III.3. Fundamental Domain for $\Gamma\backslash \mathcal{H}^{r_1}\times \mathbb{H}^{r_2}$

Let $i_\eta : \Gamma_\eta\backslash F_\eta \hookrightarrow \Gamma\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2})$ the induced injection

by the Lemma above, an injection

& $\mathrm{Im}(i_\eta)$ meets only along the bdy

$\mathbb{E} := \{ z\in \mathcal{H}^{r_1}\times\mathbb{H}^{r_2} : \mathrm{Re}\, z_{r_1} \in \Sigma,\ \mathrm{Im}\, J_{r_1} \in \mathbb{R}_{>0}\cdot \mathbb{S}\}$ ($\mathfrak{a}\mathfrak{b}^{-2}$ ↔ $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$; $\mathbb{S} = U_F^2\backslash \mathbb{S}$ ← norm one hypersurface)

Thm (Siegel: totally real; Weng: general)

(i) $D_\eta = A_\eta^{-1}\cdot\mathbb{E}\cap F_\eta$ fund. domain for $\Gamma_\eta\backslash F_\eta$ (where $A_\eta^{-1}\cdot\mathbb{E}$)

(ii) $\exists\ \gamma_1,\dots,\gamma_h = \#Cl_F \in SL(\mathcal{O}_F\oplus\mathfrak{a})$ s.t. (a) $\bigcup_{i=1}^{h} \Gamma(D_{\eta_i})$ is connected

(b) $\mathcal{D}_\Gamma :=$ (the union above) is a fundamental domain for $\Gamma\backslash(\mathcal{H}^{r_1}\times\mathbb{H}^{r_2})$.

IV. Rank Two s.stable $\mathcal{O}_F$-lattices

IV.1. Hayashi's Result: Algebraic Characterization

Thm. (Hayashi) (i) Any rank 1 sublattice of $\Lambda = (\mathcal{O}_F\oplus\mathfrak{a}, g_\Lambda)$ is contained in $\mathfrak{b}^{-1}\binom{x}{y}\cap\Lambda$ w/ $\binom{x}{y}\in F^2\backslash\{\binom{0}{0}\}$ & $\mathfrak{b} := \mathcal{O}_F\cdot x + \mathfrak{a}^{-1}\cdot y$.

(ii) Λ is s.stable $\Leftrightarrow$ $\forall \binom{x}{y}\in F^2\backslash\{\binom{0}{0}\}$

$$\prod_{\sigma\in S_\infty} \left\| \binom{x_\sigma}{y_\sigma} \right\|_{h_\sigma}^2 \geq N(\mathfrak{a}\mathfrak{b}^{-2}) = N(x\cdot\mathcal{O}_F + y\,\mathfrak{a}^{-1})\cdot N(y\cdot\mathcal{O}_F + \mathfrak{a} x).$$

IV.2 Stability & Distance to Cusps

Thm (Weng) $\Lambda = (\mathcal{O}_F\oplus\mathfrak{a}, g)$ w/ $g\in \Gamma\backslash\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$ ($\simeq \tau_\Lambda$) is s.stable

$$\Leftrightarrow\quad d(\eta,\tau_\Lambda) := \frac{1}{\mu(\eta,\tau_\Lambda)} \geq 1 \quad \forall \text{ cusp } \eta\in\Gamma\backslash\mathbb{P}^1(F).$$

IV.3. Moduli Space of semi-stable $\mathcal{O}_F$-lattices of rank 2.

Thm'. $\mathcal{M}_{F,2}^{\mathfrak{a}}[N(\mathfrak{a})\cdot\Delta_F] \cong \{ z\in\mathcal{D}_\Gamma : d(\eta,\tau_\Lambda)\geq 1\ \forall \eta \text{ cusp of } \Gamma\}$.

(finite vol.)

Notation: $\mathcal{D}_T := \mathcal{D}_\Gamma \backslash \cup_{i=1}^{h} \mathcal{F}_i(T)$ w/ $\mathcal{F}_i(T) = A_{\eta_i}\cdot \mathcal{F}_i(T)$

where $\mathcal{F}_i(T) := \{ z\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2} \mid \mathrm{Re}\, z_{r_1}\in\Sigma,\ \mathrm{Im}\, J_{r_1}\in\mathbb{R}_{>T}\cdot \mathbb{S} U_F^2 \}$

w/ $\Sigma = \{(t_1,\dots,t_{r_1}, s_1,\dots,s_{r_2})\in\mathbb{R}_{>0}^{r_1}\times\mathbb{R}_{>0}^{r_2} \mid \prod_i t_i \prod_j s_j^2 = 1\}$ ($=\mathbb{R}_{>0}^{r_1+r_2}$)