only if $E_0$ is stable with respect to $\omega$ in the sense of Mumford and Takemoto (see [D1], [D2], [Ko1], [Lü], [NS], [UY]). A holomorphic family $\{(\mathcal{E}_s, h_s)\}_{s\in S}$ of Hermitian holomorphic vector bundles over $X$ parametrized by a complex manifold $S$ is a Hermitian holomorphic vector bundle $(\mathcal{E}, h)$ over $X \times S$ such that $\mathcal{E}|_{X\times\{s\}} = \mathcal{E}_s$ and $h|_{X\times\{s\}} = h_s$. Furthermore, $\{(\mathcal{E}_s, h_s)\}_{s\in S}$ is said to be a family of (simple) Hermitian-Einstein vector bundles if each $(\mathcal{E}_s, h_s)$ is (simple) Hermitian-Einstein.

Let $\pi\colon X \times S \to S$ denote the projective map onto the second factor. By Grauert [G], the direct image sheaf $\pi_*\mathcal{E}$ is coherent over $S$, and is thus locally free outside a proper analytic subvariety $Z$ of $S$. Moreover, one has

$$(\pi_*\mathcal{E})_s = H^0(X, \mathcal{E}_s) \quad \text{for } s \in S\backslash Z,$$

where, by abuse of notation, we also denote the holomorphic vector bundle over

$S\backslash Z$ which underlies $\pi_*\mathcal{E}$ by the same symbol. On $S\backslash Z$, the vector bundle $\pi_*\mathcal{E}$ is endowed with a smooth Hermitian metric known as the $L^2$-metric and is given by

$$H(t,t') := \int_X \langle t,t'\rangle \frac{\omega^n}{n!}, \quad \text{for } t,t' \in H^0(X,\mathcal{E}_s),\ s \in S\backslash Z.$$

Here $\langle t,t'\rangle$ denotes $h_s(t,t')$. We shall also use the same symbol to denote the pointwise inner product on $\mathcal{E}_s$-valued differential forms induced by $h_s$ and $g$. It is well-known that the Hermitian-Einstein metric on a simple vector bundle is unique up to a positive constant factor (see e.g. [UY]). Thus for a family of simple Hermitian-Einstein vector bundles $(\mathcal{E} \to X \times S, h)$, $h$ is unique up to a smooth positive function $\lambda(s)$ on $S$, and the associated $L^2$-metric $H$ on $\pi_*\mathcal{E}$ is also unique up to the same function $\lambda(s)$ on $S\backslash Z$.

For $s \in S$, we let $\rho$: $T_sS \to H^1(X, \mathrm{End}(\mathcal{E}_s))$ denote the Kodaira-Spencer map, and let $\eta_v \in \mathcal{A}^{0,1}(X, \mathrm{End}(\mathcal{E}_s))$ denote the harmonic representative of $\rho(v)$ in $H^1(X, \mathrm{End}(\mathcal{E}_s))$. We let $G_{\mathcal{E}_s}$ (resp. $G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}$) denote the Green's operator on $\mathcal{E}_s$ (resp. $\mathrm{End}(\mathcal{E}_s)$)-valued differential forms associated to the $\bar\partial$-Laplacian $\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on $\mathcal{E}_s$ (resp. $\mathrm{End}(\mathcal{E}_s)$). Here $\eta_v, G_{\mathcal{E}_s}, G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}$ are with respect to the Hermitian metrics $h_s$ and $g$. Denote the curvature form of $(\mathcal{E}, h)$ by $\Omega \in \mathcal{A}^{1,1}(X\times S, \mathrm{End}(\mathcal{E}))$. We will use the symbol $[\ ,\ ]$ to denote Lie brackets of $\mathcal{E}_s$-valued differential forms, and we will adopt the Einstein notation of summing up indices which appear in both subscripts and superscripts.

We are now ready to state our main result as follows:

THEOREM 1. *Let $(\mathcal{E} \to X \times S, h)$ be a holomorphic family of simple Hermitian-Einstein vector bundles of rank $r$ over an $n$-dimensional compact Kähler manifold $X$ and parametrized by a complex manifold $S$. Let $Z \subset S$ be the proper analytic subvariety such that the associated direct image sheaf $\pi_*\mathcal{E}$ is locally free over $S\backslash Z$. Then the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S\backslash Z, \mathrm{End}(\pi_*\mathcal{E}))$ of the $L^2$-metric $H$ on $\pi_*\mathcal{E}$ is given by*

$$\Theta_{t\bar{t'}u\bar{v}} = \Psi_{u\bar{v}} \cdot H(t,t') + \int_X \langle G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Lambda[\eta_u, \overline{\eta_v}])t, t'\rangle \frac{\omega^n}{n!} - \int_X \langle G_{\mathcal{E}_s}(\eta_u t), \eta_v t'\rangle \frac{\omega^n}{n!}, \tag{1.1.1}$$

*for $s \in S\backslash Z, u, v \in T_sS$ and $t, t' \in H^0(X, \mathcal{E}_s)$, where $\Psi \in \mathcal{A}^{1,1}(S\backslash Z)$ is given by*

$$\Psi(s)_{u\bar{v}} := \frac{1}{r\cdot \mathrm{Vol}(X,\omega)} \int_X \mathrm{Tr}(\Omega_{u\bar{v}}(x,s)) \frac{\omega^n(x)}{n!}. \tag{1.1.2}$$

*Here* Tr *denotes the trace and* $\mathrm{Vol}(X,\omega)$ *denotes the volume of $X$ with respect to $\omega$.*

*Remark*. When $h$ (and hence $H$) is multiplied by a smooth positive function $\lambda(s)$ on $S$, the first term of the curvature tensor $\Theta$ of $H$ in (1.1.1) is modified by $-\partial\bar{\partial}\log\lambda \cdot H(t,t')$, while the second and the third terms of (1.1.1) remain unchanged.

