III. Periods.

III.1. Non-Abelian Zeta & Eisenstein Series

For simplicity, $F=\mathbb{Q}$.

$$\hat{\zeta}_{\mathbb{Q},n}(s) = \int_{\bigcup_{T\geq 0}\mathcal{M}_{\mathbb{Q},n}[T]} (e^{h^0(\mathbb{Q},\Lambda)}-1)\,(e^{-s})^{\deg(\Lambda)}\,d\mu(\Lambda)$$

$$= \int_0^{\infty} T^{s}\frac{dT}{T}\int_{\mathcal{M}_{\mathbb{Q},n}[1]} (e^{\underbrace{h^0(\mathbb{Q},T^{\frac{1}{n}}\cdot\Lambda)}_{\log\left(\sum_{x\in\Lambda}\exp(-\pi\|x\|^2\cdot T^{\frac{2}{n}})\right)}}-1)\,d\mu(\Lambda) \qquad \text{using } \mathcal{M}_{\mathbb{Q},n}[T]\to\mathcal{M}_{\mathbb{Q},n}[1],\ \Lambda\mapsto T^{\frac{1}{n}}\Lambda.$$

Mellin transform

$$\overset{\searrow}{=} \frac{n}{2}\cdot\Gamma_{\mathbb{R}}(ns)\cdot\int_{\mathcal{M}_{\mathbb{Q},n}[1]}\left(\sum_{x\in\Lambda\setminus\{0\}}\|x\|^{-ns}\right)d\mu_0(\Lambda)$$

w/ $\Gamma_{\mathbb{R}}(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$

$$= \frac{n}{2}\int_{\mathcal{M}_{\mathbb{Q},n}[1]}\hat{E}(\Lambda,\tfrac{n}{2}s)\,d\mu(\Lambda)$$

w/ $\Gamma(s):=\int_0^\infty e^{-t}t^s\frac{dt}{t}$.

w/ $\hat{E}(\Lambda;s):=\Gamma_{\mathbb{R}}(2s)\cdot\sum_{x\in\Lambda\setminus\{0\}}\|x\|^{-2s}$.

III.2. Arthur-Lafforgue truncation

$G=SL_n$ $\quad \phi: G(\mathbb{Z})\backslash G(\mathbb{R})/\underbrace{K=SO(n)}\to\mathbb{C}$ $\quad T\in V$ $\quad \hat{\tau}_P:=\{H\in\mathfrak{a}_P \mid \langle\varpi_\alpha,H\rangle\geq 0\ \forall\alpha\in\Delta_P\}$

Arthur's truncation: For $T\gg 0$ i.e., T s.t. $\langle T,\alpha\rangle\gg 0$ $(\forall\alpha\in\Delta)$ | Obtuse weyl chamber

$$(\Lambda^T\phi)(g):=\sum_{P:\text{standard}}(-1)^{rk(P)}\sum_{\delta\in P(\mathbb{Z})\backslash G(\mathbb{Z})}\phi_P(\delta g)\cdot\hat{\tau}_P(H_P(\delta g)-T).$$

| w/ P: standard parabolic subgroups

w/ $H_P: P(\mathbb{R})\to\mathfrak{a}_P$

$H_P(g)=\log_M m_P(g)$ ($P=MN$ Levi decomposition)

s.t. $\langle H_P(g),\lambda\rangle = p^\lambda := \prod_i |\chi_i(p)|^{s_i}$

$\lambda=\sum s_i\lambda_i\in\mathfrak{a}^*_{P,\mathbb{C}}$

using Iwasawa decomposition

Thm (Lafforgue: function field; Weng: # field)

$$(\Lambda^0\mathbb{1})(g)\neq 0 \iff (\mathbb{Z}^n, g^t g) \text{ s.stable of rank } n \text{ \& volume } 1.$$

$g \leftrightarrow g^t g$

Cor.

$$\int_{G(\mathbb{Z})\backslash G(\mathbb{R})/K}(\Lambda^0\phi)(g)\,dg = \int_{\mathcal{M}_{\mathbb{Q},n}[1]}\phi(g)\,d\mu(g)$$

← Arthur's period.

$\because$) (i) $\Lambda^T\circ\Lambda^T=\Lambda^T$

(ii) Λ^T: self-adjoint by Arthur but w/ the help of parabolic reduction & stability (Last lecture).