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Zeta Functions for G_2 and Their Zeros

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The exceptional group G_2 has two maximal parabolic subgroups P_{long}, P_{short} corresponding to the so-called long root and short root. In this paper, the second named author introduces two zeta functions associated with (G_2, P_{long}) and $(G_2, P_{\mathrm{short}})$, respectively, and the first named author proves that these zetas satisfy the Riemann hypothesis.

1 Introduction

Associated with a number field F is the genuine high-rank zeta function $\xi_{F,r}(s)$ for every fixed $r \in \mathbb{Z}_{>0}$. Being natural generalizations of (completed) Dedekind zeta functions, these functions satisfy canonical properties for zetas as well. Namely, they admit meromorphic continuations to the whole complex s-plane, satisfy the functional equation $\xi_{F,r}(1-s) = \xi_{F,r}(s)$, and have only two singularities, all simple poles, at $s = 0, 1$. Moreover, we expect that the Riemann hypothesis holds for all zetas $\xi_{F,r}(s)$, namely, all zeros of $\xi_{F,r}(s)$ lie on the central line $\mathrm{Re}(s) = 1/2$.

Recall that $\xi_{F,r}(s)$ is defined by

$$\xi_{F,r}(s) := (|\Delta_F|)^{\frac{rs}{2}} \int_{\mathcal{M}_{F,r}} \left(e^{h^0(F,\Lambda)} - 1\right)(e^{-s})^{\deg(\Lambda)}\, d\mu(\Lambda), \quad \mathrm{Re}(s) > 1,$$

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where Δ_F denotes the discriminant of F, $\mathcal{M}_{F,r}$ denotes the moduli space of semistable $\mathcal{O}_F$-lattices of rank r (here $\mathcal{O}_F$ denotes the ring of integers), $h^0(F,\Lambda)$ and $\deg(\Lambda)$ denote the 0-th geo-arithmetic cohomology and the Arakelov degree of the lattice Λ, respectively, and $d\mu(\Lambda)$ denotes a certain Tamagawa type measure on $\mathcal{M}_{F,r}$. Defined using high-rank lattices, these zetas then are expected to be naturally related with nonabelian aspects of number fields. For details, see [23–25] for basic theory, and [11, 24] for the Riemann hypothesis arguments.

On the other hand, algebraic groups associated with $\mathcal{O}_F$-lattices are general linear group GL and special linear group SL. A natural question then is whether principal lattices associated with other reductive groups G and their associated zeta functions can be introduced and studied. In this paper, we work with the exceptional group G_2. In contrast with a geo-arithmetic method used for high-rank zetas [23, 25], the one adopted in this paper is rather analytic [1, 2, 8].

For simplicity, take F to be the field $\mathbb{Q}$ of rationals. Then, via a Mellin transform, the high-rank zeta $\xi_{\mathbb{Q},r}(s)$ can be written as

$$\xi_{\mathbb{Q},r}(s) = \int_{\mathcal{M}_{\mathbb{Q},r}[1]} \widehat{E}(\Lambda, s)\, d\mu(\Lambda), \quad \mathrm{Re}(s) > 1,$$

where $\mathcal{M}_{\mathbb{Q},r}[1]$ denotes the moduli space of $\mathbb{Z}$-lattices of rank r and volume 1, and $\widehat{E}(\Lambda, s)$ the completed Epstein zeta functions associated with Λ. Note that $\mathcal{M}_{\mathbb{Q},r}[1]$ may be viewed as a compact subset in $SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)$ and Epstein zeta functions may be written as the relative Eisenstein series $E^{SL(r)/P_{r-1,1}}(\mathbf{1}; s; g)$ associated with the constant function $\mathbf{1}$ on the maximal parabolic subgroup $P_{r-1,1}$ corresponding to the partition $r = (r-1)+1$ of $SL(r)$, we have

$$\begin{aligned}\xi_{\mathbb{Q},r}(s) &= \int_{\mathcal{M}_{\mathbb{Q},r}[1] \subset SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\Lambda, s)\, d\mu(g) \\ &= \int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \mathbf{1}_{\mathcal{M}_{\mathbb{Q},r}[1]}(g) \cdot \widehat{E}(\mathbf{1}; s; g)\, d\mu(g),\end{aligned}$$

where $\mathbf{1}_{\mathcal{M}_{\mathbb{Q},r}[1]}(g)$ denotes the characteristic function of the compact subset $\mathcal{M}_{\mathbb{Q},r}[1]$.

In doing so, by integrating over intrinsically defined and hence arithmetically meaningful compact subsets $\mathcal{M}_{\mathbb{Q},r}[1]$ of $SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)$, instead of the ill-defined integrations

$$\int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \hat{E}(\mathbf{1}; s; g)\, d\mu(g),$$

we get well-defined genuine nonabelian zetas $\xi_{\mathbb{Q},r}(s)$.

In parallel, to remedy the divergence of integration

$$\int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\mathbf{1};s;g)\,d\mu(g),$$

in theories of automorphic forms and trace formula, Rankin, Selberg, and Arthur introduced an analytic truncation for smooth functions $\phi(g)$ over $SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)$. Simply put, Arthur's analytic truncation is a device to get rapidly decreasing functions from slowly increasing functions by cutting off slow growth parts near all type of cusps uniformly. Being truncations near cusps, a rather large, or better, sufficiently regular, new parameter T must be introduced. In particular, when applying to Eisenstein series $\widehat{E}(\mathbf{1};s;g)$ and to $\mathbf{1}$ on $SL(r,\mathbb{R})$, we get the truncated function $\Lambda^T\widehat{E}(\mathbf{1};s;g)$ and $(\Lambda^T\mathbf{1})(g)$, respectively. Consequently, by using basic properties on Arthur's truncation (see Section 2), we obtain the following well-defined integrations:

$$\begin{aligned}\int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \Lambda^T\widehat{E}(\mathbf{1};s;g)\,d\mu(g) &= \int_{SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} (\Lambda^T\mathbf{1})(g)\cdot\widehat{E}(\mathbf{1};s;g)\,d\mu(g)\\ &= \int_{\mathfrak{F}(T)\subset SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)} \widehat{E}(\mathbf{1};s;g)\,d\mu(g),\end{aligned}$$

where $\mathfrak{F}(T)$ is the compact subset in (a fundamental domain of) $SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)$ whose characteristic function is given by $(\Lambda^T\mathbf{1})(g)$.

As such, we find an analytic way to understand our high-rank zetas, provided that the above analytic discussion for sufficiently positive parameter T can be further strengthened so as to work for smaller T, in particular, for $T=0$ as well. In general, it is very difficult [1–3]. Fortunately, in the case of SL, this can be achieved based on an intrinsic geo-arithmetic result, called the Micro-Global Bridge [23, 25], an analog of the following basic principle in Geometric Invariant Theory for instability: A point is not GIT stable, then there is a parabolic subgroup which destroys the stability. All in all, the upshot is that we have

$$\mathbf{1}_{\mathcal{M}_{\mathbb{Q},r}[1]} \equiv \Lambda^0\mathbf{1}, \quad \text{or the same,} \quad \mathcal{M}_{\mathbb{Q},r}[1]=\mathfrak{F}(0).$$

In other words, *the moduli spaces of rank r semistable lattices of volume one coincide with the compact subsets* $\mathfrak{F}(0)\subset SL(r,\mathbb{Z})\backslash SL(r,\mathbb{R})/SO(r)$. Consequently, we have

$$\xi_{\mathbb{Q},r}(s) = \left(\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \Lambda^T\widehat{E}(\mathbf{1};s;g)\,d\mu(g)\right)\Bigg|_{T=0}.$$

This then leads to evaluation of the special Eisenstein periods

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \Lambda^T \widehat{E}(\mathbf{1};s;g)\,d\mu(g),$$

and more generally, the evaluation of *Eisenstein periods*

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \Lambda^T E(\phi;\lambda;g)\,d\mu(g),$$

where K is a certain maximal compact subgroup of a reductive group G, ϕ is a P-level automorphic forms with P parabolic, and $E(\phi;\lambda;g)$ is the relative Eisenstein series from P to G associated with ϕ [12].

Unfortunately, in general, it is quite difficult to find a close formula for Eisenstein periods. But, when ϕ is cuspidal, then the corresponding Eisenstein period can be calculated, thanks to the work of [8] (see also [22, 28]) an advanced version of Rankin–Selberg and Zagier method.

Back to high-rank zeta functions, the bad news is that this powerful calculation cannot be applied directly, since in the specific Eisenstein series, i.e., the classical Epstein zeta used, the function $\mathbf{1}$ corresponding to ϕ in general picture, on the maximal parabolic $P_{r-1,1}$ is only L^2, far from being cuspidal. To overcome this technical difficulty, we partially also motivated by our earlier work on the so-called abelian part of high-rank zeta functions [20, 22] and Venkov's trace formula for $SL(3)$ [19], introduce Eisenstein series $E^{G/B}(\mathbf{1};\lambda;g)$ associated with the constant function $\mathbf{1}$ on $P_{1,1,\dots,1}$, the Borel, into our study, since

(1) being over the Borel, the constant function $\mathbf{1}$ is cuspidal. So the associated Eisenstein period $\omega_{\mathbb{Q}}^{G;T}(\lambda)$ can be evaluated; and
(2) $E(\mathbf{1};s;g)$ used in high-rank zetas can be realized as residues of $E^{G/B}(\mathbf{1};\lambda;g)$ along with suitable singular hyperplanes, a result already known to Selberg and Langlands. See, e.g., Diehl [4].

In fact, for (1), we have

$$\omega_{\mathbb{Q}}^{G;T}(\lambda) = \sum_{w\in W}\left(\frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\prod_{\alpha>0,w\alpha<0}\frac{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}\right).$$

(See Section 2 for details and unknown notations.) And for (2), we first know that is true for $SL(3)$ only, with the use of classical Koecher zeta functions (see, e.g., [25] for details). In believing (2) holds for general $SL(r)$, we seek the help from Henry H. Kim, among others. This proves to be quite fruitful: not only in [9], we can offer a general formula for volume of truncated domain $\mathfrak{F}(T)$ in the case of split, semisimple groups, which then offers an alternative proof for Siegel-Langlands' well-known formula on volume of fundamental domains [14]; but he brings us the paper of Diehl [4], which deals with Siegel–Eisenstein series associated with the group Sp, from which (2) is exposed by a certain extra effort [27].

With all this, it is clear that there are various difficulties in introducing and studying new zetas associated with reductive groups G geo-arithmetrically, starting from principal lattices and following the outline above for high-rank zetas associated with SL. So, we decide to adopt an analytic method by focusing on the period $\omega_{\mathbb{Q}}^{G}(\lambda)$ defined by

$$\omega_{\mathbb{Q}}^{G}(\lambda) := \sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\prod_{\alpha>0,w\alpha<0}\frac{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\xi_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}\right),\quad \mathrm{Re}\,\lambda\in\mathcal{C}^+.$$

Such a period, as said above, may be understood formally as the evaluation of the Eisenstein period

$$\int_{G(F)\backslash G(\mathbb{A})/K}\Lambda^T E(\mathbf{1};\lambda;g)d\mu(g)$$

at $T=0$, even T originally is supposed to be sufficiently positive. Simply put, the period $\omega_{\mathbb{Q}}^{G}(\lambda)$ essentially comes from a regularized integration process concerning constant terms of the associated Eisenstein series $E^{G/B}(\mathbf{1};\lambda;g)$, as a by-product of an advanced version of the famous Rankin–Selberg and Zagier method.

The period $\omega_{\mathbb{Q}}^{G}(\lambda)$ of G over $\mathbb{Q}$ is of rank(G) variables. To get a single variable zeta out from it, totally rank$(G)-1$ (linearly independent) singular hyperplanes need be chosen properly. This is done for SL and Sp in [26, 27], thanks to the paper of [4]. In fact, [4] deals with Sp only. But due to the fact that positive definite matrices are naturally associated with $\mathbb{Z}$-lattices and Siegel upper spaces, SL can also be treated successfully with extra care. Simply put, for each $G=SL(r)$ (or $=Sp(2n)$), within the framework of classical Eisenstein series, there exists *only one* choice of rank$(G)-1$ singular hyperplanes $H_1=0$, $H_2=0,\ldots,H_{\mathrm{rank}(G)-1}=0$. Moreover, after taking residues along with them, that is,

$$\mathrm{Res}_{H_1=0,\,H_2=0,\ldots,H_{\mathrm{rank}(G)-1}=0}\,\omega_{\mathbb{Q}}^{G}(\lambda),$$

with suitable normalizations, we can get a new zeta $\xi_{G;\mathbb{Q}}(s)$ for G. Examples for $SL(4,5)$ and $Sp(4)$ show that all these new zetas satisfy the functional equation $\xi_{G;\mathbb{Q}}(1-s)=\xi_{G;\mathbb{Q}}(s)$, and numerical tests (by MS) give supportive evidence for the RH as well. For details, see [27].

At this point, the role played in new zetas $\xi_{G;\mathbb{Q}}(s)$ by maximal parabolic subgroups has not yet emerged. It is only after the study done for G_2 that we understand such a key role. Nevertheless, what we do observe from these discussions on SL and Sp is as follows: all singular hyperplanes are taken from only a single term appeared in the period $\omega_{\mathbb{Q}}^{G}(\lambda)$, to be more precise, the term corresponding to $w = \mathrm{Id}$, the Weyl element identity. In other words, singular hyperplanes are taken from the denominator of the expression

$$\frac{1}{\prod_{\alpha\in\Delta_0}\langle\lambda-\rho,\alpha^\vee\rangle}.$$

(Totally, there are rank(G) factors, among which we have carefully chosen rank(G) − 1 for $G = SL$, Sp.) In particular, for the exceptional G_2, being a rank two group and hence an obvious choice for our next test, this reads as

$$\frac{1}{\left\langle\lambda-\rho,\alpha_{\mathrm{short}}^\vee\right\rangle\left\langle\lambda-\rho,\alpha_{\mathrm{long}}^\vee\right\rangle}$$

where α_{short}, α_{long} denote the short and long roots of G_2, respectively. So two possibilities,

(a) $\mathrm{Res}_{\langle\lambda-\rho,\alpha_{\mathrm{short}}^\vee\rangle=0}\,\omega_{\mathbb{Q}}^{G_2}(\lambda)$, leading to $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}(s)$ after suitable normalization; and
(b) $\mathrm{Res}_{\langle\lambda-\rho,\alpha_{\mathrm{long}}^\vee\rangle=0}\,\omega_{\mathbb{Q}}^{G_2}(\lambda)$, leading to $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{short}}}(s)$ after suitable normalization.

Here, we have used the fact that there exists a natural one-to-one and onto correspondence between collection of conjugation classes of maximal parabolic groups and simple roots. This is the essence of Definition and Proposition in Section 3, dealing with very important cases of a general construction for zetas associated with reductive groups and their maximal parabolic subgroups [27].

As expected, similar to high-rank zetas, these newly obtained zetas $\xi_{\mathbb{Q}}^{G_2/P}(s)$ for G_2 over $\mathbb{Q}$ prove to be canonical as well. In particular, we have the following.

Theorem. Let $P = P_{\mathrm{long}}$ or P_{short} and $\xi_{\mathbb{Q}}^{G_2/P}(s)$ be the associated zeta functions. Then
(1) $\xi_{\mathbb{Q}}^{G_2/P}(s)$ are meromorphic, and admit only finite singularities, four for each, to be more precise;
(2) $\xi_{\mathbb{Q}}^{G_2/P}(s)$ satisfy the standard functional equation

$$\xi_{\mathbb{Q}}^{G_2/P}(1-s) = \xi_{\mathbb{Q}}^{G_2/P}(s);$$

(3) All zeros of $\xi_{\mathbb{Q}}^{G_2/P}(s)$ lie on the central line $\mathrm{Re}\,(s) = 1/2$. □

Remark. With all this said for new zetas, we now point out a difference between high-rank zetas $\xi_{\mathbb{Q},r}(s)$ and new zetas $\xi_{\mathrm{SL}(r);\mathbb{Q}}(s) := \xi_{\mathbb{Q}}^{G/P}(s)$ attached to $(G,P) = (\mathrm{SL}(r), P_{r-1,1})$. Roughly speaking, starting from Eisenstein series $E^{G/B}(\mathbf{1};\lambda;g)$, $\xi_{\mathbb{Q},r}(s)$ corresponds to

(Res $\to \int$) ordered construction, and new zeta functions $\xi_{\mathrm{SL}(r),\mathbb{Q}}(s)$ correspond to ($\int \to$ Res)-ordered construction. Here, "(Res $\to \int$)-ordered" means that we first take the residues then take the integration; similarly, "($\int \to$ Res)-ordered" means that we first take the integration then take the residues. We have $\xi_{\mathbb{Q},2}(s) = \xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$, since no need taking residue. However, in general, there is a discrepancy between $\xi_{\mathbb{Q},r}(s)$ and $\xi_{\mathrm{SL}(r),\mathbb{Q}}(s)$, because of the obstruction for the exchanging of $\int$ and Res. For example, $\xi_{\mathbb{Q},3}(s)$ has only two singularities at $s = 0, 1$, but $\xi_{\mathrm{SL}(3),\mathbb{Q}}(s)$ has four singularities at $s = 0$, $\frac{1}{3}$, $\frac{2}{3}$, 1. Simply put, though new zetas $\xi_{\mathbb{Q}}^{GL(r)/P_{r-1,1}}(s) = \xi_{SL(r),\mathbb{Q}}(s)$ are closely related with high-rank zetas $\xi_{\mathbb{Q},r}(s)$ but are quite different indeed [27]. Nevertheless, we expect that the distribution of zeros for $\xi_{\mathrm{SL}(r),\mathbb{Q}}(s)$ is quite regular as well as for $\xi_{\mathbb{Q},r}(s)$. In fact, we have the Riemann hypothesis for $\xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$ (since $\xi_{\mathbb{Q},2}(s) = \xi_{\mathrm{SL}(2),\mathbb{Q}}(s)$), for $\xi_{\mathrm{SL}(3),\mathbb{Q}}(s)$, and for $\xi_{\mathrm{Sp}(4),\mathbb{Q}}(s)$ [11, 17, 18, 24]. All this in turn suggests that the study of new zetas $\xi_F^{G/P}(s)$ is not only interesting itself but also suggestive of the study of other zetas, including Dedekind zeta functions. □

This paper is organized as follows. In Sections 2 and 3, due to LW, we introduce various periods associated with automorphic forms using Arthur's analytic truncations (Section 2), and define zeta functions associated with G_2 and its maximal parabolic subgroups (Section 3). In Sections 4–6, due to MS, we give a proof of the corresponding Riemann hypothesis.