(1.2) Next we consider a holomorphic family of (ample) Hermitian holomorphic line bundles $(\mathcal{L} \to X \times S, h) = \{(\mathcal{L}_s, h_s)\}_{s\in S}$ over a compact Kähler manifold $(X,\omega)$ and parametrized by a complex manifold $S$, such that the first Chern forms $c_1(\mathcal{L}_s, h_s)$ satisfy the following condition: there exists $k \in \mathbb{R}^+$ such that

$$c_1(\mathcal{L}_s, h_s) = \frac{k}{2\pi}\omega \quad \text{for each } s \in S. \tag{1.2.1}$$

Note that if $(X,\omega)$ is of semi-positive Ricci curvature, then each $\mathcal{L}_s - K_X$ is ample, and by Kodaira vanishing theorem, $H^i(X,\mathcal{L}_s) = 0$ for $i > 0$. Then by a result of Grauert [G], $\pi_*\mathcal{L}$ is locally free over $S$. Then Theorem 1 gives rise to the following:

THEOREM 2. *Let $(\mathcal{L} \to X \times S, h)$ be a holomorphic family of ample Hermitian line bundles with fixed positive first Chern form equal to $(k/2\pi)\omega$ for some $k \in \mathbb{R}^+$ as above. Suppose furthermore that $(X,\omega)$ is of semi-positive Ricci curvature. Then the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S, \mathrm{End}(\pi_*\mathcal{L}))$ of the $L^2$-metric $H$ on $\pi_*\mathcal{L}$ is given by*

$$\Theta_{t\bar{t}'u\bar{v}} = \Xi_{u\bar{v}} \cdot H(t,t') \tag{1.2.2}$$

*for $s \in S$, $u, v \in T_sS$ and $t,t' \in H^0(X,\mathcal{L}_s)$, where $\Xi \in \mathcal{A}^{1,1}(S)$ is given by*

$$\Xi(s)_{u\bar{v}} := \frac{1}{\mathrm{Vol}(X,\omega)}\int_X \left(\Omega(x,s)_{u\bar{v}} - \frac{1}{k}\langle\eta_u,\eta_v\rangle(x)\right)\frac{\omega^n(x)}{n!}. \tag{1.2.3}$$

*Here $\Omega \in \mathcal{A}^{1,1}(X \times S)$ and $\eta_u, \eta_v \in \mathcal{A}^{0,1}(X)$ are defined similarly as in (1.1). In particular, the Hermitian connection of $H$ on $\pi_*\mathcal{L}$ is projectively flat.*

As one will see in the proof of Theorem 2, the pointwise inner product $\langle\eta_u,\eta_v\rangle(x)$ in (1.2.3) is actually constant independent of $x$.

*Remark* 1.2.1. (1.2.2) implies that with respect to any Hermitian metric on $S$, the trace $\Lambda\Theta$ is pointwise proportional to the identity on $\pi_*\mathcal{L}$, i.e., $\Lambda\Theta = \lambda(s) \cdot \mathrm{Id}_{\pi_*\mathcal{L}}$ for some smooth function $\lambda$ on $S$.

(1.3) Let $L$ be an ample holomorphic line bundle over a compact complex manifold $X$. Denote the Picard variety of $X$ by $\mathrm{Pic}^0(X)$. There exists a line bundle known as the Poincaré line bundle $\mathcal{P}$ over $X \times \mathrm{Pic}^0(X)$ such that $\mathcal{P}|_{X\times\{0\}} = \mathcal{O}_X$ and $\mathcal{P}|_{X\times\{s\}}$ is the line bundle represented by $s \in \mathrm{Pic}^0(X)$. $\mathcal{P}$ is unique up to the pull-back of a line bundle over $\mathrm{Pic}^0(X)$ (see e.g. [La, Chapter 4]). Denote by $p\colon X \times \mathrm{Pic}^0(X) \to X$ (resp. $\pi\colon X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$) the projection onto the first (resp. second) factor. The Picard bundle $\mathcal{W}(L)$ over $\mathrm{Pic}^0(X)$ associated to $L$

is defined as follows:

$$\mathcal{W}(L) := \pi_*(\mathcal{P} \otimes p^*L). \tag{1.3.1}$$

Recall also that if the first Chern class $c_1(X)$ of $X$ in $H^2(X, \mathbb{Z})$ can be represented by a semi-positive (1,1)-form, then one simply says $c_1(X) \geq 0$. Recall also that a vector bundle is said to be poly-stable with respect to a Kähler form $\omega$ if it is a direct sum of $\omega$-stable subbundles with the same slope. Then Theorem 2 gives rise to the following:

THEOREM 3. *Let $X$ be a compact complex manifold with $c_1(X) \geq 0$.*

(i) *Let $\mathcal{L} \to X \times S$ be a holomorphic family of ample line bundles over $X$ and parametrized by a compact Kähler manifold $S$. Then $\pi_*\mathcal{L}$ is poly-stable with respect to any Kähler form on $S$.*

(ii) *In particular, the Picard bundle $\mathcal{W}(L)$ over* $\mathrm{Pic}^0(X)$ *associated to an ample line bundle $L$ is poly-stable with respect to any Kähler form on* $\mathrm{Pic}^0(X)$.

Notice that since $c_1(X) \geq 0$, one can deduce as in (1.2) that $\pi_*\mathcal{L}$ (and thus $\mathcal{W}(L)$) is necessarily locally free. Also Theorem 2 and Theorem 3 apply to a wide class of manifolds that include Calabi-Yau manifolds and abelian varieties.

(1.4) Let $A$ be an abelian variety. Then its Picard variety is just its dual abelian variety $\hat{A}$. In this case, one fixes the isomorphism class of the Poincaré line bundle $\mathcal{P}$ over $A \times \hat{A}$ by requiring additionally that $\mathcal{P}|_{\{0\}\times\hat{A}} = \mathcal{O}_{\hat{A}}$. Let $L$ be an ample line bundle over $A$, then we get a line bundle $\mathcal{P} \otimes p^*L$ on $A \times \hat{A}$. Denote the associated Picard bundle over $\hat{A}$ by $\mathcal{W}(L)$ as in (1.3.1). It is well-known that as a line bundle over an abelian variety, there exists a Hermitian metric $h$ on $\mathcal{P} \otimes p^*L$ (unique up to a positive constant multiple) such that its curvature form is invariant under translations of $A \times \hat{A}$ (see e.g. [F, Chapter 2, §2] or [MB, Chapter 2]). Kempf proved the following:

THEOREM 4. [Ke2] *With respect to any translation-invariant Kähler metric on $\hat{A}$, the Hermitian-Einstein metric on the (stable) Picard bundle $\mathcal{W}(L)$ is equal to the $L^2$-metric induced by $h$ and any flat Kähler metric on $A$.*

Thus unlike Theorem 2, no conformal change is necessary here. We will give a short alternative proof of Theorem 4 using Theorem 2. Kempf's original proof of Theorem 4 is rather algebraic, whereas our approach is more differential-geometric in nature.

## 2. Curvature of the $L^2$-metric.

(2.1) Throughout §2, we will follow the notations in (1.1). Let $(\mathcal{E} \to X \times S, h)$ be a holomorphic family of simple Hermitian-Einstein vector bundles as in Theorem 1. Let $m = \dim_{\mathbb{C}} S$ and $n = \dim_{\mathbb{C}} X$. For $s \in S$ and $v \in T_sS$, recall from (1.1) that $\eta_v \in \mathcal{A}^{0,1}(X, \mathrm{End}(\mathcal{E}_s))$ denotes the harmonic representative of the Kodaira-Spencer class of $\rho(v)$. Recall also from (1.1) the curvature tensor $\Omega \in \mathcal{A}^{1,1}(X \times S, \mathrm{End}(\mathcal{E}))$ of $(\mathcal{E}, h)$. We shall need the following proposition, which also plays a crucial role in the computation of the curvature of the Weil-Petersson metric by Schumacher-Toma [ST]:

PROPOSITION 2.1. [O], [ST, Proposition 1] *With respect to local coordinates* $s = (s^i)_{1\le i\le m}$ *for* $S$, $z = (z^\alpha)_{1\le\alpha\le n}$ *for* $X$ *and coordinate tangent vector* $\partial/\partial s^i \in T_sS$, *one has*

$$\eta_{\partial/\partial s^i}(z) = \Omega_{i\bar{\alpha}}(z,s)d\overline{z}^\alpha \in \mathrm{End}((\mathcal{E}_s)_z) \otimes \overline{T^*_zX} \quad \textit{for } z \in X. \tag{2.1.1}$$

*Remark*. Proposition 2.1 can be interpreted as follows: For $s \in S$ and $v \in T_sS$, extend $v$ arbitrarily to a vector field $\tilde{v}$ on an open neighborhood $U$ of $s$ in $S$. Then lift $\tilde{v}$ horizontally to a vector field on $X \times U \subset X \times S$, which we denote by the same symbol. Then the relative differential form (in the $X$ direction) associated to the contraction $\tilde{v} \cup \Omega \in \mathcal{A}^{0,1}(X \times U, \mathrm{End}(\mathcal{E}))$ restricts to $\eta_v$ on $X \times \{s\}$.

