Higher-rank zeta functions and SL_n-zeta functions for curves

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In earlier papers L.W. introduced two sequences of higher-rank zeta functions associated to a smooth projective curve over a finite field, both of them generalizing the Artin zeta function of the curve. One of these zeta functions is defined geometrically in terms of semistable vector bundles of rank *n* over the curve and the other one group-theoretically in terms of certain periods associated to the curve and to a split reductive group *G* and its maximal parabolic subgroup *P*. It was conjectured that these two zeta functions coincide in the special case when $G = SL_n$ and *P* is the parabolic subgroup consisting of matrices whose final row vanishes except for its last entry. In this paper we prove this equality by giving an explicit inductive calculation of the group-theoretically defined zeta functions in terms of the original Artin zeta function (corresponding to $n = 1$) and then verifying that the result obtained agrees with the inductive determination of the geometrically defined zeta functions found by Sergey Mozgovoy and Markus Reineke in 2014.

nonabelian zeta function | curves over finite fields | special permutations | zeta functions | zeta functions for SL_n

In refs. 1 and 2, a nonabelian zeta function $\zeta_{X,n}(s) = \zeta_{X/\mathbb{F}_q,n}(s)$ was defined for any smooth projective curve X over a finite field $\mathbb{F}_q$ and any integer $n \ge 1$ by

$$\zeta_{X,n}(s) = \sum_{[V]} \frac{|H^0(X, V) \smallsetminus \{0\}|}{|\mathrm{Aut}(V)|}\, q^{-\deg(V)s} \qquad (\Re(s) > 1), \qquad [1]$$

where the sum is over the moduli stack of $\mathbb{F}_q$-rational semistable vector bundles V of rank n on X with degree divisible by n. Using the Riemann–Roch, duality, and vanishing theorems for semistable bundles, it was shown that $\zeta_{X,n}(s)$ agrees with the usual Artin zeta function $\zeta_X(s)$ of $X/\mathbb{F}_q$ if $n = 1$; that it has the form $P_{X,n}(T)/(1-T)(1-q^nT)$ for some polynomial $P_{X,n}(T)$ of degree $2g$ in T, where g is the genus of X and $T = q^{-ns}$; and that it satisfies the functional equation

$$\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)\,, \qquad \text{where} \quad \widehat{\zeta}_{X,n}(s) := q^{n(g-1)s} \cdot \zeta_{X,n}(s).$$

It was also conjectured that $\zeta_{X,n}(s)$ satisfies the Riemann hypothesis (i.e., that all of its zeros have real part 1/2). In a companion paper (3), explicit formulas for $\zeta_{X,n}(s)$ and a proof of the Riemann hypothesis were given for the case when $g = 1$.

On the other hand, in refs. 2 and 4, a different approach to zeta functions for curves led to the so-called group zeta function $\widehat{\zeta}_X^{G,P}(s)$ of $X/\mathbb{F}_q$, associated to a connected split algebraic reductive group G and its maximal parabolic subgroup P. The precise definition, which is based on the theory of periods, is recalled in *Section 2*. In this paper, we are interested in the special case when $G = SL_n$ and $P = P_{n-1,1}$, the subgroup of SL_n consisting of matrices whose final row vanishes except for its last entry, and we then write simply $\widehat{\zeta}_X^{SL_n}(s)$ for $\widehat{\zeta}_X^{G,P}(s)$. Our main result is a proof of the following theorem, which was conjectured in ref. 2 ("special uniformity conjecture").

Theorem 1. *The zeta functions $\widehat{\zeta}_{X,n}(s)$ and $\widehat{\zeta}_X^{SL_n}(s)$ coincide for all $n \ge 1$.*

Theorem 1 should be regarded as a joint result of L.W. and D.Z. and of Sergey Mozgovoy and Markus Reineke (5), because the proof proceeds by comparing a formula for $\widehat{\zeta}_X^{SL_n}(s)$ established here with a formula for $\widehat{\zeta}_{X,n}(s)$ given in their paper. Specifically, the proof consists of three steps:

1) By analyzing the definition of $\widehat{\zeta}_X^{G,P}(s)$ for $G = SL_n$, $P = P_{n-1,1}$, we will prove an explicit formula, giving $\widehat{\zeta}_X^{SL_n}(s)$ as a linear combination of the functions $\widehat{\zeta}_X(ns-k)$ for $0 \le k < n$ with rational functions of T as coefficients. The calculation is given in *Sections 3–5*.

Significance

Almost 100 years ago, Artin defined an analog of the famous Riemann zeta function for curves (one-dimensional varieties) over a finite field. In 2005, L.W. defined two different series of "higher zeta functions" for curves over finite fields that both generalized Artin's zeta functions, one being defined geometrically and the other using advanced concepts from group representation theory, and conjectured that they always coincide. In this paper this conjecture is proved by giving a formula for one of the two series and showing that it agrees with the formula for the other series proved a few years ago by Sergey Mozgovoy and Markus Reineke.

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2) In ref. 5, as recalled in *Section 6*, using the theory of Hall algebras and wall-crossing techniques, a formula for $\widehat{\zeta}_{X,n}(s)$ of the same general shape is proved.
3) A short calculation, given in *Section 7*, shows that the two formulas agree.

The explicit formula is not very complicated, and we can state it here. Motivated by the Siegel–Weil formula for the total mass of vector bundles V of rank n and degree 0 on X (i.e., the number of such Vs, weighted by the inverse of the number of their automorphisms), and to make a proper normalization, we define numbers $\widehat{v}_k$ ($k \geq 1$) inductively by

$$\widehat{v}_k = \begin{cases} \lim_{s\to 1}(1-q^{1-s})\,\widehat{\zeta}_X(s) & \text{if } k=1\,, \\ \widehat{\zeta}_X(k)\,\widehat{v}_{k-1} & \text{if } k\geq 2\,, \end{cases} \qquad [2]$$

where $\widehat{\zeta}_X(s) = q^{s(g-1)}\zeta_X(s)$. Furthermore, as in ref. 3—where these functions were introduced for the purpose of writing down in a more structural way the nonabelian rank n zeta functions for elliptic curves over finite fields—we define rational functions $B_k(x)$ ($k\geq 0$) either inductively by the formulas

$$B_k(x) = \begin{cases} 1 & \text{if } k=0\,, \\ \sum\limits_{m=1}^{k} \widehat{v}_m\,\dfrac{B_{k-m}(q^m)}{1-q^m x} & \text{if } k\geq 1\,, \end{cases} \qquad [3]$$

or in closed form (if $k\geq 1$) by

$$B_k(x) = \sum_{p=1}^{k} \sum_{\substack{k_1,\dots,k_p>0 \\ k_1+\cdots+k_p=k}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{(1-q^{k_1+k_2})\dots(1-q^{k_{p-1}+k_p})}\cdot\frac{1}{1-q^{k_p}x}. \qquad [4]$$

Then the formula that we will establish for $\widehat{\zeta}_X^{SL_n}(s)$ can be stated as follows:

Theorem 2. *With the above notations, we have*

$$\widehat{\zeta}_X^{SL_n}(s) = q^{\binom{n}{2}(g-1)} \sum_{k=0}^{n-1} B_k(q^{ns-k})\,B_{n-k-1}(q^{k+1-ns})\,\widehat{\zeta}_X(ns-k). \qquad [5]$$

Remarks:

1) In the definition Eq. **1** of the nonabelian zeta function $\zeta_{X,n}(s)$, vector bundles used are assumed to be of degrees divisible by the rank n. This definition is motivated by a work of Drinfeld (6) on counting supercuspidal representations in rank 2 and also because if we summed over all degrees as was originally done in ref. 1, then the functional equation would still hold but the Riemann hypothesis would not.
2) The analog of *Theorem 1* for the case of number fields rather than function fields was proved by L.W. several years ago by totally different techniques, using the theory of Eisenstein series and Arthur trace formulas (combine the "global bridge" on p. 295 and the discussion on p. 305 of ref. 7 with the formulas on p. 284 of ref. 8 and on p. 197 of ref. 4).
3) A proof of *Theorem 1* for the cases $n=2$ and $n=3$ was given in ref. 5, at a time when the current paper was still in the preprint stage.