2 Various Periods

In this section, we introduce various periods associated with automorphic forms using Arthur's analytic truncation.

2.1 Automorphic forms and Eisenstein series

To facilitate our ensuing discussion, we make the following preparation. For details, see, e.g., [15] or [21].

Let F be a number field with $\mathbb{A} = \mathbb{A}_F$ its ring of adeles. Fix a connected reductive group G defined over F, denote by Z_G its center. Fix a minimal parabolic subgroup P_0 of G. Then $P_0 = M_0U_0$, where as usual we fix once and for all the Levi M_0 and the unipotent radical U_0. A parabolic subgroup P of G is called standard if $P \supset P_0$. For such groups, write $P = MU$ with $M_0 \subset M$ the standard Levi and U the unipotent radical. Denote by $\mathrm{Rat}(M)$ the group of rational characters of M, i.e, the morphism $M \to \mathbb{G}_m$ where $\mathbb{G}_m$

denotes the multiplicative group. Set

$$\mathfrak{a}_{M,\mathbb{C}}^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C}, \qquad \mathfrak{a}_{M,\mathbb{C}} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C}),$$

and

$$\mathfrak{a}_M^* := \mathrm{Re}\,\mathfrak{a}_M^* := \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad \mathfrak{a}_M := \mathrm{Re}\,\mathfrak{a}_M := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}).$$

For any $\chi \in \mathrm{Rat}(M)$, we obtain a (real) character $|\chi| : M(\mathbb{A}) \to \mathbb{R}^*$ defined by $m = (m_v) \mapsto m^{|\chi|} := \prod_{v \in S} |m_v|_v^{\chi_v}$ with $|\cdot|_v$ the v-absolute values. Set then $M(\mathbb{A})^1 := \cap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi|$, which is a normal subgroup of $M(\mathbb{A})$. Set X_M be the group of complex characters which are trivial on $M(\mathbb{A})^1$. Denote by $H_M := \log_M : M(\mathbb{A}) \to \mathfrak{a}_{M,\mathbb{C}}$ the map such that $\forall \chi \in \mathrm{Rat}(M) \subset \mathfrak{a}_{M,\mathbb{C}}^*, \langle \chi, \log_M(m) \rangle := \log(m^{|\chi|})$. Clearly,

$$M(\mathbb{A})^1 = \mathrm{Ker}(\log_M); \qquad \log_M(M(\mathbb{A})/M(\mathbb{A})^1) \simeq \mathrm{Re}\,\mathfrak{a}_M.$$

Hence, in particular, there is a natural isomorphism $\kappa : \mathfrak{a}_{M,\mathbb{C}}^* \simeq X_M$. Set

$$\mathrm{Re}X_M := \kappa\big(\mathrm{Re}\,\mathfrak{a}_M^*\big), \qquad \mathrm{Im}X_M := \kappa\big(i \cdot \mathrm{Re}\,\mathfrak{a}_M^*\big).$$

Moreover, define our working space X_M^G to be the subgroup of X_M consisting of complex characters of $M(\mathbb{A})/M(\mathbb{A})^1$ which are trivial on $Z_{G(\mathbb{A})}$.

Fix a maximal compact subgroup $\mathbb{K}$ such that for all standard parabolic subgroups $P = MU$ as above, $P(\mathbb{A}) \cap \mathbb{K} = (M(\mathbb{A}) \cap \mathbb{K})(U(\mathbb{A}) \cap \mathbb{K})$. Hence, we get the Langlands decomposition $G(\mathbb{A}) = M(\mathbb{A}) \cdot U(\mathbb{A}) \cdot \mathbb{K}$. Denote by $m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$ the map $g = m \cdot n \cdot k \mapsto M(\mathbb{A})^1 \cdot m$, where $g \in G(\mathbb{A}), m \in M(\mathbb{A}), n \in U(\mathbb{A})$ and $k \in \mathbb{K}$.

Fix Haar measures on $M_0(\mathbb{A})$, $U_0(\mathbb{A})$, $\mathbb{K}$, respectively such that

(1) the induced measure on $M(F)$ is the counting measure and the volume of the induced measure on $M(F)\backslash M(\mathbb{A})^1$ is 1. (Recall that it is a fundamental fact that $M(F)\backslash M(\mathbb{A})^1$ is of finite volume.)

(2) the induced measure on $U_0(F)$ is the counting measure and the volume of $U_0(F)\backslash U_0(\mathbb{A})$ is 1. (Recall that being unipotent radical, $U_0(F)\backslash U_0(\mathbb{A})$ is compact.)

(3) the volume of $\mathbb{K}$ is 1.

Such measures also induce Haar measures via $\log_M$ to the spaces $\mathfrak{a}_{M_0}$, $\mathfrak{a}_{M_0}^*$, etc. Furthermore, if we denote by ρ_0 the half of the sum of the positive roots of the maximal split torus T_0 of the central Z_{M_0} of M_0, then

$$f \mapsto \int_{M_0(\mathbb{A})\cdot U_0(\mathbb{A})\cdot \mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_0} dm$$

defined for continuous functions with compact supports on $G(\mathbb{A})$ defines a Haar measure dg on $G(\mathbb{A})$. This in turn gives measures on $M(\mathbb{A}), U(\mathbb{A})$, and hence on $\mathfrak{a}_M$, $\mathfrak{a}_M^*$, $P(\mathbb{A})$, etc, for all parabolic subgroups P. In particular, one checks that the following compatibility condition holds

$$\int_{M_0(\mathbb{A})\cdot U_0(\mathbb{A})\cdot \mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_0} dm = \int_{M(\mathbb{A})\cdot U(\mathbb{A})\cdot \mathbb{K}} f(mnk)\, dk\, dn\, m^{-2\rho_P} dm$$

for all continuous functions f with compact supports on $G(\mathbb{A})$, where ρ_P denotes one half of the sum of all positive roots of the maximal split torus T_P of the central Z_M of M. For later use, denote also by Δ_P the set of positive roots determined by (P, T_P) and $\Delta_0 = \Delta_{P_0}$.

Fix an isomorphism $T_0 \simeq \mathbb{G}_m^R$. Embed $\mathbb{R}_+^*$ by the map $t \mapsto (1; t)$. Then we obtain a natural injection $(\mathbb{R}_+^*)^R \hookrightarrow T_0(\mathbb{A})$ which splits. Denote by $A_{M_0(\mathbb{A})}$ the unique connected subgroup of $T_0(\mathbb{A})$ which projects onto $(\mathbb{R}_+^*)^R$. More generally, for a standard parabolic subgroup $P = MU$, set $A_{M(\mathbb{A})} := A_{M_0(\mathbb{A})} \cap Z_{M(\mathbb{A})}$, where as used above Z_* denotes the center of the group $*$. Clearly, $M(\mathbb{A}) = A_{M(\mathbb{A})} \cdot M(\mathbb{A})^1$. For later use, set also $A_{M(\mathbb{A})}^G := \{a \in A_{M(\mathbb{A})} : \log_G a = 0\}$. Then $A_{M(\mathbb{A})} = A_{G(\mathbb{A})} \oplus A_{M(\mathbb{A})}^G$.

Note that $\mathbb{K}$ and $U(F)\backslash U(\mathbb{A})$ are all compact, and $M(F)\backslash M(\mathbb{A})^1$ is of finite volume. With the Langlands decomposition $G(\mathbb{A}) = U(\mathbb{A})M(\mathbb{A})\mathbb{K}$ in mind, the reduction theory for $G(F)\backslash G(\mathbb{A})$ or, more generally, for $P(F)\backslash G(\mathbb{A})$ is reduced to that for $A_{M(\mathbb{A})}$ since $Z_G(F) \cap Z_{G(\mathbb{A})}\backslash Z_{G(\mathbb{A})} \cap G(\mathbb{A})^1$ is compact as well. As such, for $t_0 \in M_0(\mathbb{A})$ set

$$A_{M_0(\mathbb{A})}(t_0) := \left\{a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha,\ \forall \alpha \in \Delta_0\right\}.$$

Then, for a fixed compact subset $\omega \subset P_0(\mathbb{A})$, we have the corresponding Siegel set

$$S(\omega; t_0) := \left\{p \cdot a \cdot k : p \in \omega,\ a \in A_{M_0(\mathbb{A})}(t_0),\ k \in \mathbb{K}\right\}.$$

In particular, the classical reduction theory may be restated as, for big enough ω and small enough t_0, i.e, t_0^α is very close to 0 for all $\alpha \in \Delta_0$, $G(\mathbb{A}) = G(F) \cdot S(\omega; t_0)$. More generally, set

$$A^P_{M_0(\mathbb{A})}(t_0) := \left\{ a \in A_{M_0(\mathbb{A})} : a^\alpha > t_0^\alpha, \ \forall \alpha \in \Delta_0^P \right\},$$

and

$$S^P(\omega; t_0) := \left\{ p \cdot a \cdot k : p \in \omega, \ a \in A^P_{M_0(\mathbb{A})}(t_0), \ k \in \mathbb{K} \right\}.$$

Then, similarly as above, for big enough ω and small enough t_0, $G(\mathbb{A}) = P(F) \cdot S^P(\omega; t_0)$. (Here, Δ_0^P denotes the set of positive roots for $(P_0 \cap M, T_0)$.)

Fix an embedding $i_G : G \hookrightarrow SL_n$ sending g to (g_{ij}). Introducing a height function on $G(\mathbb{A})$ by setting $\|g\| := \prod_{v \in S} \sup\{|g_{ij}|_v : \forall i, j\}$. It is well known that up to $O(1)$, height functions are unique. This implies that the following growth conditions do not depend on the height function we choose.

A function $f : G(\mathbb{A}) \to \mathbb{C}$ is said to have moderate growth if there exist $c, r \in \mathbb{R}$ such that $|f(g)| \le c \cdot \|g\|^r$ for all $g \in G(\mathbb{A})$. Similarly, for a standard parabolic subgroup $P = MU$, a function $f : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to have moderate growth if there exist $c, r \in \mathbb{R}, \lambda \in \mathrm{Re}X_{M_0}$ such that for any $a \in A_{M(\mathbb{A})}, k \in \mathbb{K}, \ m \in M(\mathbb{A})^1 \cap S^P(\omega; t_0)$,

$$|f(amk)| \le c \cdot \|a\|^r \cdot m_{P_0}(m)^\lambda.$$

By contrast, a function $f : S(\omega; t_0) \to \mathbb{C}$ is said to be rapidly decreasing if there exists $r > 0$ and for all $\lambda \in \mathrm{Re}X_{M_0}$ there exists $c > 0$ such that for $a \in A_{M(\mathbb{A})}, \ g \in G(\mathbb{A})^1 \cap S(\omega; t_0)$, $|\phi(ag)| \le c \cdot \|a\| \cdot m_{P_0}(g)^\lambda$. And a function $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to be rapidly decreasing if $f|_{S(\omega; t_0)}$ is so.

Also a function $f : G(\mathbb{A}) \to \mathbb{C}$ is said to be smooth if for any $g = g_f \cdot g_\infty \in G(\mathbb{A}_f) \times G(\mathbb{A}_\infty)$, there exist open neighborhoods V_* of g_* in $G(\mathbb{A})$ and a C^∞-function $f' : V_\infty \to \mathbb{C}$ such that $f(g'_f \cdot g'_\infty) = f'(g'_\infty)$ for all $g'_f \in V_f$ and $g'_\infty \in V_\infty$.

By definition, a function $\phi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is called *automorphic* if

(i) ϕ has moderate growth;
(ii) ϕ is smooth;
(iii) ϕ is $\mathbb{K}$-finite, i.e, the $\mathbb{C}$-span of all $\phi(k_1 \cdot * \cdot k_2)$ parametrized by $(k_1, k_2) \in \mathbb{K} \times \mathbb{K}$ is finite dimensional; and
(iv) ϕ is $\mathfrak{z}$-finite, i.e, the $\mathbb{C}$-span of all $\delta(X)\phi$ parametrized by all $X \in \mathfrak{z}$ is finite dimensional. Here, $\mathfrak{z}$ denotes the center of the universal enveloping algebra

$\mathfrak{u} := \mathfrak{U}(\mathrm{Lie}G(\mathbb{A}_\infty))$ of the Lie algebra of $G(\mathbb{A}_\infty)$ and $\delta(X)$ denotes the derivative of ϕ along X.

Set $A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ be the space of automorphic forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$.

For a measurable locally L^1-function $f : U(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, define its *constant term* along with the standard parabolic subgroup $P = UM$ to be $f_P : U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ given by $g \mapsto \int_{U(F)\backslash G(\mathbb{A})} f(ng)dn$. Then an automorphic form $\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called a *cusp form* if for any standard parabolic subgroup P' properly contained in P, $\phi_{P'} \equiv 0$. Denote by $A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ the space of cusp forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$. One checks easily that

(i) all cusp forms are rapidly decreasing, and hence

(ii) there is a natural pairing

$$\langle \cdot, \cdot \rangle : A_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \times A(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \to \mathbb{C}$$

defined by $\langle \psi, \phi \rangle := \int_{Z_{M(\mathbb{A})}U(\mathbb{A})M(F)\backslash G(\mathbb{A})} \psi(g)\bar{\phi}(g)\, dg$.

For an automorphic form $\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$, define the associated *Eisenstein series* $E(\phi, \lambda) : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ by

$$E(\phi, \lambda)(g) := \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g) \cdot m_P(\delta g)^{\lambda+\rho_P}.$$

Then one checks that there is an open cone $\mathcal{C} \subset \mathrm{Re}X_M^G$ such that if $\mathrm{Re}\lambda \in \mathcal{C}$, $E(\phi, \lambda)(g)$ converges uniformly for g in a compact subset of $G(\mathbb{A})$ and λ in an open neighborhood of 0 in X_M^G. For example, if ϕ is cuspidal, we may even take $\mathcal{C}$ to be the cone $\{\lambda \in \mathrm{Re}X_M^G : \langle \lambda, \alpha^\vee \rangle > 0, \forall \alpha \in \Delta_P^G\}$. As a direct consequence, then $E(\phi, \lambda) \in A(G(F)\backslash G(\mathbb{A}))$. That is, it is an automorphic form.