We are now ready to give the proof of Theorem 1 as follows:

*Proof of Theorem* 1. Choose local coordinates $s = (s^1, \ldots, s^m)$ for $S$, where $m = \dim_{\mathbb{C}} S$, so that $\pi_*\mathcal{E}$ is locally free at $s = 0$ and of rank $p$. Choose a holomorphic trivialization $\{t_1, t_2, \ldots, t_p\}$ of $\pi_*\mathcal{E}$ over a coordinate neighborhood $U \subset S$ containing 0 such that the $L^2$-metric $H$ on $\pi_*\mathcal{E}$ satisfies

$$\frac{\partial H_{a\bar{b}}}{\partial s^i}\Big|_{s=0} = 0 \quad \text{for } 1 \le a, b \le p,\ 1 \le i \le m. \tag{2.1.2}$$

Here $H_{a\bar{b}} = H(t_a, t_b)$, and we will use the letters $a, b$ to index $\{t_1, \ldots, t_p\}$ thereafter. Note that $\{t_1(s), \ldots, t_p(s)\}$ forms a basis of $H^0(X, \mathcal{E}_s)$ for each $s \in U$. We will use the letters $i, j$ to index coordinate functions $s^1, \ldots, s^m$ on $S$ and we will use the Greek letters $\alpha, \beta$ to index coordinate functions $z^1, \ldots, z^n$ on $X$. In the sequel, covariant derivatives will be used with respect to the Kähler metric $g$ on $X$, the flat connection on $S$ and the Hermitian connection on $(\mathcal{E}, h)$. We will use the standard semi-colon notation to denote covariant derivatives, so that $t_{a;i} := \nabla_{\partial/\partial s^i} t_a$, $t_{a;\alpha} := \nabla_{\partial/\partial z^\alpha} t_a$, etc. With respect to the above trivialization, the curvature tensor $\Theta$ of $(\pi_*\mathcal{E}, H)$ at $s = 0$ is given by

$$\begin{aligned} \Theta_{a\bar{b}i\bar{j}}(0) &= -\frac{\partial^2 H_{a\bar{b}}}{\partial s^i \partial \overline{s^j}}\Big|_{s=0} \\ &= -\frac{\partial^2}{\partial s^i \partial \overline{s^j}} \int_X \langle t_a, t_b \rangle \frac{\omega^n}{n!}\Big|_{s=0} \\ &= -\int_X \langle t_{a;i\bar{j}}, t_b \rangle \frac{\omega^n}{n!}\Big|_{s=0} - \int_X \langle t_{a;i}, t_{b;j} \rangle \frac{\omega^n}{n!}\Big|_{s=0} \\ &= : I_1 + I_2. \end{aligned} \tag{2.1.3}$$

First we deal with the integral $I_2$. Denote by $H_{\mathcal{E}_0}$ the harmonic projection operator on $\mathcal{E}_0$ with respect to the Laplacian, so that at $s = 0$,

$$t_{a;i} = H_{\mathcal{E}_0}(t_{a;i}) + \Box G_{\mathcal{E}_0} t_{a;i}, \tag{2.1.4}$$

where $G_{\mathcal{E}_0}$ is as in (1.1). Then it follows from (2.1.2) that at $s = 0$,

$$\int_X \langle t_{a;i}, t_b \rangle \frac{\omega^n}{n!} = 0 \quad \text{for } 1 \le a, b \le p,\ 1 \le i \le m. \tag{2.1.5}$$

Since $\{t_b\}_{1 \le b \le p}$ forms a basis of $H^0(X, \mathcal{E}_0)$ at $s = 0$ and $H_{\mathcal{E}_0}(t_{a;i}) \in H^0(X, \mathcal{E}_0)$, it follows from (2.1.5) that

$$H_{\mathcal{E}_0}(t_{a;i}) = 0 \quad \text{for } 1 \le a \le p,\ 1 \le i \le m. \tag{2.1.6}$$

Combining (2.1.4) and (2.1.6), we have, at $s = 0$,

$$\begin{aligned} \int_X \langle t_{a;i}, t_{b;j} \rangle \frac{\omega^n}{n!} &= \int_X \langle \Box G_{\mathcal{E}_0} t_{a;i}, t_{b;j} \rangle \frac{\omega^n}{n!} \\ &= \int_X (\langle \bar\partial^* G_{\mathcal{E}_0} t_{a;i}, \bar\partial^* t_{b;j} \rangle + \langle \bar\partial G_{\mathcal{E}_0} t_{a;i}, \bar\partial t_{b;j} \rangle) \frac{\omega^n}{n!} \\ &= \int_X \langle G_{\mathcal{E}_0} \bar\partial t_{a;i}, \bar\partial t_{b;j} \rangle \frac{\omega^n}{n!} \\ &\qquad \text{(since } [G_{\mathcal{E}_0}, \bar\partial] = 0 \text{ and } \bar\partial^* t_{b;j} = 0). \end{aligned} \tag{2.1.7}$$

