II Rankin–Selberg & Zagier Method

II.1. Laplace Operator & Stokes' Theorem

$$\Delta_F = \sum_{\sigma:\mathbb{R}} \Delta_\sigma + \sum_{\tau:\mathbb{C}} \Delta_\tau \quad \text{w/} \quad \begin{cases} \Delta_\sigma = y_\sigma^2\left(\frac{\partial^2}{\partial x_\sigma^2}+\frac{\partial^2}{\partial y_\sigma^2}\right) \\ \Delta_\tau = r_\tau^2\left(\frac{\partial^2}{\partial x_\tau^2}+\frac{\partial^2}{\partial y_\tau^2}+\frac{\partial^2}{\partial r_\tau^2}\right) - r_\tau\frac{\partial}{\partial r_\tau} \end{cases}$$

hyperbolic Laplacian

$$?? \iint_{\mathcal{D}_T} \Delta_F\left(\hat{E}_{2,\Omega}(\tau,s)\right) d\mu$$

w/ $\mathcal{D}_T$: = the cpt part of $\mathcal{D}_{SL(\mathcal{O}_F\oplus\Omega)}$ by cutting-off the cuspidal ugh defined by $d(\eta,\tau_\infty) \le \frac{1}{T}$.

$$\begin{cases} \Delta_\sigma(y_\sigma^s) = s(s-1)\,y_\sigma^s \\ \Delta_\tau(r_\tau^{2s}) = 2s(2s-2)\,r_\tau^{2s} \end{cases} \Rightarrow \quad \| \quad \frac{s(s-1)}{r_1+4r_2}\iint_{\mathcal{D}_T} \hat{E}_{2,\Omega}(\tau,s)\,d\mu.$$

Stokes' Thm

$$\iint_{\mathcal{D}_T}\left(\Delta_F\hat{E}_{2,\Omega}(\tau,s)\cdot 1 - \hat{E}_{2,\Omega}(\tau,s)\,\Delta_F 1\right)d\mu$$

$$= \iint_{\partial\mathcal{D}_T} \frac{\partial\hat{E}_{2,\Omega}(\tau,s)}{\partial\nu}\,d\mu$$

w/ $\frac{\partial}{\partial\nu}$: the out normal direction & $d\mu$: induced volume form/ $\partial\mathcal{D}_T$.

II.2. Siegel's trick: Nice way to change variables.

two Main Directions $\begin{cases} \mathrm{Re}\,Z & \text{for } x_\sigma \,\&\, z_\tau: \text{ affine coordinate : easy} \\ \mathrm{Im}\,J & \text{for } y_\sigma \,\&\, r_\tau: \text{ cone coordinates : complicated.} \end{cases}$

However, Siegel found a nice way to see them systematically in the totally real case, i.e., $F \hookrightarrow \mathbb{R}$ $\forall \sigma\in S_\infty$. $(r_2=0)$.

We generalize it to general # field F.

Let $\langle \varepsilon_1,\dots,\varepsilon_{r_1+r_2-1}\rangle_{\mathbb{Z}} = U_F$. ($\exists$ by Dirichlet's unit thm)

$$\Rightarrow \begin{pmatrix} 1 & \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_{r_1+r_2-1}^{(1)}| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\varepsilon_1^{(r_1+r_2)}| & \cdots & \log|\varepsilon_{r_1+r_2-1}^{(r_1+r_2)}| \end{pmatrix} = B \quad \text{invertible.}$$

set $B^{-1} = (e_j^{(i)})_{i=0,\,j=1}^{r_1+r_2-1,\;r_1+r_2}$