Thm. TFAE: (i) $\alpha \in GL_1(\mathbb{A}) = \mathbb{I}_F$ is positive (Positivity)

(ii) $\alpha \in GL_1(\mathbb{A})$ is ample (ample)

(iii) $\lim\limits_{m\to\infty} h^1(F, \alpha^m g) = 0 \ \forall\, g \in GL_n(F)$ (Vanishing thm)

Thm (Effective Vanishing thm) (van der Geer–Schoof, Groenewegen $N=1$, Weng $N>1$)

If Λ: s. stable $\mathcal{O}_F$-lattice of rank n satisfying $\deg(\Lambda) \leq -[F:\mathbb{Q}]\cdot \frac{n\log n}{2}$

then $h^0(F.\Lambda) \leq \dfrac{3^{\mathrm{rk}_{\mathbb{Z}}(\Lambda)}}{1-\frac{\log 3}{\pi}} \cdot e^{\pi[F:\mathbb{Q}]} \cdot e^{-2\deg(\Lambda)/\mathrm{rk}_{\mathbb{Z}}(\Lambda)}$.

* More generally $X^{n+1}/\mathcal{O}_F$ arithmetic variety, $\mathcal{F}/X$: locally free

K. Sugahara – Weng: $\exists\ H^0_{ar}(X, \mathcal{F}), \ldots, H^{n+1}_{ar}(X, \mathcal{F})$

Adelic Approach / Chevalley, Parshin, Osipov.

In particular, for arithmetic surfaces $X^{1+1}/\mathcal{O}_F$,

$\widehat{\mathbb{A}^{ar}_X} \simeq \mathbb{A}^{ar}_X$ & $\exists$ subspaces $\mathbb{A}^{ar}_{X,01}$, $\mathbb{A}^{ar}_{X,02}$ & $\mathbb{A}^{ar}_{X,12}(D)$ $\forall$ Arakelov divisor D

s.t. $(\mathbb{A}^{ar}_{X,01})^\perp = \mathbb{A}^{ar}_{X,01}$, $(\mathbb{A}^{ar}_{X,02})^\perp = \mathbb{A}^{ar}_{X,02}$

& $(\mathbb{A}^{ar}_{X,12}(D))^\perp = \mathbb{A}^{ar}_{X,12}((\omega)-D)$.

Thm. (S-W) $\widehat{H^i_{ar}(X, D)} \simeq H^{2-i}_{ar}(X, (\omega)-D)$ $\quad i = 0,1,2$

for $D \subset X \leftarrow$ arithmetic surface

Key: Parshin–Beilinson $\leftarrow$ ~~Ind-Proj top.~~ Adelic cohomology.

Parshin–Osipov $\leftarrow$ Ind-Proj top

& Uniformity between S_{fin} & S_∞ above. $\leftarrow$ Weng.

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