In terms of the semi-colon notation, the commutation relation $\nabla_i \nabla_{\bar\alpha} t_a - \nabla_{\bar\alpha} \nabla_i t_a = \Omega_{i\bar\alpha} t_a$ becomes $t_{a;\bar\alpha i} - t_{a;i\bar\alpha} = \Omega_{i\bar\alpha} t_a$, which is usually called the Ricci identity. Taking $\bar\partial$ in the direction of $X$, we have

$$\begin{aligned} \bar\partial t_{a;i} &= t_{a;i\bar\alpha} d\bar z^\alpha \\ &= (t_{a;\bar\alpha i} - \Omega_{i\bar\alpha} t_a) d\bar z^\alpha \quad \text{(by Ricci identity)} \\ &= -\Omega_{i\bar\alpha} t_a d\bar z^\alpha \quad \text{(by holomorphicity of } t_a) \\ &= -\eta_i t_a \quad \text{(by Proposition 2.1)}, \end{aligned} \tag{2.1.8}$$

where $\eta_i$ denotes $\eta_{\partial/\partial s^i} \in \mathcal{A}^{0,1}(X, \mathrm{End}(\mathcal{E}_s))$. Combining (2.1.7) and (2.1.8), we have

$$I_2 = -\int_X \langle G_{\mathcal{E}_0}(\eta_i t_a), \eta_j t_b \rangle \frac{\omega^n}{n!}. \tag{2.1.9}$$

Next we deal with the first integral $I_1$ in (2.1.3). By the Ricci identity and holomorphicity of $t_a$, we have, as in (2.1.8),

$$t_{a;i\bar j} = t_{a;\bar j i} - \Omega_{i\bar j} t_a = -\Omega_{i\bar j} t_a. \tag{2.1.10}$$

As in (2.1.4), we have, for $s \in S$,

$$\Omega_{i\bar{j}} = H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Omega_{i\bar{j}}) + G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}\Box\Omega_{i\bar{j}}, \tag{2.1.11}$$

where $H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}$, $G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}$ denote the harmonic projection operator and the Green's operator on $\mathrm{End}(\mathcal{E}_s)$ respectively. Since $H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Omega_{i\bar{j}}) \in H^0(X, \mathrm{End}(\mathcal{E}_s))$ and $\mathcal{E}_s$ is simple, we have

$$H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Omega_{i\bar{j}}) = \Psi(s)_{i\bar{j}} \cdot \mathrm{Id}_{\mathcal{E}_s} \tag{2.1.12}$$

for some $\Psi(s)_{i\bar{j}} \in \mathcal{A}^{1,1}(S\backslash Z)$. For fixed $s \in S\backslash Z$, we have, by (2.1.12),

$$\begin{aligned}\Psi(s)_{i\bar{j}} &= \frac{1}{r\cdot \mathrm{Vol}(X,\omega)}\int_X \langle H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Omega_{i\bar{j}}), \mathrm{Id}_{\mathcal{E}_s}\rangle \frac{\omega^n}{n!}\\ &= \frac{1}{r\cdot \mathrm{Vol}(X,\omega)}\int_X \langle \Omega_{i\bar{j}}, \mathrm{Id}_{\mathcal{E}_s}\rangle \frac{\omega^n}{n!}\\ &\qquad \text{(since } H_{\mathcal{E}_s\otimes\mathcal{E}_s^*} \text{ is self-adjoint and } \mathrm{Id}_{\mathcal{E}_s} \text{ is harmonic),}\end{aligned}$$

which implies $\Psi$ is as given in (1.1.2). To deal with the last term in (2.1.11), we recall that from the computations in [ST, p. 106, equation (11)], one has

$$\Box\Omega_{i\bar{j}} = g^{\bar{\beta}\alpha}[\Omega_{i\bar{\beta}}, \Omega_{\alpha\bar{j}}] \in \mathrm{End}(\mathcal{E}_s), \tag{2.1.13}$$

which is a consequence of the Ricci identity, the Bianchi identity and the Hermitian-Einstein condition on $(\mathcal{E}_s, h_s)$. On the other hand,

$$\begin{aligned} g^{\bar{\beta}\alpha}[\Omega_{i\bar{\beta}}, \Omega_{\alpha\bar{j}}] &= g^{\bar{\beta}\alpha}[\Omega_{i\bar{\beta}}, \overline{\Omega_{j\bar{\alpha}}}]\\ &= \Lambda[\eta_i, \overline{\eta_j}] \quad \text{(by Proposition 2.1).}\end{aligned} \tag{2.1.14}$$

Thus

$$\begin{aligned} I_1 &= -\int_X \langle -\Omega_{i\bar{j}} t_a, t_b\rangle \frac{\omega^n}{n!} \quad \text{(by (2.1.3), (2.1.10))}\\ &= \int_X \langle (H_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Omega_{i\bar{j}}) + G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}\Box(\Omega_{i\bar{j}}))t_a, t_b\rangle \frac{\omega^n}{n!} \quad \text{(by (2.1.11))}\\ &= \Psi(s)_{i\bar{j}} H(t_a, t_b) + \int_X \langle G_{\mathcal{E}_s\otimes\mathcal{E}_s^*}(\Lambda[\eta_i, \overline{\eta_j}])t_a, t_b\rangle \frac{\omega^n}{n!}\\ &\qquad \text{(by (2.1.12), (2.1.13), (2.1.14)).}\end{aligned} \tag{2.1.15}$$

Then (1.1.1) follows immediately from (2.1.3), (2.1.9) and (2.1.15), and we have finished the proof of Theorem 1.