We end this discussion by introducing intertwining operators. For $w \in W$ the Weyl group of G, fix once and for all representative $w \in G(F)$ of w. Set $M' := wMw^{-1}$ and denote the associated parabolic subgroup by $P' = U'M'$. As usual, define the associated intertwining operator $M(w, \lambda)$ by

$$\begin{aligned}(M(w, \lambda)\phi)(g) \;&:=\; m_{P'}(g)^{w\lambda+\rho_{P'}} \\ &\times \int_{U'(F)\cap wU(F)w^{-1}\backslash U'(\mathbb{A})} \phi(w^{-1}n'g) \cdot m_P(w^{-1}n'g)^{\lambda+\rho_P}\, dn', \quad \forall g \in G(\mathbb{A}).\end{aligned}$$

2.2 Arthur's analytic truncation

Let P be a (standard) parabolic subgroup of G. Write T_P for the maximal split torus in the center of M_P and T'_P for the maximal quotient split torus of M_P. Set $\tilde{\mathfrak{a}}_P := X_*(T_P) \otimes \mathbb{R}$ and denote its real dimension by $d(P)$, where $X_*(T)$ is the lattice of 1-parameter subgroups in the torus T. Then it is known that $\tilde{\mathfrak{a}}_P = X_*(T'_P) \otimes \mathbb{R}$ as well. The two descriptions of $\tilde{\mathfrak{a}}_P$ show that if $Q \subset P$ is a parabolic subgroup, then there is a canonical injection $\tilde{\mathfrak{a}}_P \hookrightarrow \tilde{\mathfrak{a}}_Q$ and a natural surjection $\tilde{\mathfrak{a}}_Q \twoheadrightarrow \tilde{\mathfrak{a}}_P$. We thus obtain a canonical decomposition $\tilde{\mathfrak{a}}_Q = \tilde{\mathfrak{a}}_Q^P \oplus \tilde{\mathfrak{a}}_P$ for a certain subspace $\tilde{\mathfrak{a}}_Q^P$ of $\tilde{\mathfrak{a}}_Q$. In particular, $\tilde{\mathfrak{a}}_G$ is a summand of $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_P$ for all P. Set $\mathfrak{a}_P := \tilde{\mathfrak{a}}_P / \tilde{\mathfrak{a}}_G$ and $\mathfrak{a}_Q^P := \tilde{\mathfrak{a}}_Q^P / \tilde{\mathfrak{a}}_G$. Then we have

$$\mathfrak{a}_Q = \mathfrak{a}_Q^P \oplus \mathfrak{a}_P$$

and $\mathfrak{a}_P$ is canonically identified as a subspace of $\mathfrak{a}_Q$. Set $\mathfrak{a}_0 := \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^P = \mathfrak{a}_{P_0}^P$, then we also have $\mathfrak{a}_0 = \mathfrak{a}_0^P \oplus \mathfrak{a}_P$ for all P.

Dually we have spaces $\mathfrak{a}_0^*$, $\mathfrak{a}_P^*$, $(\mathfrak{a}_0^P)^*$, (where for a real space V, write V^* its dual space over $\mathbb{R}$), and hence the decompositions $\mathfrak{a}_0^* = (\mathfrak{a}_0^Q)^* \oplus (\mathfrak{a}_Q^P)^* \oplus \mathfrak{a}_P^*$.

So, $\mathfrak{a}_P^* = X(M_P) \otimes \mathbb{R}$ with $X(M_P)$ the group $\mathrm{Hom}_F(M_P, GL(1))$, i.e., a collection of characters on M_P. It is known that $\mathfrak{a}_P^* = X(A_P) \otimes \mathbb{R}$, where A_P denotes the split component of the center of M_P. Clearly, if $Q \subset P$, then $M_Q \subset M_P$ while $A_P \subset A_Q$. Thus, via restriction, the above two expressions of $\mathfrak{a}_P^*$ also naturally induce an injection $\mathfrak{a}_P^* \hookrightarrow \mathfrak{a}_Q^*$ and a surjection $\mathfrak{a}_Q^* \twoheadrightarrow \mathfrak{a}_P^*$, compatible with the decomposition $\mathfrak{a}_Q^* = (\mathfrak{a}_Q^P)^* \oplus \mathfrak{a}_P^*$.

As usual, let Δ_0 and $\widehat{\Delta}_0$ be the subsets of simple roots and simple weights in $\mathfrak{a}_0^*$ respectively. Write $\Delta_0^\vee$ (resp. $\widehat{\Delta}_0^\vee$) for the basis of $\mathfrak{a}_0$ dual to $\widehat{\Delta}_0$ (resp. Δ_0). Being the dual of the collection of simple weights (resp. of simple roots), $\Delta_0^\vee$ (resp. $\widehat{\Delta}_0^\vee$) is the set of coroots (resp. coweights).

For every P, let $\Delta_P \subset \mathfrak{a}_0^*$ be the set of nontrivial *restrictions* of elements of Δ_0 to $\mathfrak{a}_P$. Denote the dual basis of Δ_P by $\widehat{\Delta}_P^\vee$. For each $\alpha \in \Delta_P$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{a}_P$, where β is the root in Δ_0 whose restriction to $\mathfrak{a}_P$ is α. Set $\Delta_P^\vee := \{\alpha^\vee : \alpha \in \Delta_P\}$, and define the dual basis of $\Delta_P^\vee$ by $\widehat{\Delta}_P$.

More generally, if $Q \subset P$, write Δ_Q^P to denote the *subset* $\alpha \in \Delta_Q$ appearing in the action of T_Q in the unipotent radical of $Q \cap M_P$. (Indeed, $M_P \cap Q$ is a parabolic subgroup of M_P with nilpotent radical $N_Q^P := N_Q \cap M_P$. Thus, Δ_Q^P is simply the set of roots of the parabolic subgroup $(M_P \cap Q, A_Q)$. And one checks that the map $P \mapsto \Delta_Q^P$ gives a natural bijection between parabolic subgroups P containing Q and subsets of

Δ_Q.) Then $\mathfrak{a}_P$ is the subspace of $\mathfrak{a}_Q$ annihilated by Δ_Q^P. Denote by $(\widehat{\Delta}^\vee)_Q^P$ the dual of Δ_Q^P. Let $(\Delta_Q^P)^\vee := \{\alpha^\vee : \alpha \in \Delta_Q^P\}$ and denote by $\widehat{\Delta}_Q^P$ the dual of $(\Delta_Q^P)^\vee$.

Moreover, we extend the linear functionals in Δ_Q^P and $\widehat{\Delta}_Q^P$ to elements of the dual space $\mathfrak{a}_0^*$ by means of the canonical projection from $\mathfrak{a}_0$ to $\mathfrak{a}_Q^P$ given by the decomposition $\mathfrak{a}_0 = \mathfrak{a}_0^Q \oplus \mathfrak{a}_Q^P \oplus \mathfrak{a}_P$. Let $\widehat{\tau}_Q^P$ be the characteristic function of the *positive cone*

$$\left\{H \in \mathfrak{a}_0 : \langle \varpi, H\rangle > 0,\ \forall \varpi \in \widehat{\Delta}_Q^P\right\} = \mathfrak{a}_0^Q \oplus \left\{H \in \mathfrak{a}_Q^P : \langle \varpi, H\rangle > 0 \text{ for all } \varpi \in \widehat{\Delta}_Q^P\right\} \oplus \mathfrak{a}_P.$$

Denote $\widehat{\tau}_P^G$ simply by $\widehat{\tau}_P$.

Recall that an element $T \in \mathfrak{a}_0$ is called *sufficiently regular*, if $\alpha(T) \gg 0$ for any $\alpha \in \Delta_0$. Fix then a suitably regular point $T \in \mathfrak{a}_0$. If ϕ is a continuous function on $G(F)\backslash G(\mathbb{A})^1$, define *Arthur's analytic truncation* $(\Lambda^T\phi)(x)$ to be the function

$$(\Lambda^T\phi)(x) := \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(F)\backslash G(F)} \phi_P(\delta x) \cdot \hat{\tau}_P(H(\delta x) - T),$$

where

$$\phi_P(x) := \int_{N(F)\backslash N(\mathbb{A})} \phi(nx)\, dn$$

denotes the constant term of ϕ along P, and the sum is over all (standard) parabolic subgroups.

Note that all parabolic subgroups of G can be obtained from standard parabolic subgroups by taking conjugations with elements from $P(F)\backslash G(F)$. So we have

(a) $(\Lambda^T\phi)(x) = \sum_P (-1)^{\dim(A/Z)} \phi_P(x) \cdot \hat{\tau}_P(H(x) - T)$, *where the sum is over all, both standard and nonstandard, parabolic subgroups*;

(b) *If ϕ is a cusp form, then $\Lambda^T\phi = \phi$.*

Fundamental properties of Arthur's analytic truncation may be summarized as follows.

Theorem 1 (Arthur [1, 2]). For sufficiently regular T in $\mathfrak{a}_0$,

(1) Let $\phi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ be a locally L^1 function. Then

$$\Lambda^T\Lambda^T\phi(g) = \Lambda^T\phi(g)$$

for almost all g. If ϕ is also locally bounded, then the above is true for all g;
(2) Let ϕ_1, ϕ_2 be two locally L^1 functions on $G(F)\backslash G(\mathbb{A})$. Suppose that ϕ_1 is of moderate growth and ϕ_2 is rapidly decreasing. Then

$$\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \overline{\Lambda^T\phi_1(g)} \cdot \phi_2(g)\, dg = \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \overline{\phi_1(g)} \cdot \Lambda^T\phi_2(g)\, dg;$$

(3) Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$, and $r,\ r'$ be two positive real numbers. Then there exists a finite subset $\{X_i : i = 1, 2, \dots, N\} \subset \mathcal{U}$, the universal enveloping algebra of $\mathfrak{g}_\infty$, such that the following is satisfied: Let ϕ be a smooth function on $G(F)\backslash G(\mathbb{A})$, right invariant under K_f, and let $a \in A_{G(\mathbb{A})},\ g \in G(\mathbb{A})^1 \cap S$. Then

$$|\Lambda^T\phi(ag)| \leq \|g\|^{-r} \sum_{i=1}^{N} \sup\{|\delta(X_i)\phi(ag')|\, \|g'\|^{-r'} : g' \in G(\mathbb{A})^1\},$$

where S is a Siegel domain with respect to $G(F)\backslash G(\mathbb{A})$. □

2.3 Arthur's periods

Fix a sufficiently regular $T \in \mathfrak{a}_0$ and let ϕ be an automorphic form of G. Then, $\Lambda^T\phi$ is rapidly decreasing, and hence integrable. In particular, the integration

$$A(\phi; T) := \int_{G(F)\backslash G(\mathbb{A})} \Lambda^T\phi(g)\, dg$$

makes sense. We claim that $A(\phi; T)$ can be written as an integration of the original automorphic form ϕ over a certain compact subset.

To start with, note that for Arthur's analytic truncation Λ^T, we have $\Lambda^T \circ \Lambda^T = \Lambda^T$. Hence,

$$\begin{aligned} A(\phi; T) &= \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda^T\phi\ d\mu(g) \\ &= \int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \Lambda^T(\Lambda^T\phi)(g)\ d\mu(g). \end{aligned}$$

Moreover, by the self-adjoint property, for the constant function $\mathbf{1}$ on $G(\mathbb{A})$,

$$\begin{aligned}\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} &\mathbf{1}(g)\cdot\Lambda^T\big(\Lambda^T\phi\big)(g)\,d\mu(g)\\ &=\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}(\Lambda^T\mathbf{1})(g)\cdot(\Lambda^T\phi)(g)\,d\mu(g)\\ &=\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}\Lambda^T(\Lambda^T\mathbf{1})(g)\cdot\phi(g)\,d\mu(g),\end{aligned}$$

since $\Lambda^T\phi$ and $\Lambda^T\mathbf{1}$ are rapidly decreasing. Therefore, using $\Lambda^T\circ\Lambda^T=\Lambda^T$ again, we arrive at

$$A(\phi;T)=\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}\Lambda^T\mathbf{1}(g)\cdot\phi(g)\,d\mu(g). \tag{$*$}$$

To go further, let us give a much more detailed study of Arthur's analytic truncation for the constant function $\mathbf{1}$. Introduce the truncated subset $\Sigma(T)$ of the space $Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})$ by

$$\Sigma(T):=\big\{g\in Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A}):\Lambda^T\mathbf{1}(g)=1\big\}.$$

Proposition 1 (Arthur [3]). For sufficiently regular $T\in\mathfrak{a}_0$, $\Lambda^T\mathbf{1}$ is the characteristic function of a compact subset of $Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})$. In particular, $\Sigma(T)$ is compact. □

Consequently, $\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}\Lambda^T\phi(g)\,d\mu(g)=$

$$\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}\Lambda^T\mathbf{1}(g)\cdot\phi(g)\,d\mu(g)=\int_{\Sigma(T)}\phi(g)\,d\mu(g).$$

That is to say, we have obtained the following.

Proposition 2. For a sufficiently regular $T\in\mathfrak{a}_0$ and an automorphic form ϕ on $G(F)\backslash G(\mathbb{A})$,

$$\int_{\Sigma(T)}\phi(g)\,d\mu(g)=\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})}\Lambda^T\phi(g)\,d\mu(g).$$

□

It is because of this result that we call $\int_{G(F)\backslash G(\mathbb{A})^1}\Lambda^T\phi(g)\,d\mu(g)$ the *Arthur period* for ϕ.

2.4 Eisenstein periods

Let P be a (standard) parabolic subgroup of G with Levi decomposition $P = MU$ and $\phi \in A(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ an M-level automorphic form. Then the associated Eisenstein series $E(\phi;\lambda)(g) := \sum_{\delta\in P(F)\backslash G(F)} \phi(\delta g)\cdot m_P(\delta g)^{\lambda+\rho_P} \in A(G(F)\backslash G(\mathbb{A})$ is a G-level automorphic form. Thus, for a sufficiently positive $T \in \mathfrak{a}_0$, we obtain a well-defined Arthur period

$$\int_{Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})} \wedge^T E(\phi;\lambda)(g)\,d\mu(g).$$

Due to the obvious importance, we call such an Arthur period an *Eisenstein period*.

In general, Eisenstein periods are quite difficult to be evaluated. However, if ϕ is cuspidal, we have the following result of [8], an advanced version of the Rankin–Selberg and Zagier method.

Theorem 2 [8]. Fix a sufficiently positive $T \in \mathfrak{a}_0^+$. Let $P = MU$ be a parabolic subgroup and ϕ a P-level cusp form. Then the Eisenstein period $\int_{G(F)\backslash G(\mathbb{A})} \Lambda^T E(\lambda,\phi)(g)\,dg$ is equal to
(1) 0 if $P \neq P_0$ is not minimal; and
(2) $\mathrm{Vol}(\{\sum_{\alpha\in\Delta_0} a_\alpha\alpha^\vee : a_\alpha \in [0,1)\}) \times \sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \int_{M_0(F)\backslash M_0(\mathbb{A})^1\times K} (M(w,\lambda)\phi)(mk)\,dm\,dk$,
if $P = P_0 = M_0U_0$ is minimal. □

2.5 Periods for G over F

Now, we focus on the expression

$$\sum_{w\in W} \frac{e^{\langle w\lambda-\rho,T\rangle}}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \times \int_{M_0(F)\backslash M_0(\mathbb{A})^1\times K} (M(w,\lambda)\phi)(mk)\,dm\,dk, \tag{$*$}$$

for a cusp form ϕ at the level of the Borel. *Motivated by our study of high-rank zetas* [23, 25–27], we make the following two simplifications:

(1) Take $T = 0$. Recall that in the discussion so far, T is assumed to be sufficiently positive. However, $(*)$ makes sense even when $T = 0$; and
(2) Take $\phi \equiv \mathbf{1}$, the constant function one on the Borel. Recall that in general for a standard $P = MU$, the constant function $\mathbf{1}$ is only L^2 on M. But for the Borel, $\mathbf{1}$ is cuspidal.

With all these preparations, we are ready to introduce our first main definition.

Definition 1. The *period* $\omega_F^G(\lambda)$ *of* G *over* F is defined by

$$\omega_F^G(\lambda) := \sum_{w \in W} \left(\frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \times M(w, \lambda) \right),$$

where $M(w, \lambda)$ denotes the quantity

$$m_{P'}(e)^{w\lambda + \rho_{P'}} \cdot \int_{U'(F) \cap wU(F)w^{-1} \backslash U'(\mathbb{A})} m_P(w^{-1}n')^{\lambda + \rho_P} dn'$$

where $M' := wMw^{-1}$ and $P' = U'M'$ denote the associated parabolic subgroup. □

In particular, for $G = G_2$, by the Gindikin–Karpelevich formula [13], we have

$$M(w, \lambda) = \prod_{\alpha > 0,\, w\alpha < 0} \frac{\xi(\langle \lambda, \alpha^\vee \rangle)}{\xi(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

Here, $\xi(s) := \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$ with $\zeta(s)$ the Riemann zeta function [5]. Consequently,

$$\omega_{\mathbb{Q}}^{G_2}(\lambda) := \sum_{w \in W} \left(\frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \times \prod_{\alpha > 0,\, w\alpha < 0} \frac{\xi(\langle \lambda, \alpha^\vee \rangle)}{\xi(\langle \lambda, \alpha^\vee \rangle + 1)} \right). \qquad (**)$$

3 Zetas for G_2

In this section, we introduce zeta functions associated with (G_2, P_{short}) and (G_2, P_{long}) using the period of G_2 introduced in Section 2.

3.1 Period for G_2 over $\mathbb{Q}$

Let G be the exceptional group G_2. It is simply connected and adjoint. Fix a maximal split torus T in G and a Borel subgroup B containing T. Then we obtain two simple roots, the short root α and the long root β. So, $\Delta_0 = \{\alpha, \beta\}$ and all positive roots are given by

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

Denote by $P_{\text{long}} = P_\beta = P_1$ and $P_{\text{short}} = P_\alpha = P_2$ the maximal standard parabolic subgroups attached to $\Delta_0\backslash\{\beta\}$ and $\Delta_0\backslash\{\alpha\}$, respectively (see, e.g., [7]).

