② $c\neq 0$.

**Lemma.** $\forall (c,d)\in(\mathcal{O}_F\oplus\mathfrak{a})A/U_F$ w/ $c\neq 0$ & $w\in\mathfrak{a}\mathfrak{b}^{-2}$, we have

$(c, cw+d)\in(\mathcal{O}_F\oplus\mathfrak{a})A/U_F$, $c\neq 0$.

$\Rightarrow \begin{pmatrix} * & * \\ c & d\end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1\end{pmatrix}=\begin{pmatrix} * & * \\ c & cw+d\end{pmatrix} \Rightarrow$ if $R :=$ a system of representatives of $\begin{pmatrix} * & * \\ c & d\end{pmatrix}$ modulo the right action of $A^{-1}\Gamma_{\eta}'\cdot A$

then
$$a_{w'}(\mathrm{Im}\,J(\tau),s)=\frac{1}{\mathrm{vol}(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\sum_{\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in R}\frac{e^{2\pi i\langle w',\frac{d}{c}\rangle}}{N(c)^{2s}}\int_{\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}}\left(\frac{N(\mathrm{Im}\,J(\tau))}{\|c\tau\|^2}\right)^s\cdot e^{-2\pi i\langle w',\mathrm{Re}\,Z(\tau)\rangle}\cdot\prod_{\sigma:\mathbb{R}}dx_\sigma\cdot\prod_{\tau:\mathbb{C}}dx_\tau\, dy_\tau$$

(a) $w'=0$

(a.i) $$\int_{\mathbb{R}}\left(\frac{y}{x^2+y^2}\right)^s dx=\frac{1}{y^s}\int_{\mathbb{R}}\left(\frac{1}{(\frac{x}{y})^2+1}\right)^s d\frac{x}{y}\cdot y = y^{1-s}\cdot\pi^{\frac12}\frac{\Gamma(s-\frac12)}{\Gamma(s)}$$

(a.ii) $$\int_{\mathbb{C}}\left(\frac{r}{|z|^2+r^2}\right)^{2s}dx\,dy=r^{2-2s}\int_{\mathbb{C}}\frac{dx\,dy}{(1+|z|^2)^{2s}}=r^{2-2s}\cdot\frac{\pi}{2s-1}$$

(b) $w'\neq 0$

(b.i) $$\int_{\mathbb{R}}\left(\frac{y}{x^2+y^2}\right)^s e^{-2\pi i|w'|\cdot x}dx=y^{1-s}\cdot\int_{\mathbb{R}}\frac{1}{(1+t^2)^s}\cdot e^{-2\pi i|w'|yt}\,dt$$
$$=2\pi^s\cdot|w'|^{s-\frac12}\cdot y^{\frac12}\cdot\frac{1}{\Gamma(s)}\cdot K_{s-\frac12}(2\pi|w'|y).$$

(b.ii) $$\int_{\mathbb{C}}\left(\frac{r}{|z|^2+r^2}\right)^{2s}\cdot e^{-2\pi i|w'|\cdot x}dx\,dy=\frac{2\pi^{2s}\cdot|w'|^{2s-1}}{\Gamma(2s)}\cdot r\cdot K_{2s-1}(2\pi|w'|\cdot r).$$

**Thm** (Weng)
$$E_{2,\mathfrak{a}}(A;\tau,s)=\zeta([\mathfrak{a}^{-1}\mathfrak{b}],2s)\cdot N(\mathfrak{a}\mathfrak{b}^{-1})^{-2s}\cdot N(\mathrm{Im}\,J(\tau))^s$$
$$+\frac{1}{\mathrm{vol}(\mathfrak{a}\mathfrak{b}^{-2})}\sum_{\begin{pmatrix} * & * \\ c & cw+d\end{pmatrix}\in R}\frac{1}{N(c)^{2s}}\cdot(\pi^{\frac12})^{r_1}\left(\frac{\Gamma(s-\frac12)}{\Gamma(s)}\right)^{r_1}\cdot\left(\frac{\pi}{2s-1}\right)^{r_2}\cdot N(\mathrm{Im}\,J(\tau))^{1-s}$$
$$+\frac{1}{\mathrm{vol}(\mathfrak{a}\mathfrak{b}^{-2})}\sum_{\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in R}\frac{e^{2\pi i\langle w',\frac{d}{c}\rangle}}{N(c)^{2s}}\cdot N(\mathrm{Im}\,J(\tau))^{\frac12}\cdot N(w')^{s-\frac12}$$
$$\times\left(\frac{2\pi^s}{\Gamma(s)}\right)^{r_1}\prod_{\sigma:\mathbb{R}}K_{s-\frac12}(2\pi|w'|_\sigma y_\sigma)$$
$$\cdot\left(\frac{2\pi^{2s}|w'|^{2s-1}}{\Gamma(2s)}\right)^{r_2}\cdot\prod_{\tau:\mathbb{C}}K_{2s-1}(2\pi|w'|_\tau r_\tau).$$