Example. $P = \mathfrak{a}$: fractional ideal of F. equipped w/ the innerproduct on $E \otimes \mathbb{R}$. (induced from F_σ)

$$\Rightarrow \underset{\substack{\Downarrow \\ (\mathfrak{a}, \#)}}{\mathrm{Vol}_{Leb}(\bar{\mathfrak{a}})} = 2^{-r_2} \cdot \big(\underset{\substack{\uparrow \\ \text{Norm of } \mathfrak{a}}}{N(\mathfrak{a})} \cdot |\Delta_F|^{\frac{1}{2}}\big)$$

$\mathfrak{a} \subseteq F \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})$ ← standard metric

Def. Canonical volume of Λ:

$$\mathrm{vol}(\Lambda) = \mathrm{Vol}_{can}(\Lambda) := 2^{r_2 \cdot \mathrm{rk}_{O_F}(\Lambda)} \cdot \mathrm{Vol}_{Leb}(\Lambda)$$

$$\Rightarrow \mathrm{vol}(\bar{\mathfrak{a}}) = N(\mathfrak{a}) \cdot |\Delta_F|^{\frac{1}{2}} \quad \& \quad \mathrm{vol}(\bar{O}_F) = |\Delta_F|^{\frac{1}{2}}.$$

<u>Arakelov-Riemann-Roch Thm.</u>

$$-\log(\mathrm{vol}(\Lambda)) =: \chi(F, \Lambda) = \deg(\Lambda) - \frac{\mathrm{rk}_{O_F}(\Lambda)}{2} \cdot \log(\Delta_F).$$

Def. An O_F-lattice Λ is semi-stable

if $\forall \Lambda_1 \le \Lambda \quad \mathrm{vol}(\Lambda_1)^{\mathrm{rk}_{O_F}(\Lambda)} \geq \mathrm{vol}(\Lambda)^{\mathrm{rk}_{O_F}(\Lambda_1)}$

$$\Leftrightarrow \forall \Lambda_1 \le \Lambda \quad \frac{\deg(\Lambda_1)}{\mathrm{rk}_{O_F}(\Lambda_1)} \le \frac{\deg(\Lambda)}{\mathrm{rk}_{O_F}(\Lambda)}.$$ ← (Analogue of (Mumford) for bdls / curves.

$$\Leftrightarrow \mathrm{Vol}_{Leb}(\Lambda_1)^{\mathrm{rk}_{O_F}(\Lambda)} \geq \mathrm{vol}_{Leb}(\Lambda)^{\mathrm{rk}_{O_F}(\Lambda_1)}.$$

III. Moduli Spaces of Semi-Stable O_F-lattices

$\mathcal{M}_{F,n}$: moduli space of semi-stable O_F-lattices Λ of rank n (O_F-rank) $= \{O_F\text{-lattices of rank } n\} / \sim_{\mathrm{Isom}}$

$\mathcal{M}_{F,n}[T]$: subspace of $\mathcal{M}_{F,n}$ consisting of Λ of volume T.

Facts. (i) $\mathcal{M}_{F,n} = \coprod_{[\mathfrak{a}] \in Cl_F} \mathcal{M}_{F,n}^{\mathfrak{a}}$ ← consisting of Λ s.t. $P_\Lambda \simeq O_F^{n-1} \oplus \mathfrak{a}$

($[\mathfrak{a}] \in Cl_F$ ← finite decomposition)

(ii) $\mathcal{M}_{F,n} = \coprod_{T \in \mathbb{R}_{>0}} \mathcal{M}_{F,n}[T]$

($T \in \mathbb{R}_{>0}$ ← continuous decomposition)

(iii) $\mathcal{M}_{F,n}$ & $\mathcal{M}_{F,n}[T]$ admit natural metric structures, corresponding measures satisfying $d\mu = d\mu_T \times \frac{dT}{T}$

(iv) $\mathcal{M}_{F,n}[T]$: compact & $\mathcal{M}_{F,n}[T] \simeq \mathcal{M}_{F,n}[T']$ $\forall T, T'$