Choose a parametrization $t : \mathbb{Q}^* \times \mathbb{Q}^* \to T, (a,b) \mapsto t(a,b)$ defined by $\alpha(t(a,b)) = ab^{-1},\ \beta(t(a,b)) = a^{-1}b^2$. Then the actions of remaining positive roots are given by

$$\begin{aligned}&(\alpha+\beta)(t(a,b)) = b, \qquad (2\alpha+\beta)(t(a,b)) = a,\\ &(3\alpha+\beta)(t(a,b)) = a^2b^{-1}, \qquad (3\alpha+2\beta)(t(a,b)) = ab,\end{aligned}$$

and the corresponding coroots are given by

$$\begin{aligned}&\alpha^\vee(x) = t(x,x^{-1}), \qquad \beta^\vee(x) = t(1,x), \qquad (\alpha+\beta)^\vee(x) = t(x,x^2),\\ &(2\alpha+\beta)^\vee(x) = t(x^2,x), \quad (3\alpha+\beta)^\vee(x) = t(x,1), \quad (3\alpha+2\beta)^\vee(x) = t(x,x).\end{aligned}$$

Let $X(T)$ be the character group of T and $\mathfrak{a}_{\mathbb{C}}^* = X(T)\otimes\mathbb{C}$ its complexification. We introduce coordinates in $\mathfrak{a}_{\mathbb{C}}^*$ with respect to the basis $2\alpha+\beta,\ \alpha+\beta$. Thus, point $(z_1,z_2) \in \mathbb{C}^2$ corresponds to the character $\lambda = z_1(2\alpha+\beta) + z_2(\alpha+\beta)$. (The coordinate is chosen to make $\lambda(t(a,b)) = |a|^{z_1}|b|^{z_2}$ take the simplest form.) As such, then $\rho := \rho_B := 5\alpha+3\beta$ and $\mathcal{C}^+$ of the positive Weyl chamber in $\mathfrak{a}_{\mathbb{C}}^*$ is given by

$$\begin{aligned}\mathcal{C}^+ :=&\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}\langle\lambda,\gamma^\vee\rangle > 0,\ \forall\gamma > 0\}\\ =&\{z_1(2\alpha+\beta) + z_2(\alpha+\beta) \mid \mathrm{Re} z_1 > \mathrm{Re} z_2 > 0\}.\end{aligned}$$

For a positive root γ, denote by w_γ the reflection defined by γ, i.e., the reflection on the space $\mathfrak{a}_{\mathbb{C}}^*$ which reflects γ to $-\gamma$. And denote by $\sigma(\omega)$ the rotation through ω with center at the origin. Then it is well known that the Weyl group of G_2 is given by

$$W = \left\{e, w_\alpha, w_\beta, w_{3\alpha+\beta}, w_{2\alpha+\beta}, w_{3\alpha+2\beta}, w_{\alpha+\beta}, \sigma\left(\frac{\pi}{3}\right), \sigma\left(\frac{2\pi}{3}\right), \sigma(\pi), \sigma\left(\frac{4\pi}{3}\right), \sigma\left(\frac{5\pi}{3}\right)\right\}.$$

Moreover, by a direct calculation, we have the following table on $w\lambda$ and $\{\gamma > 0 \mid w\gamma < 0\}$:

	$w\lambda; \lambda = (z_1, z_2)$	$\{\gamma > 0 \mid w\gamma < 0\}$
e	(z_1, z_2)	$-$
w_α	(z_2, z_1)	α
w_β	$(z_1 + z_2, -z_2)$	β
$w_{3\alpha+\beta}$	$(-z_1, z_1 + z_2)$	$\alpha, 3\alpha + \beta, 2\alpha + \beta$
$w_{2\alpha+\beta}$	$(-z_1 - z_2, z_2)$	$\alpha, 3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta$
$w_{3\alpha+2\beta}$	$(- - z_2, -z_1)$	$3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta, \beta$
$w_{\alpha+\beta}$	$(z_1, - - z_1 - z_2)$	$3\alpha + 2\beta, \alpha + \beta, \beta$
$\sigma\left(\frac{\pi}{3}\right)$	$(-z_2, z_1 + z_2)$	$\alpha + \beta, \beta$
$\sigma\left(\frac{2\pi}{3}\right)$	$(-z_1 - z_2, z_1)$	$2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta, \beta$
$\sigma(\pi)$	$(-z_1, -z_2)$	$\alpha, 3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta, \alpha + \beta, \beta$
$\sigma\left(\frac{4\pi}{3}\right)$	$(z_2, -z_1 - z_2)$	$\alpha, 3\alpha + \beta, 2\alpha + \beta, 3\alpha + 2\beta$
$\sigma\left(\frac{5\pi}{3}\right)$	$(z_1 + z_2, -z_1)$	$\alpha, 3\alpha + \beta.$

Also, by definition, we see that

$$\langle \lambda, \alpha^\vee \rangle = z_1 - z_2, \qquad \langle \lambda, \beta^\vee \rangle = z_2, \qquad \langle \lambda, (3\alpha + \beta)^\vee \rangle = z_1,$$
$$\langle \lambda, (2\alpha + \beta)^\vee \rangle = 2z_1 + z_2, \ \langle \lambda, (3\alpha + 2\beta)^\vee \rangle = z_1 + z_2, \ \langle \lambda, (\alpha + \beta)^\vee \rangle = z_1 + 2z_2,$$

for $\lambda = (z_1, z_2)$, since

$$\lambda(t(x, x^{-1})) = x^{z_1} x^{-z_2} = x^{z_1 - z_2}, \quad \lambda(t(1, x)) = 1^{z_1} x^{z_2} = x^{z_2},$$
$$\lambda(t(x, 1)) = x^{z_1} 1^{z_2} = x^{z_1}, \quad \lambda(t(x^2, x)) = x^{2z_1} x^{z_2} = x^{2z_1 + z_2},$$
$$\lambda(t(x, x)) = x^{z_1} x^{z_2} = x^{z_1 + z_2}, \quad \lambda(t(x, x^2)) = x^{z_1} x^{2z_2} = x^{z_1 + 2z_2}.$$

Hence, by tedious elementary calculations, which we decide to omit, we have the following:

(a) for $\langle w\lambda, \alpha^\vee\rangle - 1$ and $\langle w\lambda, \beta^\vee\rangle - 1$,

	$w\lambda; \lambda = (z_1, z_2)$	$\langle w\lambda, \alpha^\vee\rangle - 1$	$\langle w\lambda, \beta^\vee\rangle - 1$
e	(z_1, z_2)	$z_1 - z_2 - 1$	$z_2 - 1$
w_α	(z_2, z_1)	$z_2 - z_1 - 1$	$z_1 - 1$
w_β	$(z_1 + z_2, -z_2)$	$z_1 + 2z_2 - 1$	$-z_2 - 1$
$w_{3\alpha+\beta}$	$(-z_1, z_1 + z_2)$	$-2z_1 - z_2 - 1$	$z_1 + z_2 - 1$
$w_{2\alpha+\beta}$	$(-z_1 - z_2, z_2)$	$-z_1 - 2z_2 - 1$	$z_2 - 1$
$w_{3\alpha+2\beta}$	$(- - z_2, -z_1)$	$z_1 - z_2 - 1$	$-z_1 - 1$
$w_{\alpha+\beta}$	$(z_1, - - z_1 - z_2)$	$2z_1 + z_2 - 1$	$-z_1 - z_2 - 1$
$\sigma\left(\frac{\pi}{3}\right)$	$(-z_2, z_1 + z_2)$	$-z_1 - 2z_2 - 1$	$z_1 + z_2 - 1$
$\sigma\left(\frac{2\pi}{3}\right)$	$(-z_1 - z_2, z_1)$	$-2z_1 - z_2 - 1$	$z_1 - 1$
$\sigma(\pi)$	$(-z_1, -z_2)$	$-z_1 + z_2 - 1$	$-z_2 - 1$
$\sigma\left(\frac{4\pi}{3}\right)$	$(z_2, -z_1 - z_2)$	$z_1 + 2z_2 - 1$	$-z_1 - z_2 - 1$
$\sigma\left(\frac{5\pi}{3}\right)$	$(z_1 + z_2, -z_1)$	$2z_1 + z_2 - 1$	$-z_1 - 1,$

and

(b) for $\displaystyle\prod_{\gamma>0, w\gamma<0} \frac{\xi(\langle\lambda,\gamma^\vee\rangle)}{\xi(\langle\lambda,\gamma^\vee\rangle + 1)}$,

	$\prod_{\gamma>0,w\gamma<0} \frac{\xi(\langle\lambda,\gamma^\vee\rangle)}{\xi(\langle\lambda,\gamma^\vee\rangle+1)}$
e	1
w_α	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}$
w_β	$\frac{\xi(z_2)}{\xi(z_2+1)}$
$w_{3\alpha+\beta}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \frac{\xi(z_1)}{\xi(z_1+1)} \frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}$
$w_{2\alpha+\beta}$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \frac{\xi(z_1)}{\xi(z_1+1)} \frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}$
$w_{3\alpha+2\beta}$	$\frac{\xi(z_1)}{\xi(z_1+1)} \frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \frac{\xi(z_2)}{\xi(z_2+1)}$
$w_{\alpha+\beta}$	$\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \frac{\xi(z_2)}{\xi(z_2+1)}$
$\sigma\left(\frac{\pi}{3}\right)$	$\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \frac{\xi(z_2)}{\xi(z_2+1)}$
$\sigma\left(\frac{2\pi}{3}\right)$	$\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \frac{\xi(z_2)}{\xi(z_2+1)}$
$\sigma(\pi)$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \frac{\xi(z_1)}{\xi(z_1+1)} \frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \frac{\xi(z_2)}{\xi(z_2+1)}$
$\sigma\left(\frac{4\pi}{3}\right)$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \frac{\xi(z_1)}{\xi(z_1+1)} \frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}$
$\sigma\left(\frac{5\pi}{3}\right)$	$\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \frac{\xi(z_1)}{\xi(z_1+1)}.$

Or put them in a better form, we have

$$
\begin{array}{cc}
 & \frac{1}{\langle w\lambda,\alpha^\vee\rangle-1}\frac{1}{\langle w\lambda,\beta^\vee\rangle-1}\cdot\prod_{\gamma>0,w\gamma<0}\frac{\xi(\langle\lambda,\gamma^\vee\rangle)}{\xi(\langle\lambda,\gamma^\vee\rangle+1)} \\
e & \frac{1}{z_1-z_2-1}\frac{1}{z_2-1} \\
w_\alpha & \frac{1}{z_2-z_1-1}\frac{1}{z_1-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)} \\
w_\beta & \frac{1}{z_1+2z_2-1}\frac{1}{-z_2-1}\cdot\frac{\xi(z_2)}{\xi(z_2+1)} \\
w_{3\alpha+\beta} & \frac{1}{-2z_1-z_2-1}\frac{1}{z_1+z_2-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\frac{\xi(z_1)}{\xi(z_1+1)}\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)} \\
w_{2\alpha+\beta} & \frac{1}{-z_1-2z_2-1}\frac{1}{z_2-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\frac{\xi(z_1)}{\xi(z_1+1)}\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)} \\
w_{3\alpha+2\beta} & \frac{1}{z_1-z_2-1}\frac{1}{-z_1-1}\cdot\frac{\xi(z_1)}{\xi(z_1+1)}\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}\frac{\xi(z_2)}{\xi(z_2+1)} \\
w_{\alpha+\beta} & \frac{1}{2z_1+z_2-1}\frac{1}{-z_1-z_2-1}\cdot\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}\frac{\xi(z_2)}{\xi(z_2+1)} \\
\sigma\left(\frac{\pi}{3}\right) & \frac{1}{-z_1-2z_2-1}\frac{1}{z_1+z_2-1}\cdot\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}\frac{\xi(z_2)}{\xi(z_2+1)} \\
\sigma\left(\frac{2\pi}{3}\right) & \frac{1}{-2z_1-z_2-1}\frac{1}{z_1-1}\cdot\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}\frac{\xi(z_2)}{\xi(z_2+1)} \\
\sigma(\pi) & \frac{1}{-z_1+z_2-1}\frac{1}{-z_2-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\frac{\xi(z_1)}{\xi(z_1+1)}\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)}\frac{\xi(z_1+2z_2)}{\xi(z_1+2z_2+1)}\frac{\xi(z_2)}{\xi(z_2+1)} \\
\sigma\left(\frac{4\pi}{3}\right) & \frac{1}{z_1+2z_2-1}\frac{1}{-z_1-z_2-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\frac{\xi(z_1)}{\xi(z_1+1)}\frac{\xi(2z_1+z_2)}{\xi(2z_1+z_2+1)}\frac{\xi(z_1+z_2)}{\xi(z_1+z_2+1)} \\
\sigma\left(\frac{5\pi}{3}\right) & \frac{1}{2z_1+z_2-1}\frac{1}{-z_1-1}\cdot\frac{\xi(z_1-z_2)}{\xi(z_1-z_2+1)}\frac{\xi(z_1)}{\xi(z_1+1)}.
\end{array}
$$

By taking summation for all terms appeared, we then obtain the period $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$ for G_2 over $\mathbb{Q}$.

3.2 Zetas for G_2 over $\mathbb{Q}$

Motivated by our study of high-rank zeta functions in [23, 25], and a new type of zetas for $SL(n)$ and $Sp(2n)$ in [26, 27], as described in the introduction, we can obtain two zeta functions for G_2 over $\mathbb{Q}$ from the period $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$, by taking residues along singular hyperplanes corresponding to (two) maximal parabolic subgroups.

3.2.1 *The zeta for G_2/P_{long}*

Recall that P_{long} corresponds to $\{\alpha\}=\Delta_0\backslash\{\beta\}$. Consequently, from the period $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$ of G_2 over $\mathbb{Q}$, in order to introduce a zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s)$ for G_2/P_{long}, we first take the residue along with the singular hyperplane $z_1-z_2=1$ of $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$, corresponding to $\langle\lambda-\rho,\alpha^\vee\rangle=0$, and set $z_2=s$ (then $z_1=1+s$ and $z_2-z_1=-1,\ 2z_1+z_2=3s+2,\ z_1+z_2=2s+1,\ z_1+2z_2=3s+1,\ z_1-1=s, z_2+1=s+1$). In such a way, we get the following (single variable) period $\omega_{\mathbb{Q}}^{G_2/\text{Plong}}(s)$ associated with G_2/P_{long}

over $\mathbb{Q}$:

$$
\begin{aligned}
\omega_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}(s) := & \frac{1}{s-1} + \frac{1}{-2}\frac{1}{s}\cdot\frac{1}{\xi(2)} + 0 + \frac{1}{-3s-3}\frac{1}{2s}\cdot\frac{1}{\xi(2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(3s+2)}{\xi(3s+3)} \\
& + \frac{1}{-3s-2}\frac{1}{s-1}\cdot\frac{1}{\xi(2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(3s+2)}{\xi(3s+3)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(3s+1)}{\xi(3s+2)} \\
& + \frac{1}{-s-2}\cdot\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(3s+2)}{\xi(3s+3)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(3s+1)}{\xi(3s+2)}\frac{\xi(s)}{\xi(s+1)} + 0 + 0 + 0 \\
& + \frac{1}{-2}\frac{1}{-s-1}\cdot\frac{1}{\xi(2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(3s+2)}{\xi(3s+3)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(3s+1)}{\xi(3s+2)}\frac{\xi(s)}{\xi(s+1)} \\
& + \frac{1}{3s}\frac{1}{-2s-2}\cdot\frac{1}{\xi(2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(3s+2)}{\xi(3s+3)}\frac{\xi(2s+1)}{\xi(2s+2)} \\
& + \frac{1}{3s+1}\frac{1}{-s-2}\cdot\frac{1}{\xi(2)}\frac{\xi(s+1)}{\xi(s+2)}.
\end{aligned}
$$

Multiplying with $\xi(2)\cdot\xi(s+2)\xi(2s+2)\xi(3s+3)$, we then get

$$
\begin{aligned}
\xi_{\mathbb{Q},o}^{G_2/P_{\mathrm{long}}}(s) = & \frac{1}{s-1}\xi(2)\cdot\xi(s+2)\xi(2s+2)\xi(3s+3) \\
& - \frac{1}{s+2}\xi(2)\cdot\xi(s)\xi(2s+1)\xi(3s+1) \\
& - \frac{1}{2s}\cdot\xi(s+2)\xi(2s+2)\xi(3s+3) \\
& + \frac{1}{2(s+1)}\cdot\xi(s)\xi(2s+1)\xi(3s+1) \\
& - \frac{1}{3s+3}\frac{1}{2s}\cdot\xi(s+1)\xi(2s+2)\xi(3s+2) \\
& - \frac{1}{3s}\frac{1}{2s+2}\cdot\xi(s+1)\xi(2s+1)\xi(3s+2) \\
& - \frac{1}{3s+2}\frac{1}{s-1}\cdot\xi(s+1)\xi(2s+1)\xi(3s+1) \\
& - \frac{1}{3s+1}\frac{1}{s+2}\cdot\xi(s+1)\xi(2s+2)\xi(3s+3).
\end{aligned}
$$

