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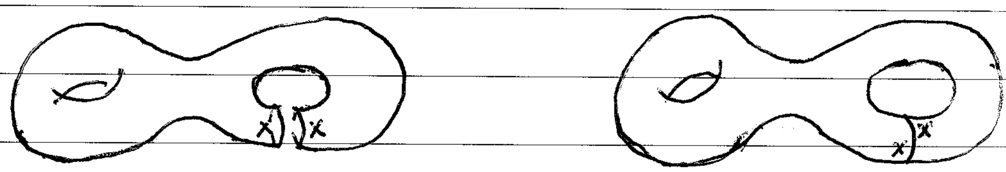
Twist-bulge deformations for convex projective surfaces

Guided by finding explicit information from derivatives of basic quantities

for hyperbolic surfaces have Fenchel-Nielsen twist t_α



α $\qquad\qquad\qquad$ α

where for length l_β of a second geodesic

$$t_\alpha l_\beta = \sum_{p\in\alpha\cap\beta} \cos\theta_p$$

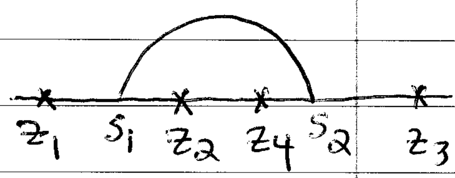
$$t_\alpha t_\beta l_\gamma = \sum_{\alpha\cap\gamma\times\beta\cap\gamma} \frac{e^{l_1}+e^{l_2}}{2(e^{l_\gamma}-1)} \sin\theta_p \sin\theta_q - \sum_{\alpha\cap\beta\times\beta\cap\gamma} \frac{e^{m_1}+e^{m_2}}{2(e^{l_\beta}-1)} \sin\theta_r \sin\theta_s$$

formulas and duality $\omega^{WP}(\ ,t_\alpha) = \frac{1}{2}dl_\alpha$ give twist bracket $[T_\alpha, T_\beta]$, linear & quadratic identities, earthquake convexity and $d\omega = 0$

Twist derivative is calculated by first treating case of a single twist and then carrying out the sum for the deck group

Cross ratio $(z_1, z_2, z_3, z_4) = \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$

single line twist derivative



$$t_{(s_1 s_2)} \log(z_1, z_2, z_3, z_4) = \sum_{j=1}^{4} \chi_L(z_j)\Big[(z_{\sigma(j)}, s_1, s_2, z_j) - (z_{\tau(j)}, s_1, s_2, z_j)\Big]$$

χ_L is characteristic fn of $\overrightarrow{s_1 s_2}$ left half plane
permutations $\sigma = (13)(24)$ and $\tau = (14)(23)$

spectral radius of B

Good fortune – $\sum_n t(B^{-n}(s_1 s_2)) \log(Bt, t, r_B, a_B)$ is a telescoping sum and result is $2(a_B, s_1, s_2, r_B) - 1 = \cos \overline{a_B r_B} \cap \overline{s_1 s_2}$

student Terence Long has been working on corresponding calculation for 2-diml convex projective structures

Hitchin component of Hom$(\pi_1(S), PSL(3;\mathbb{R}))$
Following Bonahon-Dreyer Parameterizing Hitchin Components

for $SL(2;\mathbb{R})$ acting on $\mathbb{R}^2$ then acts on space of symmetric 2-tensors gives homomorphism $PSL(2;\mathbb{R}) \to PSL(3;\mathbb{R})$

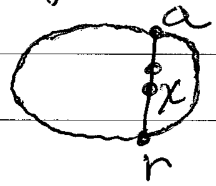
Hitchin component $Hitch_3 \subset Hom(\pi_1(S), PSL(3;\mathbb{R}))$ containing image of Teichmüller space under $PSL(2;\mathbb{R}) \to PSL(3;\mathbb{R})$

Theorem (Hitchin) $Hitch_n \cong \mathbb{R}^{2(g-1)(n^2-1)}$

Bonahon-Dreyer start with a pants decomposition and subdivide each pants by choosing two infinite spirals
count parameters for geodesics / lines, triangles and subtract for equations

Convex projective geometry for a Hitchin representation $\pi_1(S)$ acts on convex domain in $\mathbb{RP}^2$
dichotomy – domain is either a conic and the representation comes from a uniformization $\pi_1(S) \to PSL(2;\mathbb{R})$ or boundary is $C^{1,\alpha}$ not C^2 (Benzecri)

Projective quantities to work with



have cross ratio of four points on a line
gives Hilbert metric

$$\frac{(a-r)\,dx}{(x-r)(a-r)}$$

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$A \in PSL(n;\mathbb{R})$ acts on lines in $\mathbb{R}^n$, also acts on k-planes in $\mathbb{R}^n$, so in fact acts on flags

Defn A flag $F^{(0)} \subset F^{(1)} \subset \ldots \subset F^{(n-1)} \subset \mathbb{R}^n$ is a chain of subspaces with proper inclusions, so $\dim F^{(k)} = k$

Can form invariants by taking ratios of elements of $\Lambda^a F^{(a)}$

Example Labourie's cross ratio

consider quadruple (φ, ψ, u, v) — φ, ψ: lines in dual; u, v: lines in $\mathbb{R}^n$

$$b(\varphi, \psi, u, v) = \frac{(\varphi(u))(\psi(v))}{(\varphi(v))(\psi(u))}$$ homogeneous of degree zero in each quantity

