III.3 Advanced Rankin-Selberg & Zagier Method.

Def. Langlands' Eisenstein series ∀ φ: M-level automorphic fcn (P=M as variety!)

$$E^{G/P}(\varphi,\lambda;g) := \hat{\sum_{\delta\in P(\mathbb{Z})\backslash G(\mathbb{Z})}} m_P(\delta g)^{\lambda+\rho_P}\cdot\varphi(\delta g),\quad \mathrm{Re}\,\lambda\in \underset{\substack{\uparrow\\ \text{positive Weyl}\\ \text{(acute) Chamber}}}{\mathcal{C}_P^+}$$

$\{H\in\mathfrak{a}_P : \langle\alpha,H\rangle>0\ \forall\alpha\in\Delta_P\}$.

Thm (Weng)

$$\int_{\mathcal{M}_{\mathbb{Q},n}[1]} \underset{\substack{\uparrow\\ \text{Eisenstein}\\ \text{Period.}}}{E^{G/B}(\mathbb{1},\lambda;g)}\,dg = n\cdot\underbrace{\hat{\sum_{w\in W}}\frac{1}{\prod_{\alpha\in\Delta}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\substack{\alpha>0\\ w\alpha<0}}\frac{\hat\zeta_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle)}{\hat\zeta_{\mathbb{Q}}(\langle\lambda,\alpha^\vee\rangle+1)}}_{\substack{\shortparallel\\ \omega_{\mathbb{Q}}^{SL_n}(\lambda)}}$$

$\omega_{\mathbb{Q}}^{SL_n}(\lambda)$ ∼ several variable. Period of SL_n for $\mathbb{Q}$.

Proof.
- JLR's work on Eisenstein Period ⇝ Intertwining operator
- Previous thm
- Gindikin-Karpelevich. ⇝ Realizing the intertwining operator

III.4. Epstein, Koecher, Siegel zetas & Siegel-Eisenstein series versus Langlands' Eis.

- For positive definite symmetric matrix Y w/ $|Y|=1$ & $s=(s_1,\dots,s_n)$, set power function $p_{-s}(Y)=\prod_{j=1}^{n}|Y_j|^{-s_j}$ $\quad Y\ \left(\begin{array}{c|c} Y_j & \\ \hline & \end{array}\right)$

  & Siegel's Eisenstein series

$$E_{(n)}(s|Y) := \hat{\sum_{\gamma\in \underset{\substack{\shortparallel\\ SL\cap B}}{\Gamma_n}\backslash \underset{\substack{\shortparallel\\ B}}{P_{1,\dots,1}}}} p_{-s}(Y[\gamma])\qquad \mathrm{Re}(s_i)>1$$

- set $a_i=|Y_i|/|Y_{i-1}|$ ⇒ $p_{-s}(Y)=a_1^{-(s_1+s_2+\dots+s_{n-1})}a_2^{-(s_2+\dots+s_{n-1})}\cdots a_{n-1}^{-s_{n-1}}$.

Prop. (1) $\underset{\substack{\uparrow\\ \text{Langlands}\\ \text{(Modern)}}}{E(\mathbb{1},\lambda;g)} = \underset{\substack{\uparrow\\ \text{Siegel}\\ \text{(classical)}}}{E_{(n)}(s|Y)}$ w/ $\lambda=(z_1,\dots,z_n)$ & $\sum z_i=0$

$$\begin{cases} 2s_1=1+(z_1-z_2)\\ 2s_2=1+(z_2-z_3)\\ \quad\vdots\\ 2s_{n-1}=1+(z_{n-1}-z_n)\end{cases}\qquad \underline{Y:=g^T\cdot g.}$$

(2) Set $2ns-n+1=z_1-z_2$

$$\underset{\substack{\uparrow\\ \text{Epstein}\\ \text{zeta}}}{E(\Lambda(g);s)} = \mathrm{Res}_{z_2-z_3=1,\dots,z_{n-1}-z_n=1} E(\mathbb{1};z_1,\dots,z_n)(g).$$

in fact: an Eisenstein series

⇒ **Thm** (Weng) (i) $\underset{\substack{\uparrow\\ \text{(Special Uniformity of Zetas)}}}{\hat\zeta_{\mathbb{Q},n}(s)} \underset{\uparrow}{\doteq} \hat\zeta_{\mathbb{Q}}^{SL_n\,|\,G/P}(s)$! Ki-Komori-Suzuki / & Langlands

up to constant factor

(ii) Outside a finite box, the RH holds.