II. Arithmetic Cohomology Theory.

F: # field, $\mathcal{O}_F$: ring of integers, $S = S_{fin} \cup S_\infty$ set of places of F.

F_v: v-completion of F $\forall v \in S$. $\mathcal{O}_v$: ring of integers $v \in S_{fin}$.

$\mathbb{A} = \mathbb{A}_F := \prod_v' (F_v, \mathcal{O}_v)$: adele ring $GL_n(\mathbb{A})$: associated general linear group.

ω_F: dualizing adelic element of F locally $\partial_v =: (\pi_v^{\delta_v})$ then $\omega_F \doteq (\pi_v^{-\delta_v})$.
$\uparrow$ local different.

$g = (g_\mathfrak{p}; g_\sigma) \in GL_n(\mathbb{A})$, $\mathfrak{p} \in S_{fin}$, $\sigma \in S_\infty$.

* Auxiliary Topological Space $\mathbb{A}^n(g)$

(i) set theoretically: $\mathbb{A}^n(g) := \left\{ x \in \mathbb{A}^n \;\middle|\; \begin{array}{l} \exists\, a \in F^n \text{ s.t. } x_\sigma = a \;\; \forall \sigma \in S_\infty \\ g_\mathfrak{p} \cdot x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^n,\; g_\mathfrak{p} \cdot a \in \mathcal{O}_\mathfrak{p}^n \;\; \forall \mathfrak{p} \in S_{fin} \end{array} \right\}$

(ii) introduce a new topological structure on $\mathbb{A}^n$ by
by keeping the finite part but (b) altering its metric at $\sigma \in S_\infty$
i.e., using the positive definite matrix $g_\sigma^t \cdot g_\sigma$ (resp. $\bar{g}_\tau^t \cdot g_\tau$) for σ: real (resp. τ: complex)
& equip $\mathbb{A}^n(g)$ w/ the induced metric structure via $\mathbb{A}^n(g) \hookrightarrow \mathbb{A}^n$.

<u>Definition</u>. Arithmetic Cohomology Groups:

$$H^0(F, g) \doteq \mathbb{A}^n(g) \cap F^n, \quad H^1(F, g) \doteq \mathbb{A}^n \big/ (\mathbb{A}^n(g) + F^n).$$

w/ induced metric structures from altered $\mathbb{A}^n$.

$\Longrightarrow$ (i) $\mathbb{A}^n(g)$ locally compact. (ii) $\left\{\begin{array}{l} H^0(F, g) \text{ discrete} \\ H^1(F, g) \text{ compact} \end{array}\right.$ $\Rightarrow$ Fourier analysis $\Rightarrow \left\{\begin{array}{l} h^0(F, g) \\ h^1(F, g) \end{array}\right.$

& h^0: van der Geer - Schoof.

<u>Duality</u> (Topological) $\widehat{H^1(F, g)} \cong H^0(F, \omega_F \otimes g^{-t})$.

(Arithmetical) $h^1(F, g) = h^0(F, \omega_F \otimes g^{-t})$.

<u>Arithmetic Riemann-Roch Thm</u>

$$h^0(F, g) - h^1(F, g) = \deg(g) - \frac{n}{2} \log |\Delta_F|.$$