One checks easily the functional equation $\xi_{\mathbb{Q},o}^{G_2/P_{\mathrm{long}}}(-1-s) = \xi_{\mathbb{Q},o}^{G_2/P_{\mathrm{long}}}(s)$. Define the first zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}(s)$ by normalizing $\xi_{\mathbb{Q},o}^{G_2/P_{\mathrm{long}}}(s)$ with a shift

$$
\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}(s) := \xi_{\mathbb{Q},o}^{G_2/P_{\mathrm{long}}}(s-1).
$$

Then we have the following

Definition & Proposition 1. The zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s)$ for (G_2, P_{long}) over $\mathbb{Q}$ given by

$$\begin{aligned}
\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s) :=& \frac{1}{s-2}\xi(2)\cdot\xi(s+1)\xi(2s)\xi(3s)\\
&-\frac{1}{s+1}\xi(2)\cdot\xi(s-1)\xi(2s-1)\xi(3s-2)\\
&-\frac{1}{2s-2}\cdot\xi(s+1)\xi(2s)\xi(3s)\\
&+\frac{1}{2s}\cdot\xi(s-1)\xi(2s-1)\xi(3s-2)\\
&-\frac{1}{(3s)(2s-2)}\cdot\xi(s)\xi(2s)\xi(3s-1)\\
&-\frac{1}{(3s-1)(s-2)}\cdot\xi(s)\xi(2s-1)\xi(3s-2)\\
&-\frac{1}{(3s-3)(2s)}\cdot\xi(s)\xi(2s-1)\xi(3s-1)\\
&-\frac{1}{(3s-2)(s+1)}\cdot\xi(s)\xi(2s)\xi(3s),
\end{aligned}$$

satisfies the standard functional equation

$$\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(1-s) = \xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s).$$

All poles of $\xi_{\mathbb{Q}}^{G_2/P_{\text{long}}}(s)$ are two simple poles $s=-1,2$ and two double poles $s=0,1$. □

3.2.2 *The zeta for G_2/P_{short}*

In parallel, recall that P_{short} corresponds to $\{\beta\} = \Delta_0\backslash\{\alpha\}$. Consequently, from the period $\omega_{\mathbb{Q}}^{G_2}(z_1,z_2)$ of G_2 over $\mathbb{Q}$, in order to introduce a zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\text{short}}}(s)$ for G_2/P_{short}, take the residue along $z_2=1$, corresponding to $\langle\lambda-\rho,\beta^\vee\rangle=0$, and set $z_1=s$. Then we get, accordingly, for the period $\omega_{\mathbb{Q}}^{G_2/P_{\text{short}}}$ the following contributions:

$$\begin{aligned}
\xi_{\mathbb{Q},o}^{G_2/P_{\text{short}}}(s) := \operatorname{Res}_{\langle\lambda+\rho_0,\beta^\vee\rangle=0}\omega_{\mathbb{Q}}^{G_2/P_2}(z_1,z_2) :=& \frac{1}{s-2}+0+\frac{1}{s+1}\frac{1}{-2}\cdot\frac{1}{\xi(2)}+0\\
&+\frac{1}{-s-3}\cdot\frac{\xi(s-1)}{\xi(s)}\frac{\xi(s)}{\xi(s+1)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(s+2)}{\xi(s+3)}\\
&+\frac{1}{s-2}\frac{1}{-s-1}\cdot\frac{\xi(s)}{\xi(s+1)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(s+2)}{\xi(s+3)}\frac{1}{\xi(2)}
\end{aligned}$$

$$+\frac{1}{2s}\frac{1}{-s-2}\cdot\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(s+2)}{\xi(s+3)}\frac{1}{\xi(2)}$$
$$+\frac{1}{-s-3}\frac{1}{s}\cdot\frac{\xi(s+2)}{\xi(s+3)}\frac{1}{\xi(2)}$$
$$+\frac{1}{-s}\frac{1}{-2}\cdot\frac{\xi(s-1)}{\xi(s)}\frac{\xi(s)}{\xi(s+1)}\frac{\xi(2s+1)}{\xi(2s+2)}\frac{\xi(s+1)}{\xi(s+2)}\frac{\xi(s+2)}{\xi(s+3)}\frac{1}{\xi(2)}+0+0.$$

Multiplying with $\xi(2)\cdot\xi(s+3)\xi(2s+2)$, and shifting from s to $s-1$, we then arrive at the second zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ for $(G_2, P_{\rm short})$ over $\mathbb{Q}$.

Definition & Proposition 2. The zeta function $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ for $(G_2, P_{\rm short})$ over $\mathbb{Q}$ given by

$$\begin{aligned}\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s) =&\frac{1}{s-3}\xi(2)\cdot\xi(s+2)\xi(2s)\\
&-\frac{1}{s+2}\xi(2)\cdot\xi(s-2)\xi(2s-1)\\
&+\frac{1}{2s-2}\cdot\xi(s-2)\xi(2s-1)\\
&-\frac{1}{2s}\cdot\xi(s+2)\xi(2s)\\
&-\frac{1}{s(s-3)}\cdot\xi(s-1)\xi(2s-1)\\
&-\frac{1}{(s-1)(s+2)}\cdot\xi(s+1)\xi(2s)\\
&-\frac{1}{(2s-2)(s+1)}\cdot\xi(s)\xi(2s)\\
&-\frac{1}{(2s)(s-2)}\cdot\xi(s)\xi(2s-1),\end{aligned}$$

satisfies the standard functional equation

$$\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(1-s) = \xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s).$$

All poles of $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ are four simple poles $s=-2,0,1,3$. □

We expect that $\xi_{\mathbb{Q}}^{G_2/P_{\rm long}}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ satisfy the RH. For this, we have the following.

Theorem 3 (Riemann Hypothesis$_{\mathbb{Q}}^{G_2/P}$).

All zeros of $\xi_{\mathbb{Q}}^{G_2/P_{\rm long}}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P_{\rm short}}(s)$ lie on thecentral line $\mathrm{Re}(s)=1/2$.

□

Remark. Zetas $\xi_{\mathbb{Q}}^{G_2/P}(s)$ are special cases of a more general construction. In [26, 27], we are able to define zeta functions $\xi_{\mathbb{Q}}^{G/P}(s)$ associated with classical semisimple groups G and their maximal parabolic subgroups P. In particular, the conjectural standard functional equation and the RH have been checked for $G = SL(2),\ SL(3),\ Sp(4)$ ([26, 27] for the FE, [11, 17, 18] for the RH). Also, numerical calculations made by MS give supportive evidences for the RH when $G = SL(4)$ or $SL(5)$. □

4 Proof of the RH for G_2: Preliminaries

To prove the RH for G_2, we prepare several auxiliary entire functions. First, we define

$$Z_1(s) := 12s^3(s-1)^3 \cdot (s+1)(3s-1)(2s-1)(3s-2)(s-2) \cdot \xi_{\mathbb{Q}}^{G_2/P_1}(s)$$

and

$$Z_2(s) := 4s^2(s-1)^2 \cdot (s+2)(s+1)(2s-1)(s-2)(s-3) \cdot \xi_{\mathbb{Q}}^{G_2/P_2}(s).$$

(Here, we use the notation $P_{\text{long}} = P_\beta = P_1$ and $P_{\text{short}} = P_\alpha = P_2$.) Then $Z_1(s)$ and $Z_2(s)$ are entire functions by the results of Section 3. We have

$$\begin{aligned} Z_1(s) = (s-1)\chi(2s)[&(s-1)(3s-2)(As-A+1)\chi(s+1)\chi(3s) \\ &- (s+1)(s-2)\chi(s)\chi(3s-1) - 2(s-1)(s-2)\chi(s)\chi(3s)] \\ &- s\,\chi(2s-1)[s(3s-1)(As-1)\chi(s-1)\chi(3s-2) \\ &+ (s+1)(s-2)\chi(s)\chi(3s-1) + 2s(s+1)\chi(s)\chi(3s-2)], \end{aligned}$$

and

$$\begin{aligned} Z_2(s) = (s-2)\chi(2s)[&(As+3)(s-1)^2\chi(s+2) \\ &- 2(s-1)(s-3)\chi(s+1) - (s+2)(s-3)\chi(s)] \\ &- (s+1)\chi(2s-1)[(As-A-3)s^2\chi(s-2) \\ &+ 2s(s+2)\,\chi(s-1) + (s+2)(s-3)\chi(s)], \end{aligned}$$

where

$$A = 2\xi(2) - 1 = \pi/3 - 1 > 0,$$
$$\chi(s) = s(s-1)\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

We find that

- $Z_1(s)$ has real zeros at $s = 0,\ 1/3,\ 2/3,\ 1$ and $s = 1/2$, because all poles of $\xi_{\mathbb{Q}}^{G_2/P_1}(s)$ are two simple poles $s = -1, 2$ and two double poles $s = 0, 1$.
- $Z_2(s)$ has real zeros at $s = -1,\ 0,\ 1,\ 2$ and $s = 1/2$, because all poles of $\xi_{\mathbb{Q}}^{G_2/P_2}(s)$ are four simple poles $s = -2, 0, 1, 3$.

Hence, the following two theorems are equivalent to the RH of $\xi_{\mathbb{Q}}^{G_2/P_1}(s)$ and $\xi_{\mathbb{Q}}^{G_2/P_2}(s)$, respectively.

Theorem 4. All zeros of $Z_1(s)$ lie on the line $\mathrm{Re}(s) = 1/2$ except for four simple zeros $s = 0,\ 1/3,\ 2/3,\ 1$. □

Theorem 5. All zeros of $Z_2(s)$ lie on the line $\mathrm{Re}(s) = 1/2$ except for four simple zeros $s = -1,\ 0,\ 1,\ 2$. □

Now we define

$$\begin{aligned}\widetilde{f}_1(s) &= (s-1)(3s-2)(As-A+1)\chi(s+1)\chi(3s)\\ &\quad - (s+1)(s-2)\chi(s)\chi(3s-1) - 2(s-1)(s-2)\chi(s)\chi(3s),\end{aligned}$$

$$\widetilde{f}_2(s) = (As+3)(s-1)^2\chi(s+2) - 2(s-1)(s-3)\chi(s+1) - (s+2)(s-3)\chi(s).$$

and

$$f_1(s) = (s-1)\,\widetilde{f}_1(s), \quad f_2(s) = (s-2)\,\widetilde{f}_2(s). \tag{1}$$

Then

$$\begin{aligned}Z_1(s) &= \chi(2s)\,f_1(s) - \chi(2s-1)\,f_1(1-s),\\ Z_2(s) &= \chi(2s)\,f_2(s) - \chi(2s-1)\,f_2(1-s).\end{aligned}$$

The proofs of Theorem 4 and Theorem 5 are divided into two steps. First, we prove that all zeros of $f_i(s)$ lie in a vertical strip $\sigma_0 < \mathrm{Re}(s) < 0$ except for finitely many exceptional zeros (Section 5). Then we obtain a nice product formula of $f_i(s)$ by a variant of Lemma 3 in [17] (Lemma 2 in Section 5; it will be proved in Section 7). Second, by using the product formula of $f_i(s)$, we prove that all zeros of $Z_i(s)$ lie on the line $\mathrm{Re}(s) = 1/2$ except for two simple zeros (Section 6). In this process, we use the result of Lagarias [10] concerning the explicit upper bound for the difference of the imaginary parts of the zeros of the Riemann zeta function. See also [18].

Before the proof, we recall the following result.

Lemma 1 [11]. Let $\xi(s)$ be the completed Riemann zeta function and $\chi(s) = s(s-1)\xi(s)$. Then we have

$$\left|\frac{\chi(2s-1)}{\chi(2s)}\right| < 1 \quad \text{for} \quad \mathrm{Re}(s) > \frac{1}{2}. \tag{2}$$

□

5 Proof of the RH for G_2: First Step

The aim of this section is to prove the following proposition.

Proposition 3. Let $f_1(s)$ and $f_2(s)$ be functions defined in (1). Then $f_i(s)$ $(i = 1, 2)$ has the product formula

$$f_i(s) = C_i\, s^{m_i}\, e^{B_i' s} \left(1 - \frac{s}{\beta_{0,i}}\right)\left(1 - \frac{s}{\rho_{0,i}}\right)\left(1 - \frac{s}{\overline{\rho}_{0,i}}\right) \cdot \Pi_i(s) \quad (B_i' \geq 0),$$

where $C_1 = f_1'(0)$, $C_2 = f_2(0)$, $\beta_{0,1} = 1$, $\beta_{0,2} = 2$, $m_1 = 1$, $m_2 = 0$, $\rho_{0,i}$ $(i = 1, 2)$ are a complex zero of $f_i(s)$ with $\Re(\rho_{0,i}) > 1/2$ and

$$\Pi_i(s) = \prod_{\substack{\beta_i < 1/2 \\ 0 \neq \beta_i \in \mathbb{R}}} \left(1 - \frac{s}{\beta_i}\right) \prod_{\substack{\beta_i < 1/2 \\ \gamma_i > 0}} \left[\left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\overline{\rho_i}}\right)\right] \quad (\rho_i = \beta_i + \sqrt{-1}\,\gamma_i).$$

Here β_i are at most finitely many real zeros of $f_i(s)$ and $\rho_i = \beta_i + i\gamma_i$ are other complex zeros of $f_i(s)$. The product $\Pi_i(s)$ converges absolutely on any compact subset of $\mathbb{C}$ if we take the product with the bracket. □

To prove the proposition, we prepare the following lemma.

Lemma 2. Let $F(s)$ be an entire function of genus zero or one. Suppose that

(i) $F(s)$ is real on the real axis,

(ii) there exists $\sigma_0 > 0$ such that all zeros of $F(s)$ lie in the vertical strip

$$\sigma_0 < \mathrm{Re}(s) < 1/2$$

except for finitely many zeros,

(iii) the zeros of $F(s)$ are finitely many in the right-half plane $\mathrm{Re}(s) \geq 1/2$,

(iv) there exists $C > 0$ such that

$$N(T) \leq C\,T \log T \quad \text{as} \quad T \to \infty, \tag{3}$$

where $N(T)$ is the number of zeros of $F(s)$ satisfying $0 \leq \Im(\rho) < T$, and

(v) $F(1-\sigma)/F(\sigma) > 0$ for large $\sigma > 0$ and

$$F(1-\sigma)/F(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty. \tag{4}$$

Then $F(s)$ has the product formula

$$F(s) = C s^m e^{B's} \prod_{0 \neq \rho \in \mathbb{R}} \left(1 - \frac{s}{\rho}\right) \prod_{\Im(\rho) > 0} \left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right)\right]$$

with

$$B' \geq 0,$$

The product in the right-hand side converges absolutely on every compact set if we take the product with the bracket. □

The most important part of this lemma is nonnegativity of B'. We will prove Lemma 2 in Section 7.

If $f_i(s)$ satisfies all conditions in Lemma 2, then we obtain Proposition 3 by applying Lemma 2 to $f_i(s)$. Condition (i) is trivial for $f_i(s)$. Under condition (ii), (iv) is easily proved by a standard argument by using well-known estimate $|\chi(s)| \leq \exp(C|s| \log |s|)$ and Jensen's formula (see Section 4.1 of [17], for example). On the other hand, we have

$$f_1(0) = 0, \quad f_1(s) = f_1'(0)\,s + O(s^2), \quad f_1'(0) \simeq -2.176 \neq 0,$$
$$f_2(0) \simeq -6.283 \neq 0.$$

Hence, it remains to prove (ii), (iii), and (v) for $f_i(s)$.