2. Zeta Functions for (*G*, *P*)

Let G be a connected split reductive algebraic group of rank r with a fixed Borel subgroup B and associated maximal split torus T (over a base field). Denote by

$$\big(V,\,\langle\cdot,\cdot\rangle,\,\Phi=\Phi^+\cup\Phi^-,\,\Delta=\{\alpha_1,\dots,\alpha_r\},\,\varpi:=\{\varpi_1,\dots,\varpi_r\},\,W\big)$$

the associated root system. That is, V is the real vector space defined as the $\mathbb{R}$ span of rational characters of T and, as usual, is equipped with a natural inner product $\langle\cdot,\cdot\rangle$, with which we identify V with its dual V^*; and $\Phi^+\subset V$ is the set of positive roots, $\Phi^- := -\Phi^+$ the set of negative roots, $\Delta\subset V$ the set of simple roots, $\varpi\subset V$ the set of fundamental weights, and W the Weyl group. By definition, the fundamental weights are characterized by the formula $\langle\varpi_i,\alpha_j^\vee\rangle=\delta_{ij}$ for $i,j=1,2,\dots,r$, where $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\,\alpha$ denotes the coroot of a root $\alpha\in\Phi$. We also define the Weyl vector ρ by $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and introduce a coordinate system on V (with respect to the base $\{\varpi_1,\dots,\varpi_r\}$ of V and the vector ρ) by writing an element $\lambda\in V$ in the form

$$\lambda = \sum_{j=1}^{r}(1-s_j)\varpi_j = \rho - \sum_{j=1}^{r} s_j\varpi_j,$$

thus fixing identifications of V and $V_{\mathbb{C}}=V\otimes_{\mathbb{R}}\mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$. In addition, for each Weyl element $w\in W$, we set $\Phi_w := \Phi^+\cap w^{-1}\Phi^-$, i.e., the collection of positive roots whose w images are negative.

As usual, by a standard parabolic subgroup, we mean a parabolic subgroup of G that contains the Borel subgroup B. From Lie theory (e.g., ref. 9), there is a one-to-one correspondence between standard parabolic subgroups P of G and subsets Δ_P of Δ. In particular, if P is maximal, we may and will write $\Delta_P=\Delta\smallsetminus\{\alpha_p\}$ for a certain unique $p=p(P)\in\{1,\dots,r\}$. For such a standard parabolic subgroup P, denote by V_P the $\mathbb{R}$ span of rational characters of the maximal split torus T_P contained in P, by V_P^* its dual space and by $\Phi_P \subset V_P$ the set of nontrivial characters of T_P occurring in the space V. Then, by standard theory of reductive groups (e.g., ref. 10), V_P admits a canonical embedding in V (and V_P^* admits a canonical embedding in V^*), which is known to be orthogonal to the fundamental weight ϖ_p, and hence Φ_P can be viewed as a subset of Φ. Set $\Phi_P^+ = \Phi^+ \cap \Phi_P$, $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P^+}\alpha$, and $c_P = 2\langle \varpi_p - \rho_P, \alpha_p^\vee\rangle$.

Now, let X be an integral regular projective curve of genus g over a finite field $\mathbb{F}_q$. In ref. 2, motivated by the study of zeta functions for number fields,[†] for a connected split reductive algebraic group G and its standard parabolic subgroup P as above (defined over the function field of X), L.W. defined the period of G for X by

$$\omega_X^G(\lambda) := \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})} \prod_{\alpha\in\Phi_w} \frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle+1)}$$

and the period of (G,P) for X by

$$\omega_X^{G,P}(s) := \mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\,\alpha\in\Delta_P}\,\omega_X^G(\lambda)\Big|_{s_p=s} = \mathrm{Res}_{s_r=0}\cdots\mathrm{Res}_{s_{p+1}=0}\mathrm{Res}_{s_{p-1}=0}\cdots\mathrm{Res}_{s_1=0}\,\omega_X^G(\lambda)\Big|_{s_p=s},$$

where s is a complex variable[‡] and where for the last equality we used the fact that $\langle\rho,\alpha^\vee\rangle = 1$ for all $\alpha\in\Delta$ and the relation that $\langle\varpi_i,\alpha_j^\vee\rangle = \delta_{ij}$ for all $i, j\in\{1,\dots,r\}$. As proved in refs. 2 and 11, the ordering of taking residues along singular hyperplanes $\langle\lambda-\rho,\alpha^\vee\rangle = 0$ for $\alpha\in\Delta_P$ does not affect the outcome, so that the definition is independent of the numbering of the simple roots.

To get the zeta function associated to (G,P) for X, certain normalizations should be made. For this purpose, write $\omega_X^G(\lambda) = \sum_{w\in W} T_w(\lambda)$, where, for each $w\in W$,

$$T_w(\lambda) := \frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})} \prod_{\alpha\in\Phi_w} \frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle+1)}.$$

The zeta function of X associated to (G,P) will be defined in terms of the residue $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\,\alpha\in\Delta_P} T_w(\lambda)$.

We care only about those elements $w\in W$ (we call them special) that give nontrivial residues, namely, those satisfying the condition that $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\,\alpha\in\Delta_P} T_w(\lambda)\not\equiv 0$. This can happen only if all singular hyperplanes are of one of the following two forms:

1) $\langle w\lambda-\rho,\alpha^\vee\rangle = 0$ for some $\alpha\in\Delta$, giving a simple pole of the rational factor $\frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})}$;
2) $\langle\lambda,\alpha^\vee\rangle = 1$ for some $\alpha\in\Phi_w$, giving a simple pole of the zeta factor $\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)$.

For special $w\in W$ and $(k,h)\in\mathbb{Z}^2$, following ref. 11 (also ref. 2) we define

$$\begin{aligned} N_{P,w}(k,h) &:= \#\{\alpha\in w^{-1}\Phi^- : \langle\varpi_p,\alpha^\vee\rangle = k,\ \langle\rho,\alpha^\vee\rangle = h\} \\ M_P(k,h) &:= \max_{w\ \text{special}} (N_{P,w}(k,h-1) - N_{P,w}(k,h)). \\ &= N_{P,w_0}(k,h-1) - N_{P,w_0}(k,h), \end{aligned} \qquad [6]$$

where w_0 is the longest element of the Weyl group and where the last equality is corollary 8.7 of ref. 12. Note that $M_P(k,h) = 0$ for almost all but finitely many pairs of integers (k,h), so it makes sense to introduce the product

$$D_X^{G,P}(s) := \prod_{k=0}^{\infty}\prod_{h=2}^{\infty} \widehat{\zeta}_X(kn(s-1)+h)^{M_P(k,h)}. \qquad [7]$$

Following refs. 2 and 4, we define the zeta function of X associated to (G,P) by

$$\widehat{\zeta}_X^{G,P}(s) := q^{(g-1)\dim N_u(B)}\cdot D^{G,P}(s)\cdot\omega_X^{G,P}(s). \qquad [8]$$

Here $N_u(B)$ denotes the nilpotent radical of the Borel subgroup B of G.

Remark: For special $w\in W$, even after taking residues, there are some zeta factors $\widehat{\zeta}_X(ks+h)$ left in the denominator of $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\,\alpha\in\Delta_P} T_w(\lambda)$. The reason for introducing the factor $D_X^{G,P}(s)$ in our normalization of the zeta functions, based on formulas in refs. 2 and 11, is to clear up all of the zeta factors appearing in the denominators associated to special Weyl elements.

3. Specializing to SL_n

From now on, we specialize to the case when G is the special linear group SL_n and P is the maximal parabolic subgroup $P_{n-1,1}$ consisting of matrices whose final row vanishes except for its last entry, corresponding to the ordered partition $(n-1)+1$ of n. Our purpose is to study the zeta function of X associated to SL_n:

$$\widehat{\zeta}_X^{SL_n}(s) := \widehat{\zeta}_X^{SL_n,\,P_{n-1,1}}(s). \qquad [9]$$

[†]For number fields, the analogs of the two functions to be introduced below are special kinds of Eisenstein periods, defined as integrals of Eisenstein series over moduli spaces of semistable lattices. For details, see ref. 4.

[‡]We warn the reader that in refs. 4, 7, and 8 a different normalization is used, with the argument of $\omega_X^{G,P}$ (and later of $\zeta_X^{G,P}$) being given by $s = c_P(s_p-1)$ $(= n(s_p-1)$ in the special case $(G,P) = (SL_n, P_{n-1,1}))$ rather than $s = s_p$ as chosen here. With the normalization used here the functional equation relates s and $1-s$ rather than s and $-n-s$.

