I.2. Fourier Expansions

· $\eta = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ a cusp. Fix $A = \begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix} \in SL_2(F)$ s.t. if $\mathfrak{b} = \alpha\cdot\mathcal{O}_F + \beta\cdot\mathfrak{a}$, $\mathfrak{b}^{-1} = \beta^*\mathcal{O}_F + \alpha^*\cdot\mathfrak{a}$.

& $A^{-1}\eta = \infty$. $A^{-1}\cdot\Gamma_\eta'\cdot A = \left\{\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix} : w \in \mathfrak{a}\mathfrak{b}^{-2}\right\}$

$\Gamma_\eta = \Gamma_\eta' \times \left\{A^{-1}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}A : \varepsilon \in U_F\right\}$.

· $E_{2,\mathfrak{a}}(\tau;s)$ is $SL(\mathcal{O}_F \oplus \mathfrak{a})$-invariant

$\Downarrow$

$E_{2,\mathfrak{a}}(A\tau;s)$ is $\mathfrak{a}\mathfrak{b}^{-2}$-invariant. i.e., $E_{2,\mathfrak{a}}(A\tau;s)$ is invariant under parallel transforms by elements in $\mathfrak{a}\mathfrak{b}^{-2}$.

$\Rightarrow$ $\exists$ Fourier expansion $E_{2,\mathfrak{a}}(A\tau,s) = \sum_{w' \in \underbrace{(\mathfrak{a}\mathfrak{b}^{-2})^\vee}_{\text{dual lattice of } \mathfrak{a}\mathfrak{b}^{-2}}} a_{w'}(\mathrm{Im}\,J(\tau);s)\cdot e^{2\pi i\langle w', \mathrm{Re}\,Z(\tau)\rangle}$

· Set $Q :=$ a fundamental parallelogram of $\mathfrak{a}\mathfrak{b}^{-2}$ in $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$

$$(*)\quad a_{w'}(\mathrm{Im}\,J(\tau);s) := \frac{1}{\mathrm{vol}(\mathfrak{a}\mathfrak{b}^{-2})}\cdot \underbrace{\sum_{\substack{(c,d)\in(\mathcal{O}_F\oplus\mathfrak{a})A/U_F\\ (c,d)\neq(0,0)}}}\int_Q \left(\frac{N(\mathrm{Im}\,J(\tau))}{\|c\tau+d\|}\right)^s\cdot e^{-2\pi i\langle w', \mathrm{Re}\,Z(\tau)\rangle}\ \underbrace{\prod_{\sigma:\mathbb{R}} dx_\sigma\cdot\prod_{\tau:\mathbb{C}} dx_\tau\, dy_\tau}_{\text{standard Lebesgue measure}}$$

Two Cases: $\begin{cases} c=0 \Rightarrow (*) = \sum\left(\dfrac{N(\mathrm{Im}\,J(\tau))}{\|d\|}\right)^2\cdot\displaystyle\int_Q \\ c\neq 0.\end{cases}$

(1) $c=0$ $\begin{cases} w'\neq 0 \Rightarrow \displaystyle\int_Q \bigcirc = 0 \\ w'=0 \Rightarrow \displaystyle\int_Q \bigcirc = \mathrm{vol}(\mathfrak{a}\mathfrak{b}^{-2})\end{cases}$

$$\Rightarrow a_0(\mathrm{Im}\,J(\tau);s) = \sum_{\substack{(0,d)\in\mathcal{O}_F\oplus\mathfrak{a}/U_F\\ d\neq 0}}\left(\frac{N(\mathrm{Im}\,J(\tau))}{\|d\|^2}\right)^s = \left(\sum_{+}\cdot N(d)^{-2s}\right) N(\mathrm{Im}\,J(\tau))^s.$$

Lem (i) $(\mathcal{O}_F\oplus\mathfrak{a})A = (\alpha\mathcal{O}_F + \beta\mathfrak{a}, \alpha^*\mathcal{O}_F + \beta^*\mathfrak{a})/U_F = (\mathfrak{b}\oplus\mathfrak{a}\mathfrak{b}^{-1})/U_F$.

(ii) $\exists$ natural bijection $(\mathfrak{a}\setminus\{0\})/U_F \longrightarrow \{\mathfrak{b}\in[\mathfrak{a}^{-1}] : \mathfrak{b} : \text{integral ideal}\}$

$\bar{a} \longmapsto \mathfrak{b} := a\cdot\mathfrak{a}^{-1}$

(iii) $\zeta([\mathfrak{a}^{-1}];s) := \sum_{\substack{\mathfrak{b}\in[\mathfrak{a}^{-1}]\\ \mathfrak{b}\triangleleft\mathcal{O}_F}} N(\mathfrak{b})^{-s} = N(\mathfrak{a})^s\cdot\sum_{a\in(\mathfrak{a}\setminus\{0\})/U_F} N(a)^{-s}$.

$\Downarrow$

Prop (Weng) $a_0(\mathrm{Im}\,J(\tau),s) = \left(N(\mathfrak{a}^{-1}\mathfrak{b})^{-2s}\cdot\zeta([\mathfrak{a}^{-1}\mathfrak{b}];2s)\right)\cdot N(\mathrm{Im}\,J(\tau))^s$.