Example for a triangle, triple of flags E, F, G, consider homogeneous degree zero ratio of wedges of elements of $\Lambda^{(a)}E^{(a)}$, $\Lambda^{(b)}F^{(b)}$, $\Lambda^{(c)}G^{(c)}$ called <u>triangle invariants</u>

Theorem (Labourie) For a Hitchin representation, every element $\pi_1(S) - \{id\}$ maps to a diagonizable element with a lift to $SL(n;\mathbb{R})$ with distinct positive eigenvalues

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Theorem (Fock-Goncharov, Labourie) For a Hitchin representation $\rho : \pi_1(S) \to PSL(n;\mathbb{R})$ the map $\partial_\infty \pi_1(S) \to \mathbb{R}P^{n-1}$ lifts to a continuous map to $Flag(\mathbb{R}^n)$ s.t.

i) $p \in \partial_\infty \pi_1(S)$ is the attracting f.p. of $\gamma \in \pi_1(S)$ then $\mathcal{F}_\rho(p)$ is the stable flag of $\rho(\gamma)$

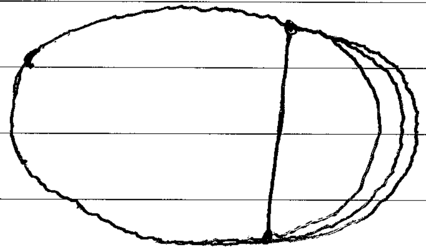
ii) $p \to \mathcal{F}_\rho(p)$ is $\pi_1(S) - \rho(\pi_1(S))$ equivariant

iii) for any two points $p, q \in \partial_\infty \pi_1(S)$ then $(\mathcal{F}_\rho(p), \mathcal{F}_\rho(q))$ is generic

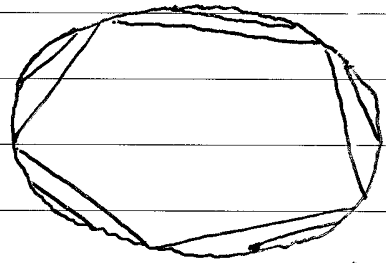
iv) $p, q, r \in \partial_\infty \pi_1(S)$ then $(\mathcal{F}_\rho(p), \mathcal{F}_\rho(q), \mathcal{F}_\rho(r))$ has all triangle invariants positive

Twist-bulge deformation

Centralizer of a generic diagonizable element has dimension 2 for $PSL(3;\mathbb{R})$



single bulge



$\pi_1(S)$ invariant lines

infinitesimal centralizer contains infinitesimal twist and infinitesimal bulge

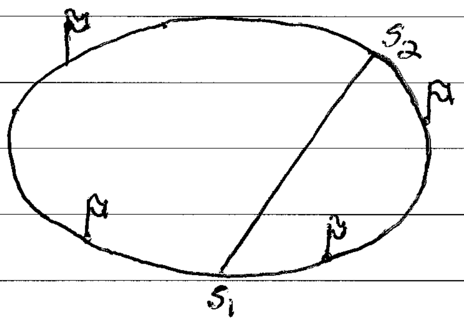
Twist-bulge of a cross ratio $(\underbrace{\varphi_1, \varphi_2}_{\text{covectors}}, \underbrace{v_3, v_4}_{\text{vectors}})$

Formulas (Terence Long)

twist bulge on $\widehat{s_1 s_2}$

infinitesimal twist-bulge on $\widehat{s_1 s_2}$

$$t(\widehat{s_1 s_2}) \log(\varphi_1, \varphi_2, v_3, v_4) = \sum_{i=1}^{2} \chi_L(i) \left[-\frac{\varphi_i L^{\widehat{s_1 s_2}} v_{\sigma(i)}}{\varphi_i v_{\sigma(i)}} + \frac{\varphi_i L^{\widehat{s_1 s_2}} v_{\tau(i)}}{\varphi_i v_{\tau(i)}} \right] + \sum_{i=3}^{4} \chi_L(i) \left[\frac{\varphi_{\sigma(i)} L^{\widehat{s_1 s_2}} v_i}{\varphi_{\sigma(i)} v_i} - \frac{\varphi_{\tau(i)} L^{\widehat{s_1 s_2}} v_i}{\varphi_{\tau(i)} v_i} \right]$$



twist-bulge derivative of log spectral radius

$$\sum_n t(B^{-n}(\widehat{s_1 s_2})) \log(Bt, t, \underbrace{r_B, a_B}_{\substack{\text{repelling attracting} \\ \text{flags}}}) = -I^{\widehat{s_1 s_2}}(r_B, a_B) + I^{\widehat{s_1 s_2}}(a_B, r_B)$$

generalization of cosine

where $I^{\widehat{s_1 s_2}}(\varphi, v) = \dfrac{\varphi(L^{\widehat{s_1 s_2}} v)}{\varphi(v)}$

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Bonahon-Dreyer parameter count

work with Hilbert metric

start with a pants decomposition with boundary lines and subdivide each pants into two spiraling triangles

centralizer of a generic diagonizable element of $PSL(n;\mathbb{R})$ has dimension $n-1$, so have $n-1$ shear dimensions on a line

count

$3g-3$ closed geodesics	$n-1$ shear parameters
$6g-6$ spirals	$n-1$ shear parameters
$4g-4$ triangles	$\frac{1}{2}(n-1)(n-2)$ parameters

equations

$3g-3$ closed geodesics	$n-1$ completeness equations