II.2. Cusps & Ideal Classes Correspondence

Thm (Maaß: $\mathfrak{a} = \mathcal{O}_F$, very classical) $\exists$ natural 1:1 correspondence

$$\pi: C_\Gamma \to Cl_F$$
$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \mapsto [\mathcal{O}_F \alpha + \mathfrak{a}\beta]$$

Proof. Chinese Remainder Thm $\Rightarrow$ $\forall$ fractional ideal $\mathfrak{b}$ of F

$\exists\, \alpha_{\mathfrak{b}}, \beta_{\mathfrak{b}} \in F$ s.t. $\mathfrak{b} = \mathcal{O}_F \cdot \alpha_{\mathfrak{b}} + \mathfrak{a}\, \beta_{\mathfrak{b}}$.

Hence $\pi^{-1}: Cl_F \to C_\Gamma$

$$[\mathfrak{b}] \mapsto \begin{bmatrix}\alpha_{\mathfrak{b}}\\ \beta_{\mathfrak{b}}\end{bmatrix}.$$

II.3. Stablizer Groups of Cusps

*. Reduce to $\infty = \begin{bmatrix}1\\ 0\end{bmatrix}$.

Lem. $\forall \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \in \mathbb{P}^1(F)$, $\exists\, M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} = \begin{pmatrix}\alpha & \alpha^* \\ \beta & \beta^*\end{pmatrix} \in SL_2(F)$ ← not in $SL(\mathcal{O}_F \oplus \mathfrak{a})$

s.t. $M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} \cdot \infty = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

* $\eta = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ a cusp. $\Gamma_\eta = \mathrm{Stab}_{SL(\mathcal{O}_F \oplus \mathfrak{a})}\, \eta = \{\gamma \in \Gamma : \gamma \cdot \eta = \eta\}$.

set $\mathfrak{b}_\eta := \mathcal{O}_F \alpha + \mathfrak{a}\beta$

Lem. (i) $M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} \cdot \Gamma_{\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]} \cdot M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}^{-1} = \Gamma_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}$

(ii) $M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}^{-1} \cdot \Gamma_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} \cdot M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} = \left\{ \begin{pmatrix}u & z\\ 0 & u^{-1}\end{pmatrix} : u \in \mathcal{O}_F^*,\ z \in \mathfrak{a} \cdot \mathfrak{b}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}^{-2} \right\}$

reason $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is called a cusp

associated lattice for the cusp $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

*. Set $\Gamma'_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] = \eta} = \left\{ M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} \cdot \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} \cdot M_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}^{-1} : z \in \mathfrak{a} \cdot \mathfrak{b}^{-2}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} \right\}$

then $\Gamma_\eta = \Gamma'_\eta \times (\mathcal{O}_F^*)^2$ $\quad \because \begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix} : z \mapsto \frac{uz}{u^{-1}} = u^2 z.$

w/ $\{u^2 : u \in \mathcal{O}_F^*\} \overset{\text{sl}}{\simeq} \left\{ A \begin{pmatrix}1 & 0\\ 0 & u^2\end{pmatrix} \cdot A^{-1} : u \in \mathcal{O}_F^* \right\}$

w/ $A = M_\eta$