Def. Rank n- Non-Abelian Zeta Function for F

$$\hat{\zeta}_{F,n}(s) := |\Delta_F|^{\frac{n}{2}\cdot s} \int_{\mathcal{M}_{F,