As usual, we realize the root system A_{n-1} associated to SL_n as follows. Denote by $\{e_1, \ldots, e_n\}$ the standard orthonormal basis of the Euclidean space $\mathbb{R}^n$. The positive roots are given by $\Phi^+ := \{e_i - e_j \mid 1 \le i < j \le n\}$, the simple roots by $\Delta = \{\alpha_1 := e_1 - e_2, \ldots, \alpha_{n-1} := e_{n-1} - e_n\}$, and the Weyl vector by $\rho = \sum_{j=1}^{n} \frac{n+1-2j}{2}\, e_j$. We identify the Weyl group W with $\mathfrak{S}_n$, the symmetric group on n letters, by the assignment $w \mapsto \sigma_w$, where $w(e_i - e_j) = e_{\sigma_w(i)} - e_{\sigma_w(j)}$. For convenience, we also write the corresponding Δ_P, Φ_P^+, ρ_P, ϖ_P, and c_P simply as Δ', Φ'^+, ρ', ϖ', and c', respectively. We have

$$\Delta' = \{\alpha_1, \ldots, \alpha_{n-2}\}, \qquad \Phi'^+ = \{e_i - e_j : 1 \le i < j \le n-1\},$$

$$\rho' = \sum_{j=1}^{n-1} \frac{n-2j}{2}\, e_j\,, \qquad \varpi' = \varpi_{n-1} = \frac{1}{n} \sum_{j=1}^{n} e_j \,-\, e_n.$$

In addition, $\langle \rho, \alpha \rangle = 1$ for all $\alpha \in \Delta$, and $\alpha^\vee = \alpha$, $\langle \rho, \alpha \rangle = 1$ for all $\alpha \in \Phi^+$. Hence

$$\rho' = \rho - \frac{n}{2}\, \varpi'\,, \qquad c' = 2\langle \varpi' - \rho', \alpha_{n-1} \rangle = n.$$

Accordingly, for positive roots $\alpha_{ij} := e_i - e_j \in \Phi^+$, we have

$$\langle \rho, \alpha_{ij} \rangle = j - i, \qquad \langle \varpi', \alpha_{ij} \rangle = \delta_{jn} - \delta_{in}, \tag{10}$$

and, for $\lambda_s := (ns - n)\varpi' + \rho$,

$$\langle \lambda_s, \alpha_{ij} \rangle = \begin{cases} j - i & \text{if } i, j \ne n, \\ ns - i & \text{if } j = n, \\ -ns + j & \text{if } i = n. \end{cases} \tag{11}$$

To write down the zeta function $\widehat{\zeta}_X^{SL_n}(s)$ explicitly, we express the multiple residues in the periods of $(SL_n, P_{n-1,1})$ as a single limit, after multiplying by suitable vanishing factors (to the period of SL_n). Indeed, since $\langle \lambda_s - \rho, \alpha_{n-1} \rangle = ns - n$, and

$$\lim_{\lambda \to \lambda_s} \left(1 - q^{-\langle \lambda - \rho, \alpha \rangle}\right) \equiv 0 \qquad (\forall\, \alpha \in \Delta'), \tag{12}$$

we have

$$\omega_X^{SL_n,\, P_{n-1,1}}(s) = \lim_{\lambda \to \lambda_s} \left(\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle}) \cdot \omega_X^{SL_n}(\lambda) \right). \tag{13}$$

Recall that $\omega_X^{SL_n}(\lambda) = \sum_{w \in W} T_w(\lambda)$. Accordingly, to pin down the nonzero contributions for the terms appearing in the limit, we should consider, for a fixed $w \in W$, the limit $\lim_{\lambda \to \lambda_s} \left(\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle}) \cdot T_w(\lambda) \right)$ or, equivalently, for a fixed $\sigma \in \mathfrak{S}_n (\simeq W)$, the function

$$L_\sigma(s) = \lim_{\lambda \to \lambda_s} \left(\frac{\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle})}{\prod_{\beta \in \Delta} (1 - q^{-\langle \sigma\lambda - \rho, \beta \rangle})} \prod_{\alpha \in \Phi^+,\, \sigma(\alpha) < 0} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha \rangle + 1)} \right). \tag{14}$$

For this limit $L_\sigma(s)$ to be nonzero, by Eq. **12**, there should be a complete cancellation of all of the factors $(1 - q^{-\langle \lambda - \rho, \alpha \rangle})$ in the numerator of the first term in Eq. **14** that vanish at $\lambda = \lambda_s$ with either

1) factors $\left(1 - q^{-\langle \sigma\lambda_s - \rho, \beta \rangle}\right)$ appearing in the denominator of the first term in Eq. **14** or else
2) the poles at $\lambda = \lambda_s$ of factors $\widehat{\zeta}_X\,(\langle \lambda, \alpha \rangle)$ appearing in the numerator of the second term in Eq. **14** for which $\langle \lambda_s, \alpha \rangle = 1$.

Since $\langle \cdot\,, \cdot \rangle$ is σ invariant, for $\alpha \in \Delta'$, by Eq. **10**, $\langle \sigma\lambda_s - \rho, \alpha \rangle = \langle \lambda_s, \sigma^{-1}\alpha \rangle - 1$. Hence, for $L_\sigma(s)$ to have a nonzero contribution to $\omega_X^{(SL_n,\, P_{n-1,1})}(s)$, the union of

$$A_\sigma := \left\{ \alpha \in \Delta' : \sigma\alpha \in \Delta \right\} \quad \text{and} \quad B_\sigma := \left\{ \alpha \in \Delta' : \sigma\alpha < 0 \right\} \tag{15}$$

must be of cardinality $n - 2$. Call such $\sigma \in \mathfrak{S}_n$ special and denote the collection of special permutations by $\mathfrak{S}_n^0$. Clearly, for $\sigma \in \mathfrak{S}_n$, we have $A_\sigma \cup B_\sigma \subset \Delta'$, and $A_\sigma \cup B_\sigma = \Delta'$ if and only if $\sigma \in \mathfrak{S}_n^0$. That is to say, the limit $L_\sigma(s)$ corresponding to the permutation $\sigma\in\mathfrak{S}_n$ can be nonzero only if σ is special, and in this case, we have $\Delta' = A_\sigma\sqcup B_\sigma$. This then completes the proof of the following:

Lemma 3. *With the notations above,*

$$\omega_X^{SL_n,\,P_{n-1,1}}(s)=\sum_{\sigma\in\mathfrak{S}_n^0} L_\sigma(s). \qquad [16]$$

Here $\sigma\in\mathfrak{S}_n^0$ if and only if $A_\sigma\cup B_\sigma=\Delta'$.

The next lemma describes $L_\sigma(s)$ for special permutations σ.

Lemma 4. *For $\sigma\in\mathfrak{S}_n^0$, set*

$$R_\sigma(s)=\prod_{\substack{1\le k\le n-1\\ \sigma^{-1}\alpha_k\notin\Delta'}}\Big(1-q^{-\langle\sigma\lambda_s-\rho,\alpha_k\rangle}\Big),\quad \widehat{\zeta}_\sigma^{[n]}(s)=\prod_{\substack{1\le i\le n-1\\ \sigma(i)>\sigma(n)}}\frac{\widehat{\zeta}_X(\langle\lambda_s,\alpha_{in}\rangle)}{\widehat{\zeta}_X(\langle\lambda_s,\alpha_{in}\rangle+1)},$$

$$\widehat{\zeta}_\sigma^{[<n]}(s):=\left.\left(\prod_{\substack{1\le k\le n-2\\ \sigma(k)>\sigma(k+1)}}\Big(1-q^{-\langle\lambda-\rho,\alpha_k\rangle}\Big)\cdot\prod_{\substack{1\le i<j\le n-1\\ \sigma(i)>\sigma(j)}}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle+1)}\right)\right|_{\lambda=\lambda_s}.$$

Then

$$L_\sigma(s)=\frac{1}{R_\sigma(s)}\cdot\widehat{\zeta}_\sigma^{[n]}(s)\cdot\widehat{\zeta}_\sigma^{[<n]}(s). \qquad [17]$$