5.1 Proof of (v)

5.1.1 *Case of $f_1(s)$*

First, we see that $f_1(1-\sigma)/f_1(\sigma)$ is positive for sufficiently large $\sigma > 0$. Using the functional equation of $\chi(s)$, we have

$$f_1(1-\sigma) = \sigma^2(3\sigma-1)(A\sigma-1)\chi(\sigma-1)\chi(3\sigma-2) \\ + \sigma(\sigma-2)(\sigma+1)\chi(\sigma)\chi(3\sigma-1) + 2\sigma^2(\sigma+1)\chi(\sigma)\chi(3\sigma-2),$$

$$f_1(\sigma) = (\sigma-1)^2(3\sigma-2)(A\sigma-A+1)\chi(\sigma+1)\chi(3\sigma) \\ - (\sigma+1)(\sigma-2)(\sigma-1)\chi(\sigma)\chi(3\sigma-1) - 2(\sigma-1)^2(\sigma-2)\chi(\sigma)\chi(3\sigma).$$

Clearly, the numerator is positive for large $\sigma > 0$. The denominator is also positive for large $\sigma > 0$, since $A > 0$ and

$$|\chi(\sigma)/\chi(\sigma+1)| < 1 \quad (\sigma > 0), \quad |\chi(3\sigma-1)/\chi(3\sigma)| < 1 \quad (\sigma > 1/3) \tag{5}$$

by replacing $2s-1$ by σ or $3\sigma-1$ in (2). Now we prove (4). We have

$$\frac{f_1(1-\sigma)}{f_1(\sigma)} = \frac{\sigma^2(3\sigma-1)(A\sigma-1)}{(\sigma-1)^2(3\sigma-2)(A\sigma-A+1)} \cdot \frac{\chi(\sigma-1)\chi(3\sigma-2)}{\chi(\sigma+1)\chi(3\sigma)} \cdot \frac{1+g(\sigma)}{1-h(\sigma)} \\ = (1+O(\sigma^{-1})) \cdot \frac{\chi(\sigma-1)\chi(3\sigma-2)}{\chi(\sigma+1)\chi(3\sigma)} \cdot \frac{1+g(\sigma)}{1-h(\sigma)},$$

where

$$g(\sigma) = \frac{(\sigma-2)(\sigma+1)}{\sigma(3\sigma-1)(A\sigma-1)} \cdot \frac{\chi(\sigma)\chi(3\sigma-1)}{\chi(\sigma-1)\chi(3\sigma-2)} + \frac{2(\sigma+1)}{(3\sigma-1)(A\sigma-1)} \cdot \frac{\chi(\sigma)}{\chi(\sigma-1)},$$

and

$$h(\sigma) = \frac{(\sigma+1)(\sigma-2)}{(\sigma-1)(3\sigma-2)(A\sigma-A+1)} \cdot \frac{\chi(\sigma)\chi(3\sigma-1)}{\chi(\sigma+1)\chi(3\sigma)} + \frac{2(\sigma-2)}{(3\sigma-2)(A\sigma-A+1)} \cdot \frac{\chi(\sigma)}{\chi(\sigma+1)}.$$

We have

$$\frac{\chi(\sigma-1)\chi(3\sigma-2)}{\chi(\sigma+1)\chi(3\sigma)} = (1+O(\sigma^{-1}))\frac{\xi(\sigma-1)\xi(3\sigma-2)}{\xi(\sigma+1)\xi(3\sigma)} \\ = (1+O(\sigma^{-1})) \cdot \pi^2 \cdot \frac{\Gamma((\sigma-1)/2)\Gamma((3\sigma-2)/2)}{\Gamma((\sigma+1)/2)\Gamma(3\sigma/2)} \frac{\zeta(\sigma-1)\zeta(3\sigma-2)}{\zeta(\sigma+1)\zeta(3\sigma)} \\ = (1+O(\sigma^{-1})) \cdot \frac{\Gamma((\sigma-1)/2)}{\Gamma((\sigma+1)/2)(3\sigma-2)} \cdot O(1)$$

for large $\sigma > 0$. Using the Stirling formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z (1 + O_\varepsilon(|z|^{-1})) \quad (|z| \geq 1,\ |\arg z| < \pi - \varepsilon),$$

we obtain

$$\frac{\chi(\sigma-1)\chi(3\sigma-2)}{\chi(\sigma+1)\chi(3\sigma)} = O(\sigma^{-2}) \quad \text{as} \quad \sigma \to +\infty. \tag{6}$$

On the other hand, by using the Stirling formula again, we have

$$g(\sigma) = O(1) + O(\sigma^{-1/2}) = O(1) \quad \text{as} \quad \sigma \to +\infty. \tag{7}$$

For $h(\sigma)$, by using (5), we have

$$h(\sigma) = O(\sigma^{-1}) \quad \text{as} \quad \sigma \to +\infty. \tag{8}$$

From (6), (7), and (8), we obtain

$$\frac{f_1(1-\sigma)}{f_1(\sigma)} = O(\sigma^{-2}) \quad \text{as} \quad \sigma \to +\infty.$$

This shows condition (v) for $f_1(s)$. □

5.1.2 *Case of* $f_2(s)$

First, we see that $f_2(1-\sigma)/f_2(\sigma)$ is positive for sufficiently large $\sigma > 0$. Using the functional equation of $\chi(s)$, we have

$$\begin{aligned} f_2(1-\sigma) &= \sigma^2(\sigma+1)(A\sigma - A - 3)\chi(\sigma-2) \\ &\quad + 2\sigma(\sigma+1)(\sigma+2)\chi(\sigma-1) + (\sigma-3)(\sigma+1)(\sigma+2)\chi(\sigma), \\ f_2(\sigma) &= (\sigma-1)^2(\sigma-2)(A\sigma+3)\chi(\sigma+2) \\ &\quad - 2(\sigma-1)(\sigma-2)(\sigma-3)\chi(\sigma+1) - (\sigma+2)(\sigma-2)(\sigma-3)\chi(\sigma). \end{aligned}$$

Clearly, the numerator is positive for large $\sigma > 0$. The denominator is also positive for large $\sigma > 0$, since $A > 0$ and

$$|\chi(\sigma+1)/\chi(\sigma+2)| < 1 \quad (\sigma > -1), \quad |\chi(\sigma)/\chi(\sigma+2)| < 1 \quad (\sigma > 0) \tag{9}$$

by (2) and $\chi(\sigma)/\chi(\sigma+2) = (\chi(\sigma+1)/\chi(\sigma+2)) \cdot (\chi(\sigma)/\chi(\sigma+1))$. We have

$$\frac{f_2(1-\sigma)}{f_2(\sigma)} = \frac{\sigma^2(\sigma+1)(A\sigma-A-3)}{(\sigma-1)^2(\sigma-2)(A\sigma+3)} \cdot \frac{\chi(\sigma-2)}{\chi(\sigma+2)} \cdot \frac{1+g(\sigma)}{1-h(\sigma)}$$
$$= (1+O(\sigma^{-1})) \cdot \frac{\chi(\sigma-2)}{\chi(\sigma+2)} \cdot \frac{1+g(\sigma)}{1-h(\sigma)},$$

where

$$g(\sigma) = \frac{2(\sigma+2)}{\sigma(A\sigma-A-3)} \cdot \frac{\chi(\sigma-1)}{\chi(\sigma-2)} + \frac{(\sigma+2)(\sigma-3)}{\sigma^2(A\sigma-A-3)} \cdot \frac{\chi(\sigma)}{\chi(\sigma-2)},$$

and

$$h(\sigma) = \frac{2(\sigma-3)}{(\sigma-1)(A\sigma+3)} \cdot \frac{\chi(\sigma+1)}{\chi(\sigma+2)} + \frac{(\sigma+2)(\sigma-3)}{(\sigma-1)^2(A\sigma+3)} \cdot \frac{\chi(\sigma)}{\chi(\sigma+2)}.$$

Using the Stirling formula, we obtain

$$\frac{\chi(\sigma-2)}{\chi(\sigma+2)} = O(\sigma^{-2}) \quad \text{as} \quad \sigma \to +\infty. \tag{10}$$

and

$$g(\sigma) = O(\sigma^{-1/2}) + O(1) = O(1) \quad \text{as} \quad \sigma \to +\infty. \tag{11}$$

Using (9), we have

$$h(\sigma) = O(\sigma^{-1}) \quad \text{as} \quad \sigma \to +\infty. \tag{12}$$

From (10), (11), and (12), we obtain

$$\frac{f_2(1-\sigma)}{f_2(\sigma)} = O(\sigma^{-2}) \quad \text{as} \quad \sigma \to +\infty.$$

This shows condition (v) for $f_2(s)$. □

5.2 Proof of (ii) and (iii)

Lemma 3. The entire function $f_1(s)$ has no zero in certain left-half plane $\mathrm{Re}(s) < \sigma_1$. □

Proof. Assume $\sigma = \mathrm{Re}(s) < 0$. We have

$$f_1(s) = -(s+1)(s-1)(s-2)\chi(s)\chi(3s-1)[1 + R_1(s) - R_2(s)],$$

where

$$R_1(s) = 2\,\frac{s-1}{s+1}\cdot\frac{\chi(3s)}{\chi(3s-1)}$$
$$R_2(s) = \frac{(s-1)(3s-2)(As-A+1)}{(s+1)(s-2)}\cdot\frac{\chi(s+1)\chi(3s)}{\chi(s)\chi(3s-1)}.$$

Clearly, the factor $(s+1)(s-1)(s-2)\chi(s)\chi(3s-1)$ has no zero in the left-half plane $\mathrm{Re}(s) < -1$. Using the functional equation, we have

$$R_1(s) = 2\,\frac{s-1}{s+1}\,\frac{\chi(1-3s)}{\chi(2-3s)} = \frac{6s(s-1)}{(3s-2)(s+1)}\,\frac{\xi(1-3s)}{\xi(2-3s)}$$
$$= \frac{6\sqrt{\pi}\,s(s-1)}{(3s-2)(s+1)}\,\frac{\Gamma((1-3s)/2)}{\Gamma((2-3s)/2)}\,\frac{\zeta(1-3s)}{\zeta(2-3s)}.$$

Therefore,

$$|R_1(s)| \le 2\sqrt{\pi}\left|\frac{s(s-1)}{\left(s-\frac{2}{3}\right)(s+1)}\right|\left|\frac{\Gamma((1-3s)/2)}{\Gamma((2-3s)/2)}\right|\zeta(1-3\sigma)\zeta(2-3\sigma).$$

If $\sigma = \mathrm{Re}(s) < 0$, $|\arg((1-3s)/2)| < \pi/2$ and $|\arg(2-3s)/2| < \pi/2$. Hence, we can apply the Stirling formula for $\mathrm{Re}(s) < 0$, and then

$$\left|\frac{\Gamma((1-3s)/2)}{\Gamma((2-3s)/2)}\right| = \sqrt{\frac{2}{3}}\,|s|^{-1/2}(1+O(|s|^{-1})) \quad (\mathrm{Re}(s) < 0).$$

On the other hand,

$$\zeta(1-3\sigma)\zeta(2-3\sigma) \to 1 \quad (\sigma \to -\infty).$$

Therefore,

$$|R_1(s)| \le \sqrt{\frac{8\pi}{3}}\cdot|s|^{-1/2}\cdot(1+O(|s|^{-1})), \tag{13}$$

if $\sigma = \mathrm{Re}(s) < 0$, and $|s|$, $|\sigma|$ are both large.

However, using the functional equation, we have

$$\begin{aligned}
R_2(s) &= \frac{(s-1)(3s-2)(As-A+1)}{(s+1)(s-2)} \cdot \frac{\chi(-s)\chi(1-3s)}{\chi(1-s)\chi(2-3s)} \\
&= \frac{3s(As-A+1)}{(s-2)} \cdot \frac{\xi(-s)\xi(1-3s)}{\xi(1-s)\xi(2-3s)} \\
&= \frac{3\pi s(As-A+1)}{(s-2)} \cdot \frac{\Gamma(-s/2)\Gamma((1-3s)/2)}{\Gamma((1-s)/2)\Gamma((2-3s)/2)} \cdot \frac{\zeta(-s)\zeta(1-3s)}{\zeta(1-s)\zeta(2-3s)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|R_2(s)| \le 3\pi A \left|\frac{s(s-1+A^{-1})}{(s-2)}\right| \left|\frac{\Gamma(-s/2)}{\Gamma((1-s)/2)}\right| \left|\frac{\Gamma((1-3s)/2)}{\Gamma((2-3s)/2)}\right| \\
\times \zeta(-\sigma)\zeta(1-\sigma)\zeta(1-3\sigma)\zeta(2-3\sigma).
\end{aligned}$$

If $\sigma = \mathrm{Re}(s) < 0$, each argument of $-s/2$, $(1-s)/2$, $(1-3s)/2$, and $(2-3s)/2$ is less than $\pi/2$. Hence, we can apply the Stirling formula for $\mathrm{Re}(s) < 0$, and then

$$\left|\frac{\Gamma(-s/2)}{\Gamma((1-s)/2)}\right| = \sqrt{2}\,|s|^{-1/2}(1 + O(|s|^{-1})) \quad (\mathrm{Re}(s) < 0).$$

$$\left|\frac{\Gamma((1-3s)/2)}{\Gamma((2-3s)/2)}\right| = \sqrt{\frac{2}{3}}\,|s|^{-1/2}(1 + O(|s|^{-1})) \quad (\mathrm{Re}(s) < 0).$$

We have

$$\zeta(-\sigma)\zeta(1-\sigma)\zeta(1-3\sigma)\zeta(2-3\sigma) \to 1 \quad (\sigma \to -\infty).$$

Therefore,

$$|R_2(s)| \le 2\sqrt{3}\pi A \cdot (1 + O(|s|^{-1})), \tag{14}$$

if $\sigma = \mathrm{Re}(s) < 0$, and $|s|$, $-\sigma$ are both large. Here,

$$2\sqrt{3}\pi A = 0.51364\ldots$$

Hence, (13) and (14) imply Lemma 3. ■

Lemma 4. The entire function $f_2(s)$ has no zero in certain left-half plane $\mathrm{Re}(s) < \sigma_2$. $\square$

Proof. Assume $\sigma = \mathrm{Re}(s) < 0$. We have

$$f_2(s) = -(s+2)(s-2)(s-3)\chi(s)[1 + R_1(s) - R_2(s)],$$

where

$$R_1(s) = 2\frac{(s-1)}{(s+2)} \cdot \frac{\chi(s+1)}{\chi(s)}, \quad R_2(s) = \frac{(As+3)(s-1)^2}{(s+2)(s-3)} \cdot \frac{\chi(s+2)}{\chi(s)}.$$

Clearly, the factor $(s+2)(s-2)(s-3)\chi(s)$ has no zero in the left-half plane $\mathrm{Re}(s) < -2$. Using the functional equation, we have

$$R_1(s) = 2\frac{(s-1)}{(s+2)} \cdot \frac{\chi(-s)}{\chi(1-s)} = 2\frac{s+1}{s+2}\frac{\xi(-s)}{\xi(1-s)} = 2\sqrt{\pi}\frac{s+1}{s+2}\frac{\Gamma(-s/2)}{\Gamma((1-s)/2)}\frac{\zeta(-s)}{\zeta(1-s)}.$$

Therefore,

$$|R_1(s)| \le 2\sqrt{\pi}\left|\frac{s+1}{s+2}\right|\left|\frac{\Gamma(-s/2)}{\Gamma((1-s)/2)}\right|\zeta(-\sigma)\zeta(1-\sigma).$$

If $\sigma = \mathrm{Re}(s) < 0$, $|\arg(-s/2)| < \pi/2$ and $|\arg(1-s)/2| < \pi/2$. Hence, we can apply the Stirling formula for $\mathrm{Re}(s) < 0$, and then

$$\left|\frac{\Gamma(-s/2)}{\Gamma((1-s)/2)}\right| = \sqrt{2}|s|^{-1/2}(1 + O(|s|^{-1})) \quad (\mathrm{Re}(s) < 0).$$

On the other hand, $\zeta(-\sigma)\zeta(1-\sigma) \to 1$ as $\sigma \to -\infty$. Therefore,

$$|R_1(s)| \le \sqrt{8\pi} \cdot |s|^{-1/2} \cdot (1 + O(|s|^{-1})), \tag{15}$$

if $\sigma = \mathrm{Re}(s) < 0$, and $|s|$, $|\sigma|$ are both large. However, using the functional equation, we have

$$\begin{aligned} R_2(s) &= \frac{(As+3)(s-1)^2}{(s+2)(s-3)} \cdot \frac{\chi(-1-s)}{\chi(1-s)} = \frac{(As+3)(s-1)(s+1)}{s(s-3)} \cdot \frac{\xi(-1-s)}{\xi(1-s)} \\ &= \pi\frac{(As+3)(s-1)(s+1)}{s(s-3)} \cdot \frac{\Gamma((-1-s)/2)}{\Gamma((1-s)/2)} \cdot \frac{\zeta(-1-s)}{\zeta(1-s)}. \end{aligned}$$

Therefore,

$$|R_2(s)| \le \pi A\left|\frac{(s+3A^{-1})(s-1)(s+1)}{s(s-3)}\right|\left|\frac{\Gamma((-1-s)/2)}{\Gamma((1-s)/2)}\right|\zeta(-1-\sigma)\zeta(1-\sigma).$$

If $\sigma = \mathrm{Re}(s) < 0$, both arguments of $(-1-s)/2$, $(1-s)/2$ are less than $\pi/2$. Hence, we can apply the Stirling formula for $\mathrm{Re}(s) < 0$, and then

$$\left|\frac{\Gamma((-1-s)/2)}{\Gamma((1-s)/2)}\right| = 2\,|s|^{-1}(1 + O(|s|^{-1})) \quad (\mathrm{Re}(s) < 0).$$

We have $\zeta(-1-\sigma)\zeta(1-\sigma) \to 1$ as $\sigma \to -\infty$. Therefore,

$$|R_2(s)| \leq 2\pi A \cdot (1 + O(|s|^{-1})), \tag{16}$$

if $\sigma = \mathrm{Re}(s) < 0$, and $|s|$, $|\sigma|$ are both large. Here,

$$2\pi A = 0.29655\ldots$$

Hence, (15) and (16) imply Lemma 4. ■

Lemma 5. The entire function $f_1(s)$ has only finitely many zeros in the right-half plane $\mathrm{Re}(s) > 1/3$. In particular, the number of zeros of $f_1(s)$ in $\mathrm{Re}(s) \geq 1/2$ is finite. □

