Day 8. Rank Two Zeta Functions & Their Zeros: Day Three

L. Weng
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Tsinghua Univ
Beijing.

I. Epstein Zeta Functions & Their Fourier Expansions

I.1 Epstein Zeta Functions & Eisenstein Series

• **Def.** Let $\Lambda = P_{\mathfrak{a}} = (\mathcal{O}_F^{(n-1)} \oplus \mathfrak{a}, g)$ be a rank n-$\mathcal{O}_F$-lattice, then its associated Epstein zeta function is defined by

$$\hat{E}_{P_{\mathfrak{a}} = \mathcal{O}_F^{(n-1)} \oplus \mathfrak{a}, g}(s) := \Gamma_{\mathbb{R}}(ns)^{r_1} \cdot \Gamma_{\mathbb{C}}(ns)^{r_2} \cdot \left(N\mathfrak{a} \cdot |\Delta_F|^{\frac{n}{2}}\right)^s \times \sum_{\substack{x \in \mathcal{O}_F^{n-1} \oplus \mathfrak{a} / U_{F,n}^+ \\ x \neq 0}} \frac{1}{\|x\|_g^{ns}}$$

w/ $U_{F,n}^+ = \{\varepsilon^n : \varepsilon \in U_F, \varepsilon^n \in U_F^+\} = U_F^+ \cap U_F^n$. $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \cdot \Gamma(\frac{s}{2})$ & $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s} \Gamma(s)$

In particular, since $U_{F,2}^+ = U_F^2$, we have

$$\hat{E}_{(\mathcal{O}_F \oplus \mathfrak{a}, g)}(s) = \Gamma_{\mathbb{R}}(2s)^{r_1} \cdot \Gamma_{\mathbb{C}}(2s)^{r_2} \cdot \left(N(\mathfrak{a}) \cdot |\Delta_F|\right)^s \cdot \sum_{\substack{x \in \mathcal{O}_F \oplus \mathfrak{a} / U_F^2 \\ x \neq 0}} \frac{1}{\|x\|_\Lambda^{2s}}.$$

• Recall that if $g = g(g) \leftrightarrow g^t \cdot g$ or $\bar{g}^t \cdot g$ w/ $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get $\forall\, (x, y) \in \mathcal{O}_F \oplus \mathfrak{a}$

$$\|(x, y)\|_\Lambda^2 = \left\|(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|^2 = \prod_{\sigma : \mathbb{R}} \left((a_\sigma x_\sigma + c_\sigma y_\sigma)^2 + (b_\sigma x_\sigma + d_\sigma y_\sigma)^2\right)$$

$$\cdot \prod_{\tau : \mathbb{C}} \left(|a_\tau x_\tau + c_\tau y_\tau|^2 + |b_\tau x_\tau + d_\tau y_\tau|^2\right)$$

w/ $\mathrm{Im}J = (\overbrace{i, \dots, i}^{r_1}; \overbrace{j, \dots, j}^{r_2}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$

$$= \left(\frac{g_\Lambda(\mathrm{Im}J)}{\|x \cdot g_\Lambda(\mathrm{Im}J) + y\|^2}\right)^{-1}$$

$N(\cdot) = N(\mathrm{Im}J(\cdot))$

& $\|x\| := N(x) := \prod_{\sigma : \mathbb{R}} |x|_\sigma \cdot \prod_{\tau : \mathbb{C}} |x|_\tau^2$ $\qquad \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$ ~~$\mapsto \varepsilon, \ \varepsilon^{-1} \ 1 \ \mapsto \varepsilon$~~

$$\hat{E}_{(\mathcal{O}_F \oplus \mathfrak{a}, g)}(s) = \Gamma_{\mathbb{R}}(2s)^{r_1} \cdot \Gamma_{\mathbb{C}}(2s)^{r_2} \cdot \left(N(\mathfrak{a}) \cdot |\Delta_F|\right)^s \cdot \underbrace{\sum_{\substack{(x,y) \in \mathcal{O}_F \oplus \mathfrak{a} / U_F \\ (x,y) \neq (0,0)}} \left(\frac{N(\mathrm{Im}J(\tau_\Lambda))}{\|x \cdot \tau_\Lambda + y\|^2}\right)^s}_{\overset{\text{Def}}{=:}\ \hat{E}_{2,\mathfrak{a}}(\tau_\Lambda; s)}.$$

Lemma: $\hat{E}_{(\mathcal{O}_F \oplus \mathfrak{a}, g)}(s) = \hat{E}_{2,\mathfrak{a}}(\tau_\Lambda; s)$.