Proof: This is obtained by regrouping the terms of Eq. **14** for special permutation $\sigma\in\mathfrak{S}_n^0$, following the discussions above. We first cancel the terms in the numerator of the first factor in Eq. **14** for $\alpha\in A_\sigma$ with the corresponding terms in the denominator for $\beta=\sigma\alpha$. The first factor $1/R_\sigma(s)$ in Eq. **17** is the value at $\lambda=\lambda_\sigma$ of the product of the remaining terms $\beta\in\Delta\smallsetminus\sigma A_\sigma$ in this denominator. The second factor $\widehat{\zeta}_\sigma^{[n]}(s)$ in Eq. **17** is the value at $\lambda=\lambda_\sigma$ of the product of the terms in the second factor in Eq. **14** for $\alpha\notin{\Phi'}^+$; i.e., $\alpha=e_i-e_n>0$. The third factor $\widehat{\zeta}_\sigma^{[<n]}(s)$ in Eq. **17**, which can also be written

$$\widehat{\zeta}_\sigma^{[<n]}(s)=\left.\left(\prod_{\alpha\in B_\sigma}(1-q^{-\langle\lambda-\rho,\alpha\rangle})\cdot\prod_{\substack{\alpha\in{\Phi'}^+\\ \sigma(\alpha)<0}}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle+1)}\right)\right|_{\lambda=\lambda_s},$$

is obtained by collecting all of the remaining zeta factors and rational factors appearing in the numerator. □

The terms occurring in $\widehat{\zeta}_\sigma^{[<n]}(s)$ are of two types: For $\alpha\in B_\sigma$ we must combine the quantities $(1-q^{-\langle\lambda-\rho,\alpha_k\rangle})$ and $\frac{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle+1)}$ before taking the limit as $\lambda\to\lambda_s$ because the first one has a zero and the second one has a pole, while in the remaining zeta quotients from the second term in Eq. **17**, corresponding to $\alpha\in{\Phi'}^+\smallsetminus B_\sigma$, we could simply substitute $\lambda=\lambda_s$ instead of taking a limit. We can say this differently as follows. By abuse of notation we write simply $\widehat{\zeta}_X(1)$ for the limit as $s\to1$ of $(1-q^{1-s})\widehat{\zeta}_X(s)$. (It should be written $\widehat{v}_1$, as defined in Eq. **2**, but the "$\widehat{\zeta}_X(1)$" notation will let us write more uniform formulas.) Then the definition of $\widehat{\zeta}_\sigma^{[<n]}(s)$ can be rewritten using the first equation in Eq. **11** as

$$\widehat{\zeta}_\sigma^{[<n]}(s)=\prod_{k\ge1}\left(\frac{\widehat{\zeta}_X(k)}{\widehat{\zeta}_X(k+1)}\right)^{m_\sigma(k)}=\prod_{k\ge1}\widehat{\zeta}_X(k)^{n_\sigma(k)}, \qquad [18]$$

where

$$m_\sigma(k)=\sum_{\substack{1\le i<j\le n-1\\ \sigma(i)>\sigma(j),\,j-i=k}}1=\#\{\alpha\in{\Phi'}^+:\sigma\alpha<0,\langle\rho,\alpha\rangle=k\} \qquad [19]$$

and

$$n_\sigma(k)=m_\sigma(k)-m_\sigma(k-1),\qquad n_\sigma(1)=m_\sigma(1)=\#B_\sigma. \qquad [20]$$

Eq. **18** gives an explicit formula for the third factor in Eq. **17**, which, as one sees, does not depend on s at all. The other two factors in Eq. **17**, which do depend on s, are computed later, in *Section 5*.

Lemmas 3 and *4* calculate the third factor $\omega_X^{G,P}(s)$ in the definition Eq. **8** of $\widehat{\zeta}_X^{G,P}(s)$ in the special case $G=SL_n$, $P=P_{n-1,1}$, but since some of the numbers $n_\sigma(k)$ in Eq. **18** may be negative, the expression for this factor may still contain some zeta values in its denominator. These zeta values in the denominator will be canceled when we include the second factor $D^{G,P}(s)$ in Eq. **8**. Our next task is therefore to evaluate this expression explicitly in the case $(G,P)=(SL_n,\,P_{n-1,1})$. Then the formulas for $D^{G,P}(s)$ and $\widehat{\zeta}_X^{G,P}(s)$ can be written explicitly as follows:

Lemma 5. *We have*

$$D^{SL_n,P_{n-1,1}}(s) = \prod_{k=2}^{n-1} \widehat{\zeta}_X(k) \cdot \widehat{\zeta}_X(ns) \tag{21}$$

and

$$\widehat{\zeta}_X^{SL_n}(s) = q^{\frac{n(n-1)}{2}(g-1)} \cdot D^{SL_n,P_{n-1,1}}(s) \cdot \omega_X^{(SL_n,P_{n-1,1})}(s). \tag{22}$$

Proof: In view of the definitions Eqs. **7** and **8**, we must show that $M_P(k,h)$ equals 1 if $k=0$ and $2 \le h < n$ or $k=1$ and $h=n$ and vanishes otherwise, which follows easily from Eq. **6** since here $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$. $\square$

4. Special Permutations

In this section we describe special permutations explicitly. Recall from *Section 3* that σ is special if and only if $A_\sigma \sqcup B_\sigma = \Delta'$, where A_σ and B_σ are defined as in Eq. **15**. This implies that σ is special if and only if $\sigma(i+1)=\sigma(i)+1$ or $\sigma(i+1)<\sigma(i)$ for all $1 \le i \le n-2$ (or equivalently, since σ is a permutation, if and only $\sigma(i+1) \le \sigma(i)+1$ for all $1 \le i \le n-2$). Denote by $t_1 > \ldots > t_m$ the distinct values of $\sigma(i)-i$ for $1 \le i \le n-2$ and by I_ν $(1 \le \nu \le m)$ the set of $i \in \{1,\ldots,n-2\}$ with $\sigma(i)-i=t_\nu$. Then σ maps I_ν onto its image $I'_\nu = \sigma(I_\nu)$ by translation by t_ν, and we have $\bigcup I_\nu = \{1,\ldots,n-1\}$ and $\bigcup I'_\nu = \{1,\ldots,n\} \smallsetminus \{a\}$, where $a=\sigma(n) \in \{1,\ldots,n\}$. It is easy to check that $I_1 < \cdots < I_m$ (in the sense that all elements of I_ν are less than all elements of $I_{\nu+1}$ if $1 \le \nu \le m-1$) and $I'_1 > \cdots > I'_m$ (in the same sense). [Indeed, let A denote the set of indexes $i \in \{1,\ldots,n-2\}$ with $\sigma(i+1)=\sigma(i)+1$. Then $\sigma(i)-i$ is constant when we pass from any $i \in A$ to $i+1$, so each set I_ν is a connected interval that is contained in A except for its right end-point i_0, which satisfies $\sigma(i_0+1)<\sigma(i_0)$, so that i_0+1 belongs to an I_μ satisfying $t_\mu < t_\nu$ and hence $\mu > \nu$. But then I_μ contains a point that is bigger than one of the points of I_ν and that has an image under σ that is smaller than the image of that point, and since all of these sets are connected intervals, this means that all of I_μ lies to the right of all of I_ν and that all of I'_μ lies to the left of all of I'_ν, proving the assertion.] These properties characterize special permutations and are illustrated in Fig. 1, in which the lengths of the intervals I_ν with I'_ν above (respectively below) a are denoted by $k_1,\ldots,k_p$ (resp. by $\ell_1,\ldots,\ell_r$), so that $\sum_{i=1}^p k_i = n-a$, $\sum_{j=1}^r \ell_j = a-1$, and $p+r=m$. We denote the corresponding special permutation by $\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$ and also define two sequences of numbers $0=K_0<K_1<\cdots<K_p=n-a$ and $0=L_0<L_1<\cdots<L_r=a-1$ by

$$K_i = k_1+\cdots+k_i \quad (1 \le i \le p), \qquad L_j = l_1+\cdots+l_j \quad (1 \le j \le r). \tag{23}$$

Remark: Denote by $\mathfrak{S}_{n,a}$ $(a=1,\ldots,n)$ the set of special permutations in $\mathfrak{S}_n$ with $\sigma(n)=a$. From the above description we find that $\mathfrak{S}_{n,a} \cong X_{n-a} \times X_{a-1}$, where X_K for $K \ge 0$ is the set of ordered partitions of K (decompositions $K=k_1+\cdots+k_p$ with all $k_i \ge 1$). Clearly the cardinality of X_K equals 1 if $K=0$ (in which case only $p=0$ can occur) and 2^{K-1} if $K \ge 1$ (the ordered partitions of K are in 1:1 correspondence with the subsets of $\{1,\ldots,K-1\}$, each such subset dividing the interval $[0,K] \subset \mathbb{R}$ into intervals of positive integral length), so $|\mathfrak{S}_{n,a}|$ equals 2^{n-2} for $a \in \{1,n\}$ and 2^{n-3} for $1<a<n$, and the whole set $\mathfrak{S}_n^0$ has cardinality $2^{n-3}(n+2)$.

