III.2. Siegel Type Distance

Guided by Siegel's work on totally real fields ⇝ we for general # field F:

- $\eta = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] \in SL(O_F \oplus \mathfrak{a})\backslash \mathbb{P}^1(F)$ a cusp $\longleftrightarrow$ $[\mathfrak{b} = O_F\cdot\alpha + \mathfrak{a}\cdot\beta] \in Cl_F$, w/ $\alpha, \beta \in O_F$. Γ_η: stablizer group of η in $\Gamma \overset{\shortparallel}{} SL(O_F\oplus\mathfrak{a})$.

& Fix $A = A_\eta = \begin{pmatrix}\alpha & \alpha^\vee \\ \beta & \beta^\vee\end{pmatrix} \in SL_2(F)$ s.t. (i) $A_\eta \cdot \infty = \eta$ (ii) $O_F\cdot\beta^\vee + \mathfrak{a}^{-1}\cdot\alpha^\vee = \mathfrak{b}^{-1}$. (always exists!)

$$\Rightarrow \quad A^{-1}\cdot\Gamma_\eta\cdot A = \left\{ \gamma = \begin{pmatrix} u & z \\ 0 & u^{-1}\end{pmatrix} \in \Gamma \;\middle|\; u \in U_F = O_F^\times,\ z \in \mathfrak{a}\mathfrak{b}^{-2} \right\}.$$

- $\tau = (z_1, \dots, z_{r_1}; P_1, \dots, P_{r_2}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ $\qquad z = x + yi,\ P = z + vj.$

set $$N(\tau) := N(\mathrm{Im}\,J(\tau)) := \prod_{i=1}^{r_1} \mathrm{Im}(z_i) \cdot \prod_{j=1}^{r_2} J(P_j)^2 = y_1 \cdots y_{r_1} \cdot (v_1 \cdots v_{r_2})^2$$

$$\Rightarrow \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F), \quad N(\mathrm{Im}\,J(\gamma\cdot\tau)) = \frac{N(\mathrm{Im}\,J(\tau))}{\| N(c\tau + d)\|^2}.$$ ← Only 2nd row of γ appears!!

Def. Reciprocal Distance

$$\mu(\underset{\text{cusp}}{\eta}, \underset{\text{Modular Point}}{\tau}) := \frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})} \cdot \frac{N(\mathrm{Im}\,J(\tau))}{\| N(-\beta\tau + \alpha)\|^2}$$

$$= N\left(\mathfrak{a}^{-1}\cdot(O_F\cdot\alpha + \mathfrak{a}\beta)^2\right) \cdot \frac{\mathrm{Im}(z_1)\cdots\mathrm{Im}(z_{r_1}) \cdot J(P_1)^2 \cdots J(P_{r_2})^2}{\prod_{i=1}^{r_1} \| -\beta^{(i)} z_i + \alpha^{(i)}\|^2 \cdot \prod_{j=1}^{r_2} \| -\beta^{(j)} P_j + \alpha^{(j)}\|^2}.$$

Lem (i) μ is well-defined

(ii) μ is $SL(O_F \oplus \mathfrak{a})$-invariant, i.e., $\mu(\gamma\eta, \gamma\tau) = \mu(\eta, \tau)$ $\forall \gamma \in SL(O_F \oplus \mathfrak{a})$

(iii) $\exists$ positive constant $C = C(F, \mathfrak{a})$ s.t.

if $\begin{cases} \mu(\eta, \tau) > C \\ \mu(\eta', \tau) > C \end{cases}$ for $\tau \in \mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$, $\eta, \eta' \in \mathbb{P}^1(F)$ then $\eta = \eta'$.

(iv) $\exists$ positive constant $T = T(F)$ s.t. $\forall \tau \in \mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$, $\exists$ cusp η of $SL(O_F\oplus\mathfrak{a})$ s.t. $\mu(\eta, \tau) > T$.

Def. The "sphere of influence" of the cusp η:

$$F_\eta := \{ \tau \in \mathcal{H}^{r_1}\times\mathbb{H}^{r_2} \mid \mu(\eta, \tau) \geq \mu(\eta', \tau)\ \forall \text{ cusp } \eta' \in \mathbb{P}^1(F) \}.$$

Lem. Let $F_\eta^\circ :=$ interior of F_η. If τ & $\gamma\tau$ both belong to F_η° then $\gamma\tau = \tau$

i.e., The action of Γ in F_η° reduces to that of Γ_η.