Day 6. Rank Two Zeta Function & Its Zeros: Part One

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I. $\hat{\zeta}_{\mathbb{Q},2}(s) = \hat{\zeta}_{\mathbb{Q}}^{SL_2}(s)$: A Toy Model

* $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} \simeq SL_2(\mathbb{R})/SO(2)$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$

$\Rightarrow$ "Fundamental Domain" $\overline{\mathcal{D}}_{SL_2(\mathbb{Z})} = \left\{ z = x+iy \in \mathcal{H} \,\middle|\, \begin{array}{l} |x| \leq 1/2 \\ y > 0, \end{array} \; x^2+y^2 \geq 1 \right\} =: D$

* $\forall \tau \in \mathcal{D}^{\circ}_{SL_2(\mathbb{Z})}$, form the associated lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ $\quad \tau = x+iy$

$\Rightarrow \mathrm{Vol}(\Lambda_\tau) = y$

& the completed Eisenstein series

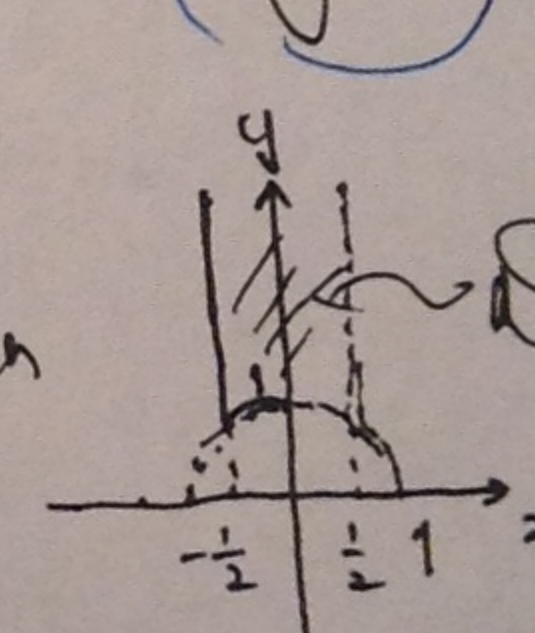
$$\hat{E}(\Lambda, s) = \hat{E}(\tau; s) := \Gamma_{\mathbb{R}}(2s) \cdot \sum_{x \in \Lambda \setminus \{0\}} \frac{1}{\|x\|^{2s}} \quad \Big| \quad \mathrm{Vol}(\sqrt{y} \cdot \Lambda_\tau) = 1$$

$$\hat{E}\left(\tfrac{1}{\sqrt{y}} \Lambda_\tau; s\right) = \Gamma_{\mathbb{R}}(2s) \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{y^s}{|m\tau+n|^{2s}} =: \hat{E}(\tau, s)$$

* Naturally, we are led to the integral $\displaystyle\int_D \hat{E}(z,s) \frac{dx \wedge dy}{y^2}$ ← Hyperbolic volume

However, this integration diverges !!!

Indeed, near the only cusp $y = \infty$, Chowla–Selberg formula



$\Rightarrow$ $\hat{E}(z,s)$ has the Fourier expansion

$$\hat{E}(z;s) = \sum_{n=-\infty}^{\infty} a_n(y,s) \cdot e^{2\pi i x}$$

w/ $$a_n(y;s) = \begin{cases} \hat{\zeta}(2s) \cdot y^s + \hat{\zeta}(2-2s) \cdot y^{1-s} & \text{if } n = 0 \\ 2|n|^{s-1/2} \sigma_{1-2s}(|n|) \sqrt{y} \cdot K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n \neq 0 \end{cases}$$

Constant term along B. $\quad \| \quad \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$.

Whittaker coef

where. $\sigma_s(n) := \sum_{d|n} d^s$, $\quad K_s(y) := \frac{1}{2}\int_0^\infty e^{-y(t+\frac{1}{t})/2} \cdot t^s \frac{dt}{t}$

} the K-Bessel function

Moreover $|K_s(y)| \leq e^{-y/2} K_{\mathrm{Re}(s)}(2)$ if $y > 4$

$\{ K_s = K_{-s}$.

$\Rightarrow$ $a_n(y,s)$, $n \neq 0$ decays exponentially & $a_0(y,s) \sim y^s$ slow growth $\displaystyle\int^\infty y^s \frac{dy}{y^2}$ diverges