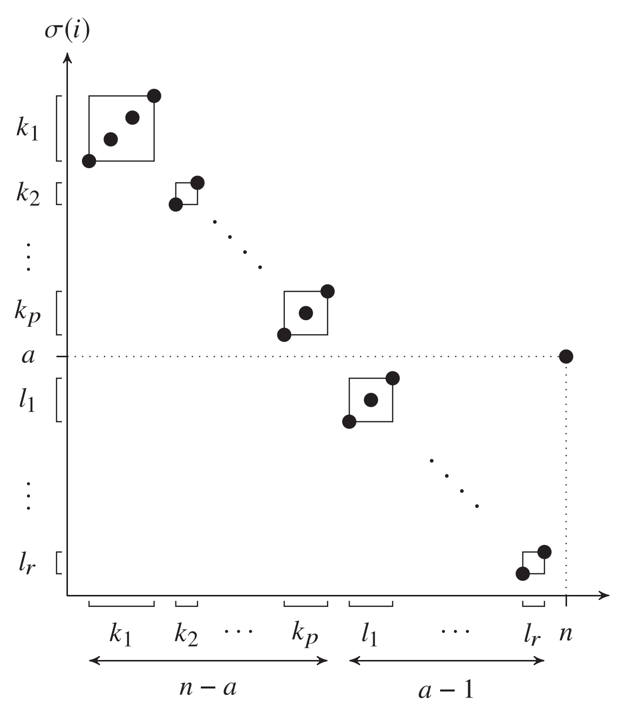

Fig. 1. The special permutation $\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$.

5. Proof of *Theorem 2*

In this section, we use the characterization of special permutations given in *Section 4* to calculate the rational factor $R_\sigma(s)$ and the zeta factors $\widehat{\zeta}_\sigma^{[n]}(s)$ and $\widehat{\zeta}_\sigma^{[<n]}(s)$ appearing in *Lemma 4* explicitly for special permutations σ. We begin with $R_\sigma(s)$.

Lemma 6. *For the special permutation $\sigma=\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$, the quantity $R_\sigma(s)$ defined in Lemma 4 is given by*

$$R_\sigma(s) = (1-q^{k_1+k_2})\cdots(1-q^{k_{p-1}+k_p})\cdot(1-q^{ns-n+a+k_p})\cdot(1-q^{-ns+n-a+l_1+1})\cdot(1-q^{l_1+l_2})\cdots(1-q^{l_{r-1}+l_r}).$$

Proof: By definition,

$$R_\sigma(s) = \prod_{\substack{1\le k\le n-1\\ \sigma^{-1}(\alpha_k)\notin\Delta'}}\Big(1-q^{-\langle\sigma\lambda_s-\rho,\alpha_k\rangle}\Big) = \prod_{\substack{1\le k\le n-1\\ \sigma^{-1}(\alpha_k)\notin\Delta'}}\Big(1-q^{1-\langle\lambda_s,\sigma^{-1}\alpha_k\rangle}\Big).$$

For each k occurring in this product, write $\sigma^{-1}(\alpha_k)=e_i-e_j=:\alpha_{ij}$. Then the condition $\alpha_{ij}\notin\Delta'$ says that the points $(i,\sigma(i)=k)$ and $(j,\sigma(j)=k+1)$ do not belong to the same square block in the picture of the graph of σ given in the last section. From that picture, we see that the ks occurring in the product, in decreasing order, together with the corresponding values of i and j, are given by the first three columns of the following table:

k	$i=\sigma^{-1}(k)$	$j=\sigma^{-1}(k+1)$	$1-\langle\lambda_s,\alpha_{ij}\rangle$
$n-K_\mu\ \ (1\le\mu<p)$	$K_{\mu+1}$	$K_{\mu-1}+1$	$k_\mu+k_{\mu+1}$
a	n	$K_{p-1}+1$	$ns-n+a+k_p$
$a-1$	$n-a+l_1$	n	$-ns+n-a+l_1+1$
$n-L_\nu\ \ (1\le\nu<r)$	$L_{\nu+1}$	$L_{\nu-1}+1$	$l_\nu+l_{\nu+1}$

The fourth column follows from Eq. **11**. The lemma follows. □

We next consider the zeta factor $\widehat{\zeta}_\sigma^{[n]}(s)$.

Lemma 7. *For the special permutation $\sigma=\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$, the zeta factor $\widehat{\zeta}_\sigma^{[n]}(s)$ of $L_\sigma(s)$ is given by*

$$\widehat{\zeta}_\sigma^{[n]}(s) = \frac{\widehat{\zeta}_X(ns-n+a)}{\widehat{\zeta}_X(ns)}.$$

Lemma 7 implies in particular that to normalize $\widehat{\zeta}_\sigma^{[n]}(s)$ we at least need to clear the denominator by multiplying by the zeta factor $\widehat{\zeta}_X(ns)$.

Proof: This is much easier. From $\lambda_s=(ns-n)\varpi+\rho$, we get $\langle\lambda_s,e_i-e_n\rangle=ns-i$. Moreover, by Fig. 1 in *Section 4*, for the special permutation $\sigma=\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$, we have

$$\{e_i-e_n\,:\,1\le i<n,\ \sigma(i)>\sigma(n)\} = \{e_1-e_n,e_2-e_n,\ldots,e_{n-a}-e_n\}.$$

Therefore, by the definition of $\widehat{\zeta}_\sigma^{[n]}(s)$ given in *Lemma 4*, we have

$$\widehat{\zeta}_\sigma^{[n]}(s) = \prod_{\substack{\alpha=e_i-e_n,\,i\le n-1\\ \sigma(i)>\sigma(n)}}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle+1)}\bigg|_{\lambda=\lambda_s} = \prod_{i=1}^{n-a}\frac{\widehat{\zeta}_X(ns-i)}{\widehat{\zeta}_X(ns-i+1)} = \frac{\widehat{\zeta}_X(ns-n+a)}{\widehat{\zeta}_X(ns)}$$

as asserted. □

Finally, we treat the zeta factor $\widehat{\zeta}_\sigma^{[<n]}(s)$. However, with the normalization stated in *Lemma 5*, to obtain the group zeta function $\widehat{\zeta}_X^{SL_n}(s)$, it suffices to investigate the product $\widehat{\zeta}_\sigma^{[<n]}(s)\cdot\prod_{i\ge2}\widehat{\zeta}_X(i)^{-n(i)}$ or, equivalently, by Eq. **18**, the product $\widehat{\zeta}_X(1)^{\#B_\sigma}\prod_{i\ge2}\widehat{\zeta}_X(i)^{n_\sigma(i)-n(i)}$, which we write as $\prod_{i\ge1}\widehat{\zeta}_X(i)^{r_\sigma(i)}$ with

$$r_\sigma(k) = \begin{cases}\#\,B_\sigma & \text{if } k=1,\\ n_\sigma(k)-n(k) & \text{if } k\ge2,\end{cases}$$

where the numbers $n(k)$ are defined, in analogy with the numbers $n_\sigma(k)$ in *Section 3* (Eqs. **19** and **20**), by

$$m(k) = \#\{\alpha>0:\langle\rho,\alpha\rangle=k\}\,,\quad n(k) = m(k)-m(k-1).$$

Clearly $m(k)=n-k$ for $1\le k\le n$ and $n(k)=-1$ for $2\le k\le n$.

Lemma 8. *For the special permutation $\sigma=\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$, we have*

$$\prod_{i\ge1}\widehat{\zeta}_X(i)^{r_\sigma(i)} = \prod_{i=1}^{p}\widehat{v}_{k_i}\cdot\prod_{j=1}^{r}\widehat{v}_{l_j}. \qquad \textbf{[24]}$$

In particular, $r_\sigma(k)\ge0$.