Proof. We have

$$f_1(s) = (s-1)^2(3s-2)(As - A + 1)\chi(s+1)\chi(3s)[1 - Q_1(s) - Q_2(s)], \tag{17}$$

where

$$Q_1(s) = \frac{(s+1)(s-2)}{(s-1)(3s-2)(As-A+1)} \cdot \frac{\chi(s)\chi(3s-1)}{\chi(s+1)\chi(3s)},$$

$$Q_2(s) = \frac{2(s-2)}{(3s-2)(As-A+1)} \cdot \frac{\chi(s)}{\chi(s+1)}.$$

The factor $(s-1)^2(3s-2)(As-A+1)\chi(s+1)\chi(3s)$ has no zero in $\mathrm{Re}(s) > 1/3$ except for $s = 2/3$ and $s = 1$. Replacing $2s-1$ by $3s-1$ or s in (2), we obtain

$$\left|\frac{\chi(3s-1)}{\chi(3s)}\right| < 1 \quad \left(\mathrm{Re}(s) > \frac{1}{3}\right), \quad \left|\frac{\chi(s)}{\chi(s+1)}\right| < 1 \quad (\mathrm{Re}(s) > 0). \tag{18}$$

Let D_1 be the region

$$D_1 := \left\{ s \in \mathbb{C} \;\middle|\; \mathrm{Re}(s) \geq \frac{1}{3},\; \left|\frac{(s+1)(s-2)}{(s-1)(3s-2)(As-A+1)}\right| + \left|\frac{2(s-2)}{(3s-2)(As-A+1)}\right| \geq 1 \right\}.$$

Then $f_1(s) \neq 0$ if $s \notin D_1$ and $\mathrm{Re}(s) \geq 1/3$, because of (17) and (18). The region D_1 is bounded, since

$$\left|\frac{(s+1)(s-2)}{(s-1)(3s-2)(As-A+1)}\right| + \left|\frac{2(s-2)}{(3s-2)(As-A+1)}\right| < 1$$

for large $|s|$. Hence, the number of zeros of $f_1(s)$ in $\mathrm{Re}(s) \geq 1/3$ is finite. ■

Lemma 6. The entire function $f_2(s)$ has only finitely many zeros in the right-half plane $\mathrm{Re}(s) > 0$. In particular, the number of zeros of $f_2(s)$ in $\mathrm{Re}(s) \geq 1/2$ is finite. □

Proof. We have

$$f_2(s) = (As+3)(s-1)^2(s-2)\chi(s+2)[1 - Q_1(s) - Q_2(s)], \tag{19}$$

where

$$Q_1(s) = \frac{2(s-3)}{(As+3)(s-1)} \cdot \frac{\chi(s+1)}{\chi(s+2)}$$

$$Q_2(s) = \frac{(s+2)(s-3)}{(As+3)(s-1)^2} \cdot \frac{\chi(s)}{\chi(s+2)}.$$

The factor $(As+3)(s-1)^2(s-2)\chi(s+2)$ has no zero in $\mathrm{Re}(s) > 0$ except for $s = 1$ and $s = 2$. Replacing $2s-1$ by $s+1$ or s in (2), we obtain

$$\begin{aligned} &\left|\frac{\chi(s+1)}{\chi(s+2)}\right| < 1 \quad \big(\mathrm{Re}(s) > -1\big), \\ &\left|\frac{\chi(s)}{\chi(s+2)}\right| = \left|\frac{\chi(s+1)}{\chi(s+2)}\right|\left|\frac{\chi(s)}{\chi(s+1)}\right| < 1 \quad (\mathrm{Re}(s) > 0). \end{aligned} \tag{20}$$

Let D_2 be the region

$$D_2 := \left\{ s \in \mathbb{C} \;\middle|\; \mathrm{Re}(s) \geq 0,\; \left|\frac{2(s-3)}{(As+3)(s-1)}\right| + \left|\frac{(s+2)(s-3)}{(As+3)(s-1)^2}\right| \geq 1 \right\}.$$

Then $f_2(s) \neq 0$ if $s \notin D_2$ and $\mathrm{Re}(s) \geq 0$, because of (19) and (20). Clearly, the region D_2 is bounded, the number of zeros of $f_2(s)$ in $\mathrm{Re}(s) \geq 0$ is finite. ■

5.3 Proof of Proposition 3

By the results in Sections 5.1 and 5.2, we can apply Lemma 2 to $f_i(s)$ $(i = 1, 2)$. Hence, the proof of Proposition 3 is completed by the following lemmas.

Lemma 7. The number of zeros of $f_1(s)$ in $\mathrm{Re}(s) \geq 1/2$ is just three. One of them is the real zero $s = 1$, and another two zeros are nonreal zeros and conjugate each other. The values of complex zeros are about $s \simeq 0.927 \pm i \cdot 2.09$. □

Lemma 8. The number of zeros of $f_2(s)$ in $\mathrm{Re}(s) \geq 1/2$ is just three. One of them is the real zero $s = 2$, and another two zeros are nonreal zeros and conjugate each other. The values of complex zeros are $s \simeq 1.17 \pm i \cdot 3.43$. □

Proof of Lemma 7 and Lemma 8. The domain $D_i \cap \{\mathrm{Re}(s) \leq 1/2\}$ is contained in the rectangle $R = [1/2, 5] \times [-10, 10]$, where D_i is the region in the proof of Lemma 7 or Lemma 8. Because of the argument principle, the number of zeros of $f(s)$ in R is given by

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f_i'}{f_i}(s) ds.$$

In particular, the value of this integral is an integer. Therefore, we can check that the value of this integral is just three by a computational way (for example, Mathematica, Maple, PARI/GP). Hence, we conclude that $f_i(s)$ has just three zeros in the rectangle R. One of them is trivial real zero of $f_1(s)$ (resp. $f_2(s)$) at $s = 1$ (resp. $s = 2$). By suitable computational way, we find an approximate value of the above two complex zeros of $f_1(s)$ (resp. $f_2(s)$) are $s \simeq 0.927 \pm i \cdot 2.09$ (resp. $s \simeq 1.17 \pm i \cdot 3.43$). ■

6 Proof of the RH for G_2: Second Step

6.1 Proof of Theorem 4 and Theorem 5

We have the following three assertions for $Z_1(s)$.

Proposition 4. $Z_1(s)$ has no zero in the right-half plane $\mathrm{Re}(s) \geq 20$. □

Proposition 5. $Z_1(s)$ has no zero in the region $1/2 < \sigma < 20$, $|t| \geq 25$. □

Proposition 6. $Z_1(s)$ has only one simple zero $s = 2/3, 1$ in the region $1/2 < \sigma < 20$, $|t| \leq 25$. □

Then, as a consequence of these results and the functional equation of $Z_1(s)$, all zeros of $Z_1(s)$ lie on the line $\mathrm{Re}(s) = 1/2$ except for simple zeros $s = 0, 1/3, 1/2, 2/3, 1$.

While we have the following three assertions for $Z_2(s)$.

Proposition 7. $Z_2(s)$ has no zero in the right-half plane $\mathrm{Re}(s) \geq 20$. □

Proposition 8. $Z_2(s)$ has no zero in the region $1/2 < \sigma < 20$, $|t| \geq 36$. □

Proposition 9. $Z_2(s)$ has only one simple zero $s = 1, 2$ in the region $1/2 < \sigma < 20$, $|t| \leq 36$. □

Then, as a consequence of these results and the functional equation of $Z_2(s)$, all zeros of $Z_2(s)$ lie on the line $\mathrm{Re}(s) = 1/2$ except for simple zeros $s = -1, 0, 1/2, 1, 2$. □

Hence, it remains to prove the above six propositions. We carry out their proof below. The hardest part is the proof of Propositions 5 and 8. To prove Propositions 5 and 8, we use the results in the first step and a result of Lagarias [10].

6.2 Proof of Proposition 4

We have

$$\begin{aligned} Z_1(s) &= (s-1)^2(3s-2)(As - A + 1)\chi(s+1)\chi(3s)\chi(2s) \\ &\quad \times (1 - R_1(s) - R_2(s) - R_3(s) + R_4(s) + R_5(s)), \end{aligned} \tag{21}$$

where

$$R_1(s) = \frac{(s+1)(s-2)}{(s-1)(3s-2)(As-A+1)} \frac{\chi(s)}{\chi(s+1)} \frac{\chi(3s-1)}{\chi(3s)},$$
$$R_2(s) = \frac{2(s-1)(s-2)}{(s-1)(3s-2)(As-A+1)} \frac{\chi(s)}{\chi(s+1)},$$
$$R_3(s) = \frac{s^2(3s-1)(As-1)}{(s-1)^2(3s-2)(As-A+1)} \frac{\chi(s-1)}{\chi(s+1)} \frac{\chi(3s-2)}{\chi(3s)} \frac{\chi(2s-1)}{\chi(2s)},$$
$$R_4(s) = \frac{s(s+1)(s-2)}{(s-1)^2(3s-2)(As-A+1)} \frac{\chi(s)}{\chi(s+1)} \frac{\chi(3s-1)}{\chi(3s)} \frac{\chi(2s-1)}{\chi(2s)},$$
$$R_5(s) = \frac{2s^2(s+1)}{(s-1)^2(3s-2)(As-A+1)} \frac{\chi(s)}{\chi(s+1)} \frac{\chi(3s-2)}{\chi(3s)} \frac{\chi(2s-1)}{\chi(2s)}.$$

Replacing $2s-1$ by s or $3s-1$ in (2), we have

$$\left|\frac{\chi(s)}{\chi(s+1)}\right| < 1 \ (\mathrm{Re}(s) > 0), \quad \left|\frac{\chi(3s-1)}{\chi(3s)}\right| < 1 \ \left(\mathrm{Re}(s) > \frac{1}{3}\right).$$

Moreover, replacing $2s-1$ by $s-1$ or $3s-2$ in (2), we have

$$\left|\frac{\chi(s-1)}{\chi(s+1)}\right| = \left|\frac{\chi(s)}{\chi(s+1)}\right| \left|\frac{\chi(s-1)}{\chi(s)}\right| < 1 \quad (\mathrm{Re}(s) > 1),$$
$$\left|\frac{\chi(3s-2)}{\chi(3s)}\right| = \left|\frac{\chi(3s-1)}{\chi(3s)}\right| \left|\frac{\chi(3s-2)}{\chi(3s-1)}\right| < 1 \quad \left(\mathrm{Re}(s) > \frac{2}{3}\right).$$

Hence, $|R_i(s)| \le C_i|s|^{-1}$ $(i = 1, 2, 4, 5)$ for $\mathrm{Re}(s) > 1$. Applying the Stirling formula to $R_3(s)$, we obtain $|R_3(s)| = (|s|^{-5/2})$ for $\mathrm{Re}(s) > 1$ as $|s| \to \infty$ in the right-half plane.

Therefore, $Z_1(s) \neq 0$ for some right-half plane $\mathrm{Re}(s) \ge \sigma_3$. Using the monotone decreasing property of $\zeta(\sigma)$ as $\sigma \to +\infty$ and the effective version of Stirling's formula [16]

$$\Gamma(s) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \left(\frac{s}{e}\right)^s \left\{1 + \Theta\left(\frac{1}{8|s|}\right)\right\} \quad (\mathrm{Re}(s) > 1),$$

where the notation $f = \Theta(g)$ means $|f| \le g$,

we have

$$|R_1(s)| \le 0.1, \quad |R_2(s)| \le 0.3, \quad |R_3(s)| \le 0.05 \quad |R_4(s)| \le 0.1, \quad |R_5(s)| \le 0.1$$

for $\mathrm{Re}(s) \ge 20$ (in fact, these bounds already hold for $\mathrm{Re}(s) \ge 10$). These estimates imply $Z_1(s) \neq 0$ for $\mathrm{Re}(s) \ge 20$ by (21), since $(s-1)^2(3s-2)(As-A+1)\chi(s+1)\chi(3s)\chi(2s)$ has no zero in the right-half plane $\mathrm{Re}(s) \ge 20$. □

6.3 **Proof of Proposition 7**

We have

$$Z_2(s) = (s-1)^2(s-2)(As+3)\chi(s+2)\chi(2s) \times (1 - R_1(s) - R_2(s) + R_3(s) - R_4(s) - R_5(s)), \tag{22}$$

where

$$R_1(s) = \frac{2(s-3)}{(s-1)(As+3)}\frac{\chi(s+1)}{\chi(s+2)},$$
$$R_2(s) = \frac{(s+2)(s-3)}{(s-1)^2(As+3)}\frac{\chi(s)}{\chi(s+2)},$$
$$R_3(s) = \frac{s^2(s+1)(As-3-A)}{(s-1)^2(s-2)(As+3)}\frac{\chi(s-2)}{\chi(s+2)}\frac{\chi(2s-1)}{\chi(2s)},$$
$$R_4(s) = \frac{2s(s+1)(s+2)}{(s-1)^2(s-2)(As+3)}\frac{\chi(s-1)}{\chi(s+2)}\frac{\chi(2s-1)}{\chi(2s)},$$
$$R_5(s) = \frac{(s-3)(s+1)(s+2)}{(s-1)^2(s-2)(As+3)}\frac{\chi(s)}{\chi(s+2)}\frac{\chi(2s-1)}{\chi(2s)}.$$

Replacing $2s-1$ by $s-a$ $(a=-1,0,1,2)$ in (2), we have

$$\left|\frac{\chi(s+1)}{\chi(s+2)}\right| < 1, \quad (\mathrm{Re}(s) > -1),$$
$$\left|\frac{\chi(s)}{\chi(s+2)}\right| = \left|\frac{\chi(s+1)}{\chi(s+2)}\right|\left|\frac{\chi(s)}{\chi(s+1)}\right| < 1, \quad (\mathrm{Re}(s) > 0),$$
$$\left|\frac{\chi(s-1)}{\chi(s+2)}\right| = \left|\frac{\chi(s+1)}{\chi(s+2)}\right|\left|\frac{\chi(s)}{\chi(s+1)}\right|\left|\frac{\chi(s-1)}{\chi(s)}\right| < 1, \quad (\mathrm{Re}(s) > 1),$$
$$\left|\frac{\chi(s-2)}{\chi(s+2)}\right| = \left|\frac{\chi(s+1)}{\chi(s+2)}\right|\left|\frac{\chi(s)}{\chi(s+1)}\right|\left|\frac{\chi(s-1)}{\chi(s)}\right|\left|\frac{\chi(s-2)}{\chi(s-1)}\right| < 1, \quad (\mathrm{Re}(s) > 2).$$

Hence, $|R_i(s)| \le C_i|s|^{-1}$ $(i=1,2,4,5)$ for $\mathrm{Re}(s) > 2$. Applying the Stirling formula to $R_3(s)$, we obtain $|R_3(s)| = (|s|^{-5/2})$ for $\mathrm{Re}(s) \gg 0$. Therefore, $Z_2(s) \neq 0$ for some right-half plane $\mathrm{Re}(s) \ge \sigma_4$. Using the monotone decreasing property of $\zeta(\sigma)$ as $\sigma \to +\infty$ and the effective version of Stirling's formula, we have

$$|R_1(s)| \le 0.3, \quad |R_2(s)| \le 0.13, \quad |R_3(s)| \le 0.15, \quad |R_4(s)| \le 0.2, \quad |R_5(s)| \le 0.1$$

for $\mathrm{Re}(s) \ge 20$. These estimates imply $Z_2(s) \neq 0$ for $\mathrm{Re}(s) \ge 20$ by (22), since $(s-1)^2(s-2)(As+3)\chi(s+2)\chi(2s)$ has no zero in the right-half plane $\mathrm{Re}(s) \ge 20$. □

6.4 Proof of Proposition 5

Let $\rho_0 = \beta_0 + i\gamma_0$ ($\gamma_0 > 0$) be the complex zero of $f_1(s)$ in Lemma 7. By Proposition 3, $f_1(s)$ has the factorization

$$f_1(s) = f_1'(0)\, e^{B_1' s}\, s(1-s)\left(1 - \frac{s}{\rho_0}\right)\left(1 - \frac{s}{\overline{\rho}_0}\right) \cdot \Pi_1(s) \quad (B_1' \geq 0),$$

where

$$\Pi_1(s) = \prod_{0 \neq \beta \in \mathbb{R}} \left(1 - \frac{s}{\beta}\right) \prod_{\substack{\rho=\beta+i\gamma \\ \beta<1/2,\, \gamma>0}} \left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\overline{\rho}}\right)\right].$$

Note that all zeros of $\Pi_1(s)$ lie in $\sigma_0 < \mathrm{Re}(s) < 1/2$ for some σ_0. We have

$$Z_1(s) = g_1(s) \cdot \left(1 - \frac{g_1(1-s)}{g_1(s)}\right) \quad (g_1(s) = f_1(s) \cdot \chi(2s)). \tag{23}$$

and

$$\left|\frac{g_1(1-s)}{g_1(s)}\right| = e^{B_1'(1-2\sigma)} \cdot \left|\frac{\Pi_1(1-s)}{\Pi_1(s)}\right| \cdot \left|\frac{s-1+\rho_0}{s-\rho_0} \cdot \frac{s-1+\overline{\rho}_0}{s-\overline{\rho}_0}\right| \cdot \left|\frac{\chi(2s-1)}{\chi(2s)}\right|. \tag{24}$$

Because $B_1' \geq 0$, we have

$$e^{B_1'(1-2\sigma)} \leq 1 \quad (\mathrm{Re}(s) > 1/2). \tag{25}$$

For the ratio $\Pi_1(1-s)/\Pi_1(s)$ in (24), we have

$$\left|\frac{\Pi_1(1-s)}{\Pi_1(s)}\right| = \prod_{\substack{\rho=\beta+i\gamma \\ \beta<1/2,\, \gamma>0}} \left(\left|\frac{1-s-\overline{\rho}}{s-\rho}\right| \cdot \left|\frac{1-s-\rho}{s-\overline{\rho}}\right|\right) < 1 \quad (\mathrm{Re}(s) > 1/2), \tag{26}$$

by term-by-term argument as in [11] by using $\beta < 1/2$ and

$$\left|\frac{1-s-\overline{\rho}}{s-\rho}\right|^2 = 1 - \frac{(2\sigma-1)(1-2\beta)}{(\sigma-\beta)^2 + (t-\gamma)^2},$$

where $\rho = \beta + i\gamma$ is a zero of $f_1(s)$. It remains to give an estimate for

$$r_1(s) := \left|\frac{s-1+\rho_0}{s-\rho_0} \cdot \frac{s-1+\overline{\rho}_0}{s-\overline{\rho}_0}\right| \cdot \left|\frac{\chi(2s-1)}{\chi(2s)}\right|. \tag{27}$$

To estimate $r_1(s)$, we use the following lemma essentially.