## 3. Family of Hermitian line bundles with fixed positive first Chern form.

(3.1) Throughout §3, we will follow the notation in (1.2). Let $(\mathcal{L} \to X \times S, h) = \{(\mathcal{L}_s, h_s)\}_{s\in S}$ be a holomorphic family of Hermitian line bundles as in Theorem 2. Recall that $X$ is endowed with a Kähler metric $g$ of semi-positive Ricci curvature, and whose Kähler form $\omega$ satisfies (1.2.1). To prove Theorem 2, we shall need the following lemmas:

LEMMA 3.1. *Let $(X, g)$ be a compact Kähler manifold of semi-positive Ricci curvature, and let $\eta \in \mathcal{A}^{0,1}(X)$ be a harmonic (0,1)-form. Then*

$$\nabla\eta = 0 \;\; \textit{and} \;\; \overline{\nabla}\eta = 0. \tag{3.1.1}$$

*Here $\nabla\eta$ (resp. $\overline{\nabla}\eta$) denotes the tensor with components $\eta_{\bar\alpha;\beta}$ (resp. $\eta_{\bar\alpha;\bar\beta}$).*

*Proof*. Since $\eta \in \mathcal{A}^{0,1}(X)$, we have $\nabla\eta = \partial\eta$. Then the first equality of (3.1.1) follows from the fact that the $\bar\partial$-Laplacian and $\partial$-Laplacian are equal on a Kähler manifold. The second equality of (3.1.1) follows from the semi-positivity of the Ricci curvature and the Bochner formula (see e.g. [S1, equation (1.3.3)]).

LEMMA 3.2. *Let $\{(\mathcal{L}_s, h_s)\}_{s\in S}$ be as in Theorem 2. For fixed $s \in S$, let $t \in H^0(X, \mathcal{L}_s)$ and let $\eta \in \mathcal{A}^{0,1}(X)$ be a harmonic (0,1)-form on $X$. Then with $k \in \mathbb{R}^+$ as in (1.2.1), we have*

$$\Box(\eta t) = k\eta t. \tag{3.1.2}$$

*Proof*. By (1.2.1), the curvature form $\Omega$ of $(\mathcal{L}_s, h_s)$ satisfies $\Omega_{\alpha\bar\beta} = kg_{\alpha\bar\beta}$, where $g$ denotes the Kähler metric on $X$. Also, since both $\eta$ and $t$ are $\bar\partial$-closed, $\bar\partial(\eta t) = \bar\partial\eta \wedge t - \eta \wedge \bar\partial t = 0$. In terms of local coordinates $(z^\alpha)$ for $X$, we write $\eta = \eta_{\bar\alpha}\overline{dz^\alpha}$. Then

$$\begin{aligned}
\Box(\eta t) &= (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)(\eta t)\\
&= \bar\partial\bar\partial^*(\eta t) \quad (\text{since } \bar\partial(\eta t) = 0)\\
&= \bar\partial(-g^{\bar\beta\alpha}\nabla_\alpha(\eta_{\bar\beta} t))\\
&= -\bar\partial(g^{\bar\beta\alpha}(\eta_{\bar\beta;\alpha} t + \eta_{\bar\beta} t_{;\alpha}))\\
&= -\bar\partial(g^{\bar\beta\alpha}\eta_{\bar\beta} t_{;\alpha}) \quad (\text{since } \eta_{\bar\beta;\alpha} = 0 \text{ by Lemma 3.1})\\
&= -(g^{\bar\beta\alpha}\eta_{\bar\beta;\bar\gamma} t_{;\alpha} + g^{\bar\beta\alpha}\eta_{\bar\beta} t_{;\alpha\bar\gamma})\overline{dz^\gamma} \quad (\text{since } g^{\bar\beta\alpha}{}_{;\bar\gamma} = 0)\\
&= -g^{\bar\beta\alpha}\eta_{\bar\beta} t_{;\alpha\bar\gamma}\overline{dz^\gamma} \quad (\text{since } \eta_{\bar\beta;\bar\gamma} = 0 \text{ by Lemma 3.1})\\
&= -g^{\bar\beta\alpha}\eta_{\bar\beta}(-\Omega_{\alpha\bar\gamma} t)\overline{dz^\gamma}\\
&\qquad (\text{by Ricci identity and holomorphicity of } t \text{ as in (2.1.10)})\\
&= kg^{\bar\beta\alpha}\eta_{\bar\beta} g_{\alpha\bar\gamma} t\,\overline{dz^\gamma} \quad (\text{by (1.2.1)})\\
&= k\eta t.
\end{aligned}$$

This finishes the proof of Lemma 3.2.