Proof: This is based on a detailed analysis of $r_\sigma(k)$. Obviously,

$$r_\sigma(1)=\#\{\alpha\in\Delta':\sigma\alpha<0\}=\#\{(i,i+1):1\le i\le n-2,\ \sigma(i)>\sigma(i+1)\}.$$

If $k\ge 2$, by definition,

$$\begin{aligned}m(k)-m_\sigma(k)&=\#\{\alpha>0:\langle\rho,\alpha\rangle=k\}-\#\{\alpha\in{\Phi'}^+:\sigma\alpha<0,\langle\rho,\alpha\rangle=k\}\\&=\#\{e_i-e_n:\langle\rho,\alpha\rangle=k\}+\#\{\alpha\in{\Phi'}^+:\sigma\alpha>0,\langle\rho,\alpha\rangle=k\}\\&=1+\#\{\alpha\in{\Phi'}^+:\sigma\alpha>0,\langle\rho,\alpha\rangle=k\},\end{aligned}$$

since, by Eq. **10**, $\{e_i-e_n:\langle\rho,\alpha\rangle=k\}=\{e_{n-k}-e_n\}$. Thus, by applying the characterization graph in *Section 4* for special permutation $\sigma(k_1,\dots,k_p;a;l_1,\dots,l_r)$, we conclude that $\alpha=\alpha_{ij}\in{\Phi'}^+$ satisfying $\sigma\alpha>0$ (or equivalently $\alpha=\alpha_{ij}$ satisfying $i<j\le n-1$ and $\sigma(i)<\sigma(j)$) if and only if i and j belong to the same block, say I_μ for some μ, associated to $\sigma(k_1,\dots,k_p;a;l_1,\dots,l_r)$, and also $\sigma(j)\in I_\mu$ (or equivalently $j+1\in I_\mu$), since otherwise $\sigma(\alpha_{ij})<0$.

Denote by $(m(k)-m_\sigma(k))_\mu$ (resp. $r_{\sigma,\mu}(k)$) the contribution to $m(k)-m_\sigma(k)$ (resp. to $r_\sigma(k)$) of the block I_μ. With the discussion above, we have

$$m(k)-m_\sigma(k)=\sum\nolimits_\mu(m(k)-m_\sigma(k))_\mu\qquad\text{and}\qquad r_\sigma(k)=\sum\nolimits_\mu r_{\sigma,\mu}(k).$$

Fix some μ and let $I_\mu:=\{a+1,a+2,\dots,a+b\}$ with $a,\ b\in\mathbb{Z}_{>0}$. Clearly, when $k=1$, $r_{\sigma,\mu}(1)=\#\{(a+b-1,a+b)\}=1$, since, for other $(i,i+1)$ s, $\sigma(i)<\sigma(i+1)$. Moreover, when $k\ge2$, by Eq. **10** and the characterization of the graph again, we have

$$\begin{aligned}(m(k)-m_\sigma(k))_\mu&=\#\{(i,j):\ i,\ j+1\in I_\mu,\ i<j,\ j=i+k\}\\&=\#\{(i,j):\ a+1\le i<j<a+b,\ j=i+k\}.\end{aligned}$$

Note that, for each fixed i (with $a+1\le i<a+b$),

$$\#\{(i,j):\ a+1\le i<j<a+b,\ j=i+k\}=\begin{cases}1 & i+k<a+b\\ 0 & i+k\ge a+b\end{cases}.$$

Hence, $(m(k)-m_\sigma(k))_\mu=b-(k+1)$. This implies that for all $k\ge1$ $r_{\sigma,\mu}(k)=(m(k-1)-m_\sigma(k-1))_\mu-(m(k)-m_\sigma(k))_\mu=1$. Consequently,

$$\prod_{i\ge1}\widehat{\zeta}_X(k)^{r_{\sigma,\mu}(k)}=\widehat{\zeta}_X(1)\,\widehat{\zeta}_X(2)\,\cdots\,\widehat{\zeta}_X(b).$$

Eq. **24** follows. □

Combining *Lemmas 5*, *6*, *7*, and *8*, we get

$$\begin{aligned}\frac{\widehat{\zeta}_X^{SL_n}(s)}{q^{\frac{n(n-1)}{2}(g-1)}}&=\prod_{i\ge2}\widehat{\zeta}_X(i)^{-n(i)}\cdot\lim_{\lambda\to\lambda_s}\left(\prod_{\alpha\in\Delta_P}(1-q^{-\langle\lambda-\rho,\alpha^\vee\rangle})\cdot\omega_X^{SL_n}(\lambda)\right)\\&=\sum_{a=1}^{n}\sum_{\substack{k_1,\dots,k_p>0\\k_1+\cdots+k_p=n-a}}\frac{\widehat{v}_{k_1}\cdots\widehat{v}_{k_p}}{(1-q^{k_1+k_2})\dots(1-q^{k_{p-1}+k_p})}\cdot\frac{1}{1-q^{ns-n+a+k_p}}\\&\quad\times\widehat{\zeta}(ns-n+a)\sum_{\substack{l_1,\dots,l_r>0\\l_1+\cdots+l_r=a-1}}\frac{1}{1-q^{-ns+n-a+1+l_1}}\cdot\frac{\widehat{v}_{l_1}\cdots\widehat{v}_{l_r}}{(1-q^{l_1+l_2})\dots(1-q^{l_{r-1}+l_r})}.\end{aligned}$$

This completes the proof of *Theorem 2*.

6. The Theorem of Mozgovoy and Reineke

In the previous three sections we have given an explicit formula for the group zeta function associated to a curve over a finite field in the case $(G,P)=(SL_n,P_{n-1,1})$. As explained in the Introduction, our main result (*Theorem 1*) will follow by comparing this formula with the explicit formula for the rank n nonabelian zeta function $\widehat{\zeta}_{X,n}(s)$ found by Mozgovoy and Reineke, namely the following:

Theorem (theorem 7.2 of ref. 5). *The function $\widehat{\zeta}_{X,n}(s)$ is given by*

$$\begin{aligned}\widehat{\zeta}_{X,n}(s)=q^{\binom{n}{2}(g-1)}\sum_{h=1}^{n-1}\sum_{\substack{n_1,\dots,n_h>0\\n_1+\cdots+n_h=n-1}}\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_h}}{\prod_{j=1}^{h-1}(1-q^{n_j+n_{j+1}})}\\\times\left(\frac{\widehat{\zeta}_X(ns)}{1-q^{-ns+n_1+1}}+\sum_{i=1}^{h-1}\frac{(1-q^{n_i+n_{i+1}})\cdot\widehat{\zeta}_X(ns-(n_1+\cdots+n_i))}{(1-q^{ns-(n_1+\cdots+n_{i-1})})(1-q^{-ns+n_1+\cdots+n_{i+1}+1})}+\frac{\widehat{\zeta}_X(ns-n+1)}{1-q^{ns-(n_1+\cdots+n_{k-1})}}\right).\end{aligned}\qquad\textbf{[25]}$$

This already looks very similar to *Theorem 2*, and the precise equality of the two formulas will be verified in *Section 7*. But since the ideas leading to the expressions for the group zeta function and for the nonabelian zeta function are very different, and since the ideas of the proof in ref. 5 are very interesting, we include a brief account of their calculation for the benefit of the interested reader. A reader who is interested only in the proof of the main result, or who is already familiar with the paper (5), can skip this section and go immediately to *Section 7*.