Lemma 9 [10]. For any real value of t there exist at least three distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$ and

$$|t - \gamma| \leq 22. \tag{28}$$

□

Proof. Suppose $|t| \geq 25$. Then there exist at least three distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ satisfying $0 < \beta \leq 1/2$ and $|t - \gamma| < 15.1$ by applying Lemma 5 in [17] to $t + 10.1$ and $t - 10.1$ (Lemma 5 in [17] is essentially Lemma 3.5 of [10]). For $|t| < 25$, estimate (28) also holds for three distinct zeros because $\xi(s)$ has zeros at $s = \pm 14.13, \pm 21.02, \pm 25.01$. ■

Using Lemma 9, we show the following.

Lemma 10. Let $\rho_0 = \beta_0 + i\gamma_0 \simeq 0.927 + i \cdot 2.09$ be the complex zero of $f_1(s)$ in Lemma 7. Let $s = \sigma + it$ with $1/2 < \sigma \leq 20$ and $t \geq 25$. Then there exist at least two distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$, $|t - \gamma| \leq 22$,

$$\left|\frac{s - 1 + \overline{\rho_0}}{s - \rho_0}\right| \cdot \left|\frac{2s - 1 - (1 - \overline{\rho})}{2s - \rho}\right| < 1, \tag{29}$$

and

$$\left|\frac{s - 1 + \rho_0}{s - \overline{\rho_0}}\right| \cdot \left|\frac{2s - 1 - (1 - \overline{\rho})}{2s - \rho}\right| < 1. \tag{30}$$

□

Proof. By squaring (29) and (30), we have

$$\frac{(\sigma + \beta_0 - 1)^2 + (t \pm \gamma_0)^2}{(\sigma - \beta_0)^2 + (t \pm \gamma_0)^2} \cdot \frac{(2\sigma + \beta - 2)^2 + (t - \gamma)^2}{(2\sigma - \beta)^2 + (t - \gamma)^2} < 1. \tag{31}$$

To prove Lemma 10, it is sufficient that (31) holds for $0 < \beta \leq 1/2$, $|t - \gamma| < 22$, $1/2 < \sigma \leq 20$, and $t \geq 25$, because of Lemma 9. To establish (31) in that conditions it suffices to show that

$$\frac{(\sigma + \beta_0 - 1)^2 + (t \pm \gamma_0)^2}{(\sigma - \beta_0)^2 + (t \pm \gamma_0)^2} \cdot \frac{\left(2\sigma - \frac{3}{2}\right)^2 + 22^2}{\left(2\sigma - \frac{1}{2}\right)^2 + 22^2} < 1,$$

by a similar reason in the later half of Section 4.3 in [17]. This inequality is equivalent to

$$(2\sigma - 1)(8(t \pm \gamma_0)^2 - P(\sigma)) > 0, \tag{32}$$

where $P(\sigma) = 8(4\beta_0 - 3)\sigma^2 - 8(4\beta_0 - 3)\sigma - 8\beta_0^2 + 3890\beta_0 - 1945$. Using the value $\beta_0 \simeq 0.927$, we see that $P(\sigma) < 3807$ for $1/2 < \sigma < 20$. On the other hand, using the value $\gamma_0 \simeq 2.09$, we see that $8(t \pm \gamma_0)^2 > 3872$ for $t \geq 25$ since $|t \pm \gamma_0| = t \pm \gamma_0 > 22$ for $t \geq 25$. Hence, (32) holds, and it implies (31). ■

Lemma 10 and $\overline{Z_1(s)} = Z_1(\overline{s})$ imply

$$|r_1(s)| < 1 \quad \text{for} \quad 1/2 < \sigma \leq 20,\ |t| \geq 25 \tag{33}$$

by taking two distinct zeros of $\xi(s)$ in that region, since other terms in $r_1(s)$ are estimated as

$$\left| \frac{2s - 1 - (1 - \overline{\rho})}{2s - \rho} \right| < 1 \quad (\mathrm{Re}(s) > 1/2),$$

where ρ is a zero of $\xi(s)$. Estimates (25), (26), and (33) show that

$$\left| \frac{g_1(1-s)}{g_1(s)} \right| < 1 \quad \text{for} \quad 1/2 < \sigma \leq 20,\ |t| \geq 25.$$

By (23), this estimate implies Proposition 5, because $g_1(s)$ has no zero in the region $1/2 < \sigma \leq 20$, $|t| \geq 25$. □

6.5 Proof of Proposition 8

Let $\rho_0 = \beta_0 + i\gamma_0$ ($\gamma_0 > 0$) be the complex zero of $f_2(s)$ in Lemma 8. By Proposition 3, $f_2(s)$ has the factorization

$$f_2(s) = f_2(0)\, e^{B_2' s} \left(1 - \frac{s}{2}\right)\left(1 - \frac{s}{\rho_0}\right)\left(1 - \frac{s}{\overline{\rho}_0}\right) \cdot \Pi_2(s) \quad (B_2' \geq 0),$$

where

$$\Pi_2(s) = \prod_{0 \neq \beta \in \mathbb{R}} \left(1 - \frac{s}{\beta}\right) \prod_{\substack{\rho = \beta + i\gamma \\ \beta < 1/2,\, \gamma > 0}} \left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\overline{\rho}}\right)\right].$$

Here, all zeros of $\Pi_2(s)$ lie in $\sigma_0 < \mathrm{Re}(s) < 1/2$ for some σ_0. We have

$$Z_2(s) = g_2(s) \cdot \left(1 - \frac{g_2(1-s)}{g_2(s)}\right) \quad (g_2(s) = f_2(s) \cdot \chi(2s)). \tag{34}$$

and

$$\left|\frac{g_2(1-s)}{g_2(s)}\right| = e^{B_2'(1-2\sigma)} \cdot \left|\frac{\Pi_2(1-s)}{\Pi_2(s)}\right| \cdot \left|\frac{s}{1-s}\frac{s-1+\rho_0}{s-\rho_0} \cdot \frac{s-1+\overline{\rho}_0}{s-\overline{\rho}_0}\right| \cdot \left|\frac{\chi(2s-1)}{\chi(2s)}\right|.$$

For $e^{B_2'(1-2\sigma)}$ and $\Pi_2(1-s)/\Pi_2(s)$, we have

$$e^{B_2'(1-2\sigma)}, \quad \left|\frac{\Pi_2(1-s)}{\Pi_2(s)}\right| < 1 \quad (\mathrm{Re}(s) > 1/2) \tag{35}$$

by a similar argument as in $f_1(s)$. It remains to give an estimate for

$$r_2(s) := \left|\frac{s}{1-s}\frac{s-1+\rho_0}{s-\rho_0} \cdot \frac{s-1+\overline{\rho}_0}{s-\overline{\rho}_0}\right| \cdot \left|\frac{\chi(2s-1)}{\chi(2s)}\right|. \tag{36}$$

Using Lemma 9, we show the following.

Lemma 11. Let $\rho_0 = \beta_0 + i\gamma_0 \simeq 1.17 + i \cdot 3.43$ be the complex zero of $f_2(s)$ in Lemma 8. Let $s = \sigma + it$ with $1/2 < \sigma \leq 20$ and $t \geq 36$. Then there exist at least three distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$, $|t - \gamma| \leq 22$,

$$\left|\frac{s-1+\overline{\rho_0}}{s-\rho_0}\right| \cdot \left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right| < 1, \tag{37}$$

$$\left|\frac{s-1+\rho_0}{s-\overline{\rho_0}}\right| \cdot \left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right| < 1, \tag{38}$$

and

$$\left|\frac{s}{s-1}\right| \cdot \left|\frac{2s-1-(1-\overline{\rho})}{2s-\rho}\right| < 1. \tag{39}$$

□

Proof. By squaring (37) and (38), we have

$$\frac{(\sigma+\beta_0-1)^2+(t\pm\gamma_0)^2}{(\sigma-\beta_0)^2+(t\pm\gamma_0)^2}\cdot\frac{(2\sigma+\beta-2)^2+(t-\gamma)^2}{(2\sigma-\beta)^2+(t-\gamma)^2}<1. \tag{40}$$

To prove Lemma 11, it is sufficient that (40) holds for $0<\beta\le 1/2$, $|t-\gamma|<22$, $1/2<\sigma\le 20$, and $t\ge 25$, because of Lemma 9. To establish (40) in that conditions it suffices to show that

$$\frac{(\sigma+\beta_0-1)^2+(t\pm\gamma_0)^2}{(\sigma-\beta_0)^2+(t\pm\gamma_0)^2}\cdot\frac{\left(2\sigma-\frac{3}{2}\right)^2+22^2}{\left(2\sigma-\frac{1}{2}\right)^2+22^2}<1.$$

This inequality is equivalent to

$$(2\sigma-1)(8(t\pm\gamma_0)^2-P(\sigma))>0, \tag{41}$$

where $P(\sigma)=8(4\beta_0-3)\sigma^2-8(4\beta_0-3)\sigma-8\beta_0^2+3890\beta_0-1945$. Using the value $\beta_0\simeq 1.17$, we see that $P(\sigma)<7777$ for $1/2<\sigma<20$. However, using the value $\gamma_0\simeq 3.43$, we see that $8(t\pm\gamma_0)^2>8192$ for $t\ge 36$ since $|t\pm\gamma_0|=t\pm\gamma_0>32$ for $t\ge 36$. Hence, (41) holds, and it implies (40).

Similarly, to establish (39), it is sufficient to show

$$(2\sigma-1)(8t^2-p(\sigma))>0,$$

where $p(\sigma)=8\sigma^2-8\sigma+1937$. Because $p(\sigma)<4977$ for $1/2<\sigma<20$ and $8t^2\ge 5000$ for $t\ge 25$, we obtain (39). ■

Lemma 11 and $\overline{Z_2(s)}=Z_2(\overline{s})$ imply

$$|r_2(s)|<1 \quad \text{for} \quad 1/2<\sigma\le 20,\ |t|\ge 25 \tag{42}$$

by taking three distinct zeros of $\xi(s)$ in that region. Estimates (35) and (42) show that

$$\left|\frac{g_2(1-s)}{g_2(s)}\right|<1 \tag{43}$$

for $1/2<\sigma\le 20$, $|t|\ge 36$. By (34), this estimate implies Proposition 8, because $g_2(s)$ has no zero in the region $1/2<\sigma\le 20$, $|t|\ge 36$. □

6.6 Proof of Propositions 6 and 9

Because the region $1/2 < \sigma \le 20$, $|t| \le 25$ or 36 is finite, we can check the assertions of Proposition 6 and Proposition 9 by using the help of computer as in the proof of Lemma 7 and Lemma 8. □

7 Proof of Lemma 2

We prove the lemma only if $F(s)$ has genus one, since if $F(s)$ has genus zero, it is easily proved by a way similar to the case of genus one. The genus one assumption is known to be equivalent to the Hadamard product factorization

$$F(s) = e^{A+Bs} s^m \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp(s/\rho) \quad (m \in \mathbb{Z}_{\geq 0}), \tag{44}$$

means the continuation of what you wrote up to $\mathbb{C}$. That is also equivalent to $\sum_{\rho} |\rho|^{-2} < \infty$. Assumption (i) implies the symmetry of the set of zeros under the conjugation $\rho \mapsto \overline{\rho}$. It follows that the set of zeros $\rho = \beta + i\gamma$, counted with multiplicity, is partitioned into blocks $B(\rho)$ comprising $\{\rho, \overline{\rho}\}$ if $\gamma > 0$ and $\{\rho\}$ if $\beta \neq 0$ and $\gamma = 0$. Each block is labeled with the unique zero in it having $\gamma \geq 0$. Using assumption (ii), we show

$$F(s) = s^m e^{A+B's} \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left(1 - \frac{s}{\rho}\right) \right), \tag{45}$$

where the outer product on the right-hand side converges absolutely and uniformly on any compact subsets of $\mathbb{C}$. This assertion holds because the block convergence factors $\exp(c(B(\rho))s)$ are given by $c(B(\rho)) = 2\beta|\rho|^{-2}$ for $\gamma > 0$. Assumption (ii) implies $|\beta - 1/2| < \sigma_0$. Hence,

$$\sum_{B(\rho)} |c(B(\rho))| \leq \sum_{0 \neq \rho:\,\mathrm{real}} |\rho|^{-1} + (2\sigma_0 + 1) \sum_{\rho} |\rho|^{-2} < \infty.$$

Thus, the convergence factors $\exp(c(B(\rho))s)$ can be pulled out of the product. Hence, we have (45) with

$$B' = B + \sum_{B(\rho)} c(B(\rho)). \tag{46}$$

Using assumptions (iii), (iv), and (v), we show

$$B' \geq 0. \tag{47}$$

By (4) in assumption (v), we have

$$\mathbb{R} \ni \log\left(\frac{F(1-\sigma)}{F(\sigma)}\right) \to -\infty \quad \text{as} \quad \sigma \to +\infty. \tag{48}$$

Using (45), we have

$$\frac{F(1-\sigma)}{F(\sigma)} = e^{B'(1-2\sigma)}\left(\frac{\sigma-1}{\sigma}\right)^m \prod_{\rho=\beta\in\mathbb{R}} \frac{\sigma-1+\beta}{\sigma-\beta} \prod_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \frac{(\sigma-1+\beta)^2+\gamma^2}{(\sigma-\beta)^2+\gamma^2}.$$

Thus,

$$\begin{aligned}\log\left(\frac{F(1-\sigma)}{F(\sigma)}\right) &= B'(1-2\sigma) + m\log\left(1-\frac{1}{\sigma}\right) + \sum_{\rho=\beta\in\mathbb{R}} \log\left(1-\frac{1-2\beta}{\sigma-\beta}\right) \\ &\quad + \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right).\end{aligned} \tag{49}$$

Note that

$$\log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right) < 0 \quad \text{for} \quad \sigma > 1/2 \tag{50}$$

if $\beta < 1/2$, and

$$\log\left(1-\frac{1}{\sigma}\right),\ \log\left(1-\frac{1-2\beta}{\sigma-\beta}\right),\ \log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right) \to 0 \quad \text{as} \quad \sigma \to +\infty$$

for any fixed $\rho = \beta + i\gamma$. By assumption (iii), (50) holds except for finitely many zeros. Hence, if we suppose $B' < 0$, (48) and (49) imply

$$\left|\sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| \geq 2|B'|\sigma \tag{51}$$

for large $\sigma > 1/2$, because the number of real zeros is also finite by assumptions (ii) and (iii). However, for large $\sigma > 1/2$, we have

$$\begin{aligned}\left|\sum_{\substack{\rho=\beta+i\gamma\\ \gamma>0}} \log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| &\le \left|\sum_{\substack{\rho=\beta+i\gamma\\ \gamma>0}} \log\left(1-\frac{(1-2\sigma_0)(2\sigma-1)}{(\sigma-1/2)^2+\gamma^2}\right)\right| \\ &\ll (2\sigma-1)\sum_{\substack{\rho=\beta+i\gamma\\ \gamma>0}} \frac{1}{(\sigma-1/2)^2+\gamma^2}.\end{aligned}$$

The sum in the right-hand side can be written as the Stieltjes integral

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2+t^2}.$$

Using (3) in assumption (iv), we have

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2+t^2} \ll \int_{\gamma_0}^{\infty} \frac{(\log t)\,dt}{(\sigma-1/2)^2+t^2} \ll \frac{\log(\sigma+\gamma_0)}{\sigma-1/2}.$$

Hence, we obtain

$$\left|\sum_{\substack{\rho=\beta+i\gamma\\ \gamma>0}} \log\left(1-\frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2+\gamma^2}\right)\right| \ll \log(\sigma+\gamma_0) \tag{52}$$

for large $\sigma > 1/2$. This contradicts (51). Thus (47) holds. □

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