# II. $O_F$-lattices

## II.1. Projective $O_F$-Modules.

$F$: # field, $O_F$: integral ring, $M$: $O_F$-module

**Def.** $M$ is projective if $\exists$ $O_F$-module $N$ s.t. $M \oplus N$ is free.

$\Rightarrow$ (i) For any exact sequence of projective $O_F$-modules $0 \to M_0 \to M_1 \to M \to 0 \Rightarrow M_1 = M \oplus M_0$.

(ii) All fractional $O_F$-ideals are projective.

(iii) rank 1 projective $O_F$-modules in $F$ are simply fractional $O_F$-modules.

$Cl_F := \{\text{fractional ideals}\}/(\text{principal ideals})$ : Class group of $F$.

$h = h_F := \# Cl_F$ · class number $< \infty$. $\Rightarrow \exists$ integral ideals $\mathfrak{a}_i$, $i = 1, \dots, h$

s.t. (i) any rank 1 projective $O_F$-module $\simeq \mathfrak{a}_i$ $\exists i$.

(ii) $\mathfrak{a}_i \not\simeq \mathfrak{a}_j$ if $i \neq j$.

$\mathfrak{a}$: running symbol.

$P_{\mathfrak{a}} := P_{n,\mathfrak{a}} := O_F^{n-1} \oplus \mathfrak{a} \hookrightarrow F^n$. $\forall$ fractional ideal $\mathfrak{a}$

**Prop.** (0) $P_{n,\mathfrak{a}}$ : rank $n$ projective $O_F$-module

(1) $\forall$ fractional ideals $\mathfrak{a}, \mathfrak{b}$, $P_{n,\mathfrak{a}} \simeq P_{n,\mathfrak{b}} \Leftrightarrow \mathfrak{a} \simeq \mathfrak{b}$.

(2) $\forall$ projective $O_F$-module $P$, $\exists\, n, \mathfrak{a}$ s.t. $P \simeq P_{n,\mathfrak{a}}$.

**Lem.** For an $O_F$-isomorphism $A: P_{n,\mathfrak{a}} \to P_{n,\mathfrak{b}}$ induced from $A \in GL_n(F)$,

$\mathfrak{b} \simeq \det(A) \cdot \mathfrak{a}$.

In particular, if $\mathfrak{a}$ & $\mathfrak{b}$ are integral, (i) $\det A \in U_F = O_F^\times$ ← the group of units in $F$.

(ii) $\mathfrak{a} \simeq \mathfrak{b}$

(iii) $A \in \mathrm{Aut}_{O_F}(P_{n,\mathfrak{a}}) =: GL(P_{n,\mathfrak{a}})$.

## II.2. Semi-stable $O_F$-lattices.

$\sigma$: Archimedean place of $F$. $F_\sigma$ = $\sigma$-completion $= \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$

$F^n \hookrightarrow (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^n$

$O_F^{n-1} \oplus \mathfrak{a}$

**Def.** An $O_F$-lattice $\Lambda = (P, g)$ w/ $P = P(\Lambda)$ a projective $O_F$-module of finite rank.

$V_\sigma := P \otimes_{O_F} F_\sigma$ & $V = P \otimes_{\mathbb{Z}} \mathbb{R} = \prod_{\sigma \in S_\infty} V_\sigma$

$[F:\mathbb{Q}] = r_1 + 2r_2$ $r_1$: # real places. $r_2$: # of complex places.