The first ingredient is that of semistable pairs and triples. Fix an integral regular projective curve X over a finite field $\mathbb{F}_q$. By a pair (E, s) over X we mean a vector bundle E on X together with a global section s of E on X. Such pairs form an $\mathbb{F}_q$-linear category, a morphism $(E, s) \to (E', s')$ being an element $(\lambda, f) \in \mathbb{F}_q \times \mathrm{Hom}_X(E, E')$ such that $f \circ s = \lambda\, s'$. A pair (E, s) is called τ semistable ($\tau \in \mathbb{R}$) if $\mu(F) \le \tau$ for any subbundle F of E and $\mu(E/F) \ge \tau$ for any subbundle F of E with $s \in H^0(X, F)$. Here, as usual, $\mu(E)$ denotes the Mumford slope of E. For $(r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ we denote by $\mathcal{M}_X^\tau(r, d)$ the moduli stack of τ-semistable pairs (E, s) of rank r and degree d. If $\tau = d/r$, then this is the same as the usual slope semistability of E, so if we write $\mathcal{M}_X(r, d)$ for the moduli space of semistable bundles of rank r and degree d, then (cf. corollary 3.7 of ref. 5)

$$\sum_{(E,s)\in\mathcal{M}_X^{d/r}(r,d)} \frac{1}{\#\mathrm{Aut}(E, s)} = \frac{1}{q-1} \sum_{E\in\mathcal{M}_X(r,d)} \frac{q^{h^0(X,E)}-1}{\#\mathrm{Aut}\, E}.$$

Next, we consider triples $\mathcal{E} = (E_0, E_1, s)$ consisting of two coherent sheaves $E_0,\ E_1$ on X and a morphism $s : E_1 \to E_0$. These triples form an abelian category which we denote by $\mathcal{A}$. The triple $\mathcal{E} = (E_0, E_1, s)$ is called μ_τ semistable if $\mu_\tau(\mathcal{F}) \le \mu_\tau(\mathcal{E})$ for any subobject $\mathcal{F}$ of $\mathcal{E}$, where

$$\mu_\tau(\mathcal{E}) := \frac{\deg E_0 + \deg E_1 + \tau \cdot \mathrm{rank}\ E_1}{\mathrm{rank}\ E_0 + \mathrm{rank}\ E_1}.$$

We also introduce $\chi(\mathcal{E}, \mathcal{F}) := \sum_{k=0}^{2} (-1)^k \dim \mathrm{Ext}^k_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. It is known that $\chi(\mathcal{E}, \mathcal{F}) = \chi(E_0, F_0) + \chi(E_1, F_1) - \chi(E_1, F_0)$, where as usual, $\chi(E, F) := \dim \mathrm{Hom}(E.F) - \dim \mathrm{Ext}^1(E, F)$. For $\alpha = (r, d),\ \beta = (r', d') \in \mathbb{Z}_{>0} \times \mathbb{Z}$, set $\chi(\alpha) = d - (g-1)r$ and $\langle \alpha, \beta \rangle := 2(rd' - r'd)$. Similarly, for $\underline{\alpha} = (\alpha, v),\ \underline{\beta} = (\beta, w)$ with $v,\ w \in Z_{\ge 0}$ we set $\langle \underline{\alpha}, \underline{\beta} \rangle := \langle \alpha, \beta \rangle - v\, \chi(\beta) + w\, \chi(\alpha)$.

The next ingredients are Hall algebras and integration maps. Let $K_0(\mathrm{St}_{\mathbb{F}_q})$ be the Grothendieck ring of finite-type stacks over $\mathbb{F}_q$ with affine stabilizers and $\mathbb{L}$ be the Lefschetz motive. We introduce the coefficient ring $R = K_0(\mathrm{St}_{\mathbb{F}_q})[\mathbb{L}^{\pm 1/2}]$ and define the quantum affine plane $\mathbb{A}_0$ to be the completion of the algebra $R[x_1, x_2^{\pm 1}]$ with the multiplication

$$x^\alpha \circ x^\beta := (-\mathbb{L}^{1/2})^{\langle \alpha, \beta \rangle} x^{\alpha+\beta}.$$

(Here the completion is defined by requiring that for $f = \sum_{\alpha\in\mathbb{N}\times\mathbb{Z}} f_\alpha x^\alpha \in \mathbb{A}_0$ and any $t \in \mathbb{R}$ there are only finitely many (r, d) with $f_{r,d} \ne 0$ and $\frac{d}{r+1} < t$.) If we further denote by $\mathcal{A}_0$ the category of coherent sheaves on X and by $H(\mathcal{A}_0)$ its associated Hall algebra, whose multiplication $[E] \circ [F]$ counts extensions from $\mathrm{Ext}^1(F, E)$, then we have a morphism of algebras

$$\begin{array}{rccc} I: & H(\mathcal{A}_0) & \longrightarrow & \mathbb{A}_0 \\ & E & \mapsto & (-\mathbb{L}^{1/2})^{\chi(E,E)} \cdot \frac{x^{\mathrm{ch}(E)}}{[\mathrm{Aut} E]}, \end{array}$$

which we call the integration map. Here $\mathrm{ch}(E) := (\mathrm{rank}\, E, \deg E)$. Similarly, if we introduce a second quantum affine plane $\mathbb{A}$ as the completion of the algebra $R[x_1, x_2^{\pm 1}, x_3]$ with the multiplication

$$x^{\underline{\alpha}} \circ x^{\underline{\beta}} := (-\mathbb{L}^{1/2})^{\langle \underline{\alpha}, \underline{\beta} \rangle} x^{\underline{\alpha}+\underline{\beta}},$$

then we have an integration map on the Hall algebra $H(\mathcal{A})$,

$$\begin{array}{rccc} \underline{I}: & H(\mathcal{A}) & \longrightarrow & \mathbb{A} \\ & \mathcal{E} & \mapsto & (-\mathbb{L}^{1/2})^{\chi(\mathcal{E},\mathcal{E})} \cdot \frac{x^{\mathrm{cl}(\mathcal{E})}}{[\mathrm{Aut}\mathcal{E}]}, \end{array}$$

where $\mathrm{cl}(E) := (\mathrm{rank}\ E_0, \deg\ E_0, \mathrm{rank}\ E_1)$. We have $\underline{I}|_{H(\mathcal{A}_0)} = I$. The map $\underline{I}$ is not an algebra morphism in general, but if $\mathrm{Ext}^2(\mathcal{F}, \mathcal{E}) = 0$, then $\underline{I}(\mathcal{E} \circ \mathcal{F}) = \underline{I}(\mathcal{E})\underline{I}(\mathcal{F})$.

The last and most important ingredient of the proof in ref. 5 is a wall-crossing formula. For $\alpha = (r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ and $\tau \in \mathbb{R}$, let

$$\mathfrak{u}(\alpha) := (-\mathbb{L}^{-1/2})^{\chi(\alpha,\alpha)+d}\, [\mathcal{M}_X(\alpha)]$$

be the motivic class of $\mathcal{M}_X(\alpha)$ counting semistable bundles E on X with $\mathrm{ch}\, E = \alpha$, and similarly set

$$\mathfrak{f}_\tau(\alpha) = (\mathbb{L}-1)(-\mathbb{L}^{-1/2})^{\chi(\alpha,\alpha)+d}[\mathcal{M}_X^\tau(\alpha)].$$

We introduce the two generating series

$$\mathfrak{u}_\tau = 1 + \sum_{\mu(\alpha)=\tau} \mathfrak{u}(\alpha)\, x^\alpha \in \mathbb{A}_0\,, \qquad \mathfrak{f}_\tau = \sum_\alpha \mathfrak{f}_\tau(\alpha)\, x^{(\alpha,1)} \in \mathbb{A}.$$

Then the rank n nonabelian zeta function for X can be expressed as

$$\zeta_{X,n}(s) = (q-1)\sum_{k\geq 0}[\mathcal{M}_X(n,kn)]q^{-sk} = q^{\frac{n(n-1)}{2}(g-1)}\sum_{k\geq 0}\mathfrak{f}_k(n,kn)q^{-ks}.$$

We can also identify the moduli stack $\mathcal{M}_X^\infty(1,d)$ with the Hilbert scheme $\mathrm{Hilb}^d X$ or with $\mathrm{Sym}^d X$, the dth symmetric product of X. Consequently,

$$\mathfrak{f}_\infty := x_1x_3\sum_{d\geq 0}[\mathrm{Sym}^d X]\, x_2^d = x_1x_3\, Z_X(x_2),$$

where $Z_X(t)$ is the Artin zeta function with $\zeta_X(s) = Z_X(q^{-s})$. (This can be interpreted as the limiting special case of $\mathfrak{f}_\tau$ as $\tau\to\infty$, since the condition of semistability with respect to τ of a pair (E,s) in the limit $\tau\to\infty$ is equivalent to the requirement that coker(s) is finite.) Finally, set

$$\mathfrak{u}_{\geq\tau} := \prod_{\tau'\geq\tau}^{\rightarrow}\mathfrak{u}_{\tau'},$$

where the product is taken in the decreasing slope order, and, for an element $\mathfrak{g} = \sum_\alpha g_\alpha x^{(\alpha,1)}\in\mathbb{A}$, set

$$\mathfrak{g}\,|_{\mu\leq\tau} := \sum_{\mu(\alpha)<\tau} g_\alpha x^{(\alpha,1)}.$$

Then, using the theory of Hall algebras and wall-crossing techniques, the main result (theorem 5.4 of ref. 5) is the identity

$$\mathfrak{f}_\tau = \left(\mathfrak{u}_{>\tau}^{-1}\circ f_\infty\circ\mathfrak{u}_{\geq\tau}\right)\Big|_{\mu\leq\tau} \qquad (\tau\in\mathbb{R}).$$

Eq. **25** is obtained from this basic formula by a somewhat involved combinatorial discussion, using a "Zagier-type formula" (i.e., one based on the combinatorics in ref. 13) for the motivic classes of moduli spaces of semistable bundles.

