III. Fundamental Domain for $SL(O_F\oplus\mathfrak{a})\backslash\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$

III.1. Fund. Domain for Γ_∞ in $\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$

• For the cusp $\eta=[\alpha:\beta]\in SL(O_F\oplus\mathfrak{a})\backslash\mathbb{P}^1(F)$, fix $\overset{A_\eta}{A}=\begin{pmatrix}\alpha&\alpha'\\ \beta&\beta'\end{pmatrix}\in SL_2(F)$ s.t.

(i) $A\cdot\infty=[\alpha:\beta]$ (ii) $\Gamma_\eta\subset A^{-1}\underbrace{SL_2(O_F\oplus\mathfrak{a})}_{=\Gamma}A$ is generated by $\begin{cases} z\mapsto z+b & \text{w/ } b\in\mathfrak{a}\mathfrak{b}^{-2}\\ z\mapsto uz & u\in O_F^{*,2}=U_F^2\end{cases}$

w/ $\mathfrak{b}=O_F\alpha+\mathfrak{a}\beta$.

* Consider the map $\mathrm{Im}J:\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}\longrightarrow\mathbb{R}^{r_1+r_2}_{>0}$

$$(z_1,\dots,z_{r_1};P_1,\dots,P_{r_2})\mapsto(\mathrm{Im}(z_1),\dots,\mathrm{Im}(z_{r_1});J(P_1),\dots,J(P_{r_2}))$$

w/ $\begin{cases} z=x+iy\Rightarrow\mathrm{Im}z:=y\\ P=z+rj\Rightarrow J(P):=r.\end{cases}$

$\Downarrow$

(i) $A^{-1}\Gamma_\eta A\backslash\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}\longrightarrow U_F^2\backslash\mathbb{R}^{r_1+r_2}_{>0}$:

a torus bundle w/ fiber = the $n=r_1+2r_2=[F:\mathbb{Q}]$-dimensional torus $(\mathbb{R}^{r_1}\times\mathbb{C}^{r_2})/\mathfrak{a}\mathfrak{b}^{-2}$.

Set J: fundamental domain $\subset\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$.

(ii) The fundamental domain for $U_F^2\backslash\mathbb{R}^{r_1+r_2}_{>0}$: Standard $\Leftrightarrow$ Dirichlet's Unit thm.

$\mathcal{S}:=\{y\in\mathbb{R}^{r_1+r_2}_{>0}: N(y):=\prod_{i=1}^{r_1+r_2}y_i=1\}$ ← the Norm One Hypersurface

$y\downarrow$ $\quad\downarrow\log$ $\qquad$ Trace 0 Hyperplane.

$(\log y_1,\dots,\log y_{r_1+r_2})\in\mathbb{R}^{r_1+r_2-1}:=\{(a_1,\dots,a_{r_1+r_2})\in\mathbb{R}^{r_1+r_2}:\sum_j a_j=0\}$

& $U_F^2\times\mathcal{S}\to\mathcal{S}$
$(\varepsilon,a)\mapsto(a_i+\log\varepsilon^{(i)})$.

Dirichlet's Unit thm $\Rightarrow$ $\log(U_F^2)\hookrightarrow\mathbb{R}^{r_1+r_2-1}$ a full rank lattice

The Cone $\mathbb{R}_{>0}\cdot\mathcal{S}_{U_F^2}\subset\mathbb{R}^{r_1+r_2}_{>0}$ $\Leftarrow$ $\mathcal{S}_{U_F^2}\subset\mathbb{R}^{r_1+r_2-1}$: fundamental parallelopiped for the quotient.

the fundamental domain of $U_F^2\backslash\mathbb{R}^{r_1+r_2}$

• $\mathrm{Re}Z(z_1,\dots,z_{r_1};P_1,\dots,P_{r_2}):=(\mathrm{Re}(z_1),\dots,\mathrm{Re}(z_{r_1});Z(P_1),\dots,Z(P_{r_2}))$

w/ $\mathrm{Re}(x+iy)=x$ & $Z(z+rj)=z$.

Prop. (triv). A fund. domain for $A_\eta^{-1}\Gamma_\eta A_\eta\backslash\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$ is given by

$$\mathbb{E}:=\{z\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}\mid \mathrm{Re}Z(z)\in J,\ \mathrm{Im}J(z)\in\mathbb{R}_{>0}\cdot\mathcal{S}_{U_F^2}\}$$

Set $\mathcal{F}_\eta:=A_\eta^{-1}\cdot\mathbb{E}$