III. Zeros of $\hat{\zeta}_{\mathbb{Q},2}(s)$.

III.1. Stabilities & Zeros of $\hat{\zeta}^{T}_{\mathbb{Q},n}(s)$

Thm (Lagarias–Suzuki, Ki) $\hat{\zeta}^{T}_{\mathbb{Q},2}(s)=0 \Rightarrow \mathrm{Re}(s)=\frac{1}{2}$.

However, numerical examples show that

$$\hat{\zeta}^{T}_{\mathbb{Q},3}(s)=0 \not\Rightarrow \mathrm{Re}(s)=\frac{1}{2}. \quad \text{for general } T.$$

But

Thm (i) $\hat{\zeta}_{\mathbb{Q},n\le 5}(s)=0 \Rightarrow \mathrm{Re}(s)=\frac{1}{2}$ $\quad$ $n=3$, Suzuki; $n=4,5$ Ki $\quad (RH)_{n\le 5}$

(ii) (WRH) Weng $\quad \hat{\zeta}_{\mathbb{Q},n}(s)=0 \Rightarrow \mathrm{Re}(s)=\frac{1}{2}$ except for a finite box.

Rk. (i) RH holds for $G=G_2, Sp_2$ by Ki

(ii) WRH holds by Ki-Komori-Suzuki for $\hat{\zeta}^{G/P}_{\mathbb{Q}}(s)$, Assuming a conjecture on parabolic Reduction, stability & the volumes

$\Rightarrow$ Stability plays a Key Rule in the study of zeros.

$$\because \hat{\zeta}_{\mathbb{Q},n}(s)=\hat{\zeta}^{1}_{\mathbb{Q},n}(s).$$

$$\mathcal{M}_{\mathbb{Q},n}[1] \longleftrightarrow \text{Stability}.$$

III.2. RH for $\hat{\zeta}_{\mathbb{Q},2}(s)$

Proof. (L–S). (i) Let $F(z)=Z(\frac{1}{2}+2iz)$ w/ $Z(s)=s(1-s)\cdot\hat{\zeta}(s)$

Then $F(z)=e^{A+Bz}\cdot\prod_{\rho:F(\rho)=0}(1-\frac{z}{\rho})\cdot\exp(\frac{z}{\rho})$ w/ $A,B\in\mathbb{R}$.

(ii) $F(z)=e^{A+B'z}\prod'_{\rho:F(\rho)=0}(1-\frac{z}{\rho})$ w/ $\prod'$ means pairs ρ w/ $1-\rho$, $\bar{\rho}$ w/ $1-\bar{\rho}$.

(iii) $\dfrac{(x_0-\rho)^2+(y_0-\frac{1}{4})^2}{(x_0-\rho)^2+(y_0+\frac{1}{4})^2}\begin{cases}<1, & y_0>0\\ >1, & y_0<0\end{cases} \Rightarrow RH.$

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