7. Proof of *Theorem 1* and Structure of the Function $\zeta_{X,n}(s)$

To complete the proof of *Theorem 1*, we verify the term-by-term equality of the sums appearing in Eqs. **5** and **25**. Clearly, the factor $q^{\binom{n}{2}(g-1)}$ is the same in both cases. Both sums have the form of a linear combination of $\widehat{\zeta}_X(ns-k)$ with $0\leq k\leq n-1$, so we have only to check the equality of the coefficients. The case $k=0$ is immediate: Since $B_0(x)$ is identically 1, the coefficient of $\widehat{\zeta}_X(ns)$ in the sum in Eq. **5** is $B_{n-1}(q^{1-ns})$, which by Eq. **4** is identical with the coefficient of $\widehat{\zeta}_X(ns)$ in the sum in Eq. **25**. (Set $p=h$, $k_i = n_{h+1-i}$.) The case $k=n-1$ is exactly similar or can be deduced from the case $k=0$ by noticing that Eq. **5** is invariant under $k\to n-1-k$, $s\to 1-s$ and Eq. **25** under $n_j\to n_{h+1-j}$, $i\to h-i$, and $s\to 1-s$. If $0<k<n-1$, then the coefficient of $\widehat{\zeta}_X(ns-k)$ in the sum in Eq. **25** can be rewritten as

$$\sum_{0<i<h<n}\ \sum_{\substack{n_1+\cdots+n_i=k\\ n_{i+1}+\cdots+n_h=n-1-k}}\left(\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_i}}{\prod_{j=1}^{i-1}(1-q^{n_j+n_{j+1}})}\cdot\frac{1}{1-q^{ns-k+n_i}}\cdot\frac{\widehat{v}_{n_{i+1}}\cdots\widehat{v}_{n_h}}{\prod_{j=i+1}^{h-1}(1-q^{n_j+n_{j+1}})}\cdot\frac{1}{1-q^{-ns+k+n_{i+1}+1}}\right),$$

and since the summations over the tuples $(n_1,\ldots,n_i)$ with sum k and the tuples $(n_{i+1},\ldots,n_h)$ with sum $n-k-1$ are independent, this equals $B_k(q^{ns-k})B_{n-k-1}(q^{k+1-ns})$ as required. This completes the comparison of Eqs. **5** and **25** and hence the proof of *Theorem 1*.

We end this paper by looking briefly at the structure of the explicit formula for the higher-rank zeta function $\zeta_{X,n}(s)$, and in particular we check that it implies the known properties of this zeta function as listed in the opening paragraph. One of these properties was the functional equation $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$, which, as we have already said, follows immediately from Eq. **5** by interchanging k and $n-k-1$ and using the known functional equation $\widehat{\zeta}_X(1-s) = \widehat{\zeta}_X(s)$. The other one concerned the form of $\zeta_{X,n}(s)$. Here it is more convenient to work with the variables $t=q^{-s}$ and $T=q^{-ns}=t^n$, writing $\zeta_X(s)$ and $\zeta_{X,n}(s)$ as $Z_X(t)$ and $Z_{X,n}(T)$, respectively, and similarly $\widehat{\zeta}_X(s) = \widehat{Z}_X(t)$ and $\widehat{\zeta}_{X,n}(s) = \widehat{Z}_{X,n}(T)$ with $\widehat{Z}_X(t) = t^{1-g}Z_X(t)$, $\widehat{Z}_{X,n}(T) = T^{1-g}Z_{X,n}(T)$. It is well known that $Z_X(t)$ has the form $P(t)/(1-t)(1-qt)$ where $P(t) = P_X(t)$ is a polynomial of degree $2g$, and the assertion is that $Z_{X,n}(T)$, which from the definition Eq. **1** is just a power series in T, has the corresponding form $P_n(T)/(1-T)(1-q^nT)$ where $P_n(T) = P_{X,n}(T)$ is again a polynomial of degree $2g$. In these terms, the formula for the rank n zeta function becomes

$$q^{-\binom{n}{2}(g-1)}\,\widehat{Z}_{X,n}(T) = \sum_{k=0}^{n-1}B_k(q^{-k}T^{-1})\,\widehat{Z}_X(q^kT)\,B_{n-k-1}(q^{k+1}T). \qquad \textbf{[26]}$$

From this it is clear that $\widehat{Z}_{X,n}(T)$ is a rational function of T and grows at most like $\mathrm{O}(T^{g-1})$ as $T\to\infty$ and like $\mathrm{O}(T^{1-g})$ as $T\to 0$, since the definition of the function $B_k(x)$ shows that it is bounded at both 0 and ∞, so the only nontrivial assertion is that $\widehat{Z}_{X,n}(T)$ has at most simple poles at $T=1$ and $T=q^{-n}$ and no other poles. From the definition of $B_k(x)$ and the properties of $\widehat{Z}_X(t)$ we see that every term in Eq. **26** has simple poles at $T=1, q^{-1},\dots,q^{-n}$ (the first factor has simple poles at q^{-i} with $0\le i<k$, the second one at $i=k$ and $i=k+1$, and the third one at $k+1<i\le n$), so the only thing that needs to be checked is that the residues at q^{-i} for $0<i<n$ sum to 0. Denote by R_i $(0\le i\le n)$ the limiting value as $T\to q^{-i}$ of the right-hand side of Eq. **26** multiplied by $1-q^iT$ and by $R_{i,k}$ the corresponding contribution from the kth term, so that $R_i=\sum_{k=0}^{n-1}R_{i,k}$. Suppose that $0<i<n$. Then for $0\le k\le i-2$ we find

$$R_{i,k}\,=\,B_k(q^{i-k})\,\widehat{Z}_X(q^{k-i})\,\widehat{v}_{i-k-1}\,B_{n-i}(q^{i-k-1})$$

and for $k=i-1$ we find

$$R_{i,i-1}\,=\,B_{i-1}(q)\,\widehat{v}_1\,B_{n-i}(1).$$

Since $\widehat{Z}_X(q^{k-i})\widehat{v}_{i-k-1}=\widehat{v}_{i-k}$, these formulas can be written uniformly as

$$R_{i,k}\,=\,B_k(q^{i-k})\,\widehat{v}_{i-k}\,B_{n-i}(q^{i-k-1})\qquad(0\le k\le i-1).$$

The formulas in the other two cases can be computed similarly, but this is not necessary since the abovementioned symmetry of the terms in Eq. **26** under $(k,T)\mapsto(n-1-k,q^{-n}T^{-1})$ implies that $R_{i,k}=-R_{n-i,n-k-1}$ and hence $R_i=S_i-S_{n-i}$ with $S_i=\sum_{k=0}^{i-1}R_{i,k}$. But the formula just proved for $R_{i,k}$ for $0\le k\le i-1$ can be rewritten as

$$R_{i,k}\,=\sum_{1\le s<r\le n}\ \sum_{\substack{n_1,\dots,n_r\ge1\\ n_1+\cdots+n_r=n\\ n_1+\cdots+n_{s-1}=k,\ n_s=i-k}}\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})},$$

so

$$S_i\,=\sum_{1\le s<r\le n}\ \sum_{\substack{n_1,\dots,n_r\ge1\\ n_1+\cdots+n_r=n\\ n_1+\cdots+n_s=i}}\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})},$$

which is visibly symmetric under $i\mapsto n-i$ by replacing n_j by n_{r+1-j} and s by $r+1-s$. This completes the proof of vanishing of R_i for $0<i<n$, and by essentially the same calculation we also get the corresponding formulas

$$R_n\,=\,-\,R_0\,=\sum_{r=1}^{n}\ \sum_{\substack{n_1,\dots,n_r\ge1\\ n_1+\cdots+n_r=n}}\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})}$$

for the two remaining coefficients R_i describing the poles of $\zeta_{X,n}(s)$.

Data Availability. There are no data associated with this paper.

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