Now we give the proof of Theorem 2 as follows:

*Proof of Theorem* 2. Let $(\mathcal{L} \to X \times S, h) = \{(\mathcal{L}_s, h_s)\}_{s\in S}$ and $\Omega$ be as in Theorem 2. On $X$, we denote by $g$ the Kähler metric of semi-positive Ricci curvature. For each $s \in S$, it follows from (1.2.1) that the curvature form $\Omega_s (= \Omega|_{\mathcal{L}_s})$ of $(\mathcal{L}_s, h_s)$ satisfies $\Lambda\Omega = k\cdot \mathrm{Id}_{\mathcal{L}_s}$, where $k \in \mathbb{R}^+$ is as in (1.2.1). In particular, each $(\mathcal{L}_s, h_s)$ is Hermitian-Einstein. By Theorem 1, the curvature $\Theta$ of the $L^2$-metric $H$ on $\pi_*\mathcal{L}$ with respect to $\{h_s\}$ and $g$ is given by (1.1.1) and (1.1.2). Note that $\mathcal{L}_s \otimes \mathcal{L}_s^* = \mathcal{O}_X$. For $s \in S$ and $u, v \in T_sS$, let $\eta_u, \eta_v \in \mathcal{A}^{0,1}(X, \mathrm{End}(\mathcal{L}_s)) = \mathcal{A}^{0,1}(X)$ be the associated harmonic forms as in (1.1). By Lemma 3.1, $\eta_u, \eta_v$ are parallel tensors and thus the pointwise inner product $\langle \eta_u, \eta_v\rangle$ is also parallel, which implies that $\langle \eta_u, \eta_v\rangle$ is a constant function on $X$. In terms of local coordinates $(z^\alpha)$ for $X$, we write $\eta_u = (\eta_u)_{\bar\alpha} d\bar\alpha$, etc. Then

$$\Lambda[\eta_u, \bar\eta_v] = g^{\bar\beta\alpha}(\eta_u)_{\bar\beta}\overline{(\eta_v)_{\bar\alpha}} = \langle \eta_u, \eta_v\rangle.$$

Hence, $\Lambda[\eta_u, \bar\eta_v]$ is also a constant function on $X$, and by definition of the Green's operator,

$$G_{\mathcal{L}_s\otimes\mathcal{L}_s^*}(\Lambda[\eta_u, \bar\eta_v]) = 0. \tag{3.1.3}$$

Let $t, t' \in H^0(X, \mathcal{L}_s)$. By Lemma 3.2, $\Box(\eta_u t) = k\eta_u t$, and thus by definition of the Green's operator, we must have $G_{\mathcal{L}_s}(\eta_u t) = (1/k)\eta_u t$. Also, we have seen earlier that $\langle \eta_u, \eta_v\rangle$ is pointwise constant on $X$. Hence,

$$\begin{aligned}\int_X \langle G_{\mathcal{L}_s}(\eta_u t), \eta_v t'\rangle \frac{\omega^n}{n!} &= \int_X \left\langle \frac{1}{k}\eta_u t, \eta_v t'\right\rangle \frac{\omega^n}{n!} \\ &= \frac{1}{k}\int_X \langle \eta_u, \eta_v\rangle\langle t, t'\rangle \frac{\omega^n}{n!} \\ &= \frac{1}{k}\langle \eta_u, \eta_v\rangle H(t, t').\end{aligned} \tag{3.1.4}$$

Combining (1.1.1), (3.1.3), (3.1.4) and with $\Psi$ as defined in (1.1.2), we have

$$\begin{aligned}\Theta_{t\bar t' u\bar v} &= \Psi_{u\bar v}\cdot H(t, t') + 0 - \frac{1}{k}\langle \eta_u, \eta_v\rangle H(t, t') \\ &= (\Psi_{u\bar v} - \frac{1}{k}\langle \eta_u, \eta_v\rangle)\cdot H(t, t') \\ &= \Xi_{u\bar v}\cdot H(t, t'),\end{aligned}$$

where $\Xi \in \mathcal{A}^{1,1}(S)$ is easily seen to be given as in (1.2.3). Thus we have completed the proof of Theorem 2.

(3.2) In this section, we are going to deduce Theorem 3 from Theorem 2.

First we recall the following well-known lemma, whose proof will be skipped here:

LEMMA 3.3. *Let* $\{\mathcal{L}_s\}_{s\in S}$ *be a smooth family of holomorphic line bundles over a compact Kähler manifold X and parametrized by a smooth manifold S. Let* $\nu$ *be a real d-closed (1,1)-form on X representing* $c_1(\mathcal{L}_s) \in H^2(X,\mathbb{Z})$ *for each* $s \in S$ *(*$c_1(\mathcal{L}_s)$ *does not vary with s since* $H^2(X,\mathbb{Z})$ *is discrete). Then there exists a smooth family of Hermitian metrics* $\{h_s\}_{s\in S}$ *on* $\{\mathcal{L}_s\}_{s\in S}$ *such that*

$$c_1(\mathcal{L}_s, h_s) = \nu \quad \text{for each } s \in S. \tag{3.2.1}$$

Now we give the proof of Theorem 3 as follows:

*Proof of Theorem* 3. First we prove (i). Let $\{\mathcal{L}_s\}_{s\in S}$ be a holomorphic family of ample line bundles over a compact Kähler manifold $X$ with $c_1(X) \geq 0$ and parametrized by $S$ as in (i). As mentioned in (i), $c_1(\mathcal{L}_s) \in H^2(X,\mathbb{Z})$ does not vary with $s$. Since each $\mathcal{L}_s$ is ample, $c_1(\mathcal{L}_s)$ is a Kähler class. By a theorem of Yau [Y], for any $d$-closed (1,1)-form $\phi$ representing $c_1(X)$, there exists in any Kähler class a Kähler metric whose Ricci form is $2\pi\phi$. Since $c_1(X) \geq 0$, the above theorem implies that there exists a Kähler metric of semi-positive Ricci curvature and such that the cohomology class $[\omega]$ of its Kähler form $\omega$ satisfies $[\omega] = c_1(\mathcal{L}_s)$. By Lemma 3.3, there exists a smooth family of Hermitian metrics $\{h_s\}_{s\in S}$ on $\{\mathcal{L}_s\}_{s\in S}$ such that $c_1(\mathcal{L}_s, h_s) = \omega$ for each $s \in S$. Then we endow $\pi_*\mathcal{L}$ with the $L^2$-metric $H$ associated to the Hermitian metrics $\{h_s\}_{s\in S}$ and the Kähler metric $\omega$. By Theorem 2, the curvature $\Theta$ of $H$ satisfies (1.2.2). Then with respect to any Kähler metric on $S$, the trace $\Lambda\Theta$ of $\Theta$ satisfies $\Lambda\Theta = \lambda(s)\cdot \mathrm{Id}_{\pi_*\mathcal{L}}$ for some smooth function $\lambda$ on $S$ (see Remark 1.2.1). Then it is known that this implies $H$ is conformally equivalent to a Hermitian-Einstein metric on $\pi_*\mathcal{L}$ (see e.g. [S2, p.16]). By a result of Kobayashi [Ko1] and Lübke [Lü], this implies that $\pi_*\mathcal{L}$ is poly-stable with respect to the Kähler form $\omega$ on $S$. This finishes the proof of (i). To prove (ii), we first observe that by construction, the Poincaré line bundle $\mathcal{P}$ forms a holomorphic family of line bundles over $X$ with zero first Chern class. Then for an ample line bundle $L$ on $X$, $\mathcal{P} \otimes p^*L$ forms a holomorphic family of ample line bundles over $X$ with first Chern class equal to $c_1(L)$, and parametrized by $\mathrm{Pic}^0(X)$. Then (ii) follows immediately from (i). Thus we have completed the proof of Theorem 3.

**4. A Theorem of Kempf.** In this section, we give an alternative proof of Theorem 4 of Kempf [Ke2] using Theorem 2 as follows:

*Proof of Theorem* 4. Observe that the volume forms of any two flat Kähler metrics over $A$ are constant multiples of each other. Thus we just need to prove Theorem 4 for one fixed flat Kähler form on $A$. As in Theorem 3(ii), $\mathcal{W}(L)$ is the direct image sheaf associated to $\mathcal{P} \otimes p^*L$ which forms a family of ample line bundles $\{\mathcal{L}_s\}_{s\in\hat{A}}$ with first Chern class equal to $c_1(L)$ and parametrized by $\hat{A} = \mathrm{Pic}^0(A)$. For the line bundle $\mathcal{P} \otimes p^*L$ over the abelian variety $A \times \hat{A}$, denote by $h$ the Hermitian metric (unique up to a constant multiple) whose curvature form $\Omega$ is invariant under translations of $A \times \hat{A}$ as in Theorem 4. Let $h_s = h|_{\mathcal{L}_s}$ for $s \in \hat{A}$. The ampleness of $L$ and the translation-invariance of $\Omega$ implies that for each $s \in \hat{A}$, $c_1(\mathcal{L}_s, h_s)$ is equal to a fixed positive-definite translation-invariant (1,1)-form $\omega$ on $A$. As a translation-invariant Kähler form on $A$, $\omega$ is necessarily flat. Then by Theorem 2, the curvature $\Theta$ of the $L^2$-metric $H$ on $\mathcal{W}(L)$ with respect to the Hermitian metric $h$ on $\mathcal{P} \otimes p^*L$ and the Kähler form $\omega$ on $A$ is given as in (1.2.2). Let $\Xi \in \mathcal{A}^{1,1}(\hat{A})$ and $\eta_u, \eta_v \in \mathcal{A}^{0,1}(A)$ be as in (1.2.3). Then it follows from the translation-invariance of $\Omega$ and (2.1.1) that $\eta_u, \eta_v$ remain unchanged under translations of $u, v \in T\hat{A}$ induced by those of $\hat{A}$. Thus by (1.2.3), $\Xi$ is also invariant under translations of $\hat{A}$. Now with respect to any translation-invariant Kähler metric on $\hat{A}$, the contraction $\Lambda\Xi$ is also translation-invariant and is thus equal to a constant $c$ on $\hat{A}$. Then by (1.2.2), $\Lambda\Theta = c \cdot \mathrm{Id}_{\mathcal{W}(L)}$, which implies $H$ is a Hermitian-Einstein metric on $\mathcal{W}(L)$, which is unique up to a multiplicative constant since Kempf [Ke2] proved the stability of $\mathcal{W}(L)$ with respect to any polarization on $\hat{A}$. This finishes the proof of Theorem 4.

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