Day 7. Rank Two Zeta Functions & Their Zeros: Part Two

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I. Upper Half Space Model.

I.1. Upper half plane ⇒ Metrics on $\mathbb{R}^2$

· $\mathcal{H} := \{ z = x+iy \in \mathbb{C} \mid x \in \mathbb{R},\ y \in \mathbb{R}_+^* \}$ the upper half plane

· The group $SL_2(\mathbb{R})$ naturally acts on $\mathcal{H}$ via $z \mapsto Mz := \frac{az+b}{cz+d}$ $\forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $z \in \mathcal{H}$.
transitively

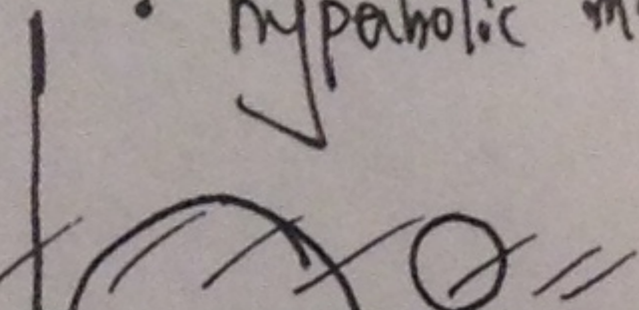
$\because \; = x^* + y^* i \Rightarrow y^* = \frac{y}{(cx+d)^2 + c^2y^2} > 0$

· $\mathrm{Stab}_{SL_2(\mathbb{R})}\, i = SO(2)$ & $SL_2(\mathbb{R})/SO(2) \simeq \mathcal{H}$, $i = (0,1)$

$g \mapsto g.i$

$0 < g^t g$: metrics$/\mathbb{R}^2$

· $\exists$ hyperbolic metric given by line element $ds^2 = \frac{dx^2+dy^2}{y^2}$ w/ volume form $d\mu = \frac{dx \wedge dy}{y^2}$



& Laplace-Beltrami operator $\Delta := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

· $\mathcal{H}$ admits the real line $\mathbb{R}$ as its boundary ⇒ Compactification = add $\mathbb{P}^1(\mathbb{R})$

& $SL_2(\mathbb{R})$'s action extends to $\mathbb{P}^1(\mathbb{R})$: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ $\forall M \in SL_2(\mathbb{R})$, as above.

I.2 Upper half space ⇒ Metrics on $\mathbb{C}^2$

· 3 dimensional hyperbolic space $\mathbb{H} := \mathbb{C} \times]0, \infty[= \{ (z, r) \mid z = x+iy \in \mathbb{C},\ r \in \mathbb{R}_+^* \}$

$= \{ (x, y, r) \mid x, y \in \mathbb{R},\ r \in \mathbb{R}_+^* \}$

· think $\mathbb{H}$ as a subset of Hamilton's quaternions w/ $1, i, j, k$ as $\mathbb{R}$-basis

$\mathbb{H} \ni P = (z, r) = (x, y, r) =: z + rj$ w/ $z = x+iy$, $j = (0,0,1)$

· $SL_2(\mathbb{C})$ acts on $\mathbb{H}$ & its bdy $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{C})$ as follows:

$P \mapsto M \cdot P := (aP+b)\cdot(cP+d)^{-1}$ $\forall P \in \mathbb{H}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$
for quaternions.

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

$\bar{g}^t g > 0$ metrics$/\mathbb{C}^2$

· The action is transitive & $\mathrm{Stab}_{SL_2(\mathbb{C})}\, j = SU(2) \Rightarrow \mathbb{H} \simeq SL_2(\mathbb{C})/SU(2)$, $j = (0,0,1)$

· Hyperbolic metric$/\mathbb{H}$ w/ line element $ds^2 = \frac{dx^2+dy^2+dr^2}{r^2}$ $\quad g\cdot j \leftarrow [g]$

& volume form $d\mu = dx \wedge dy \wedge dr / r^3$

& Laplace-Beltrami operator $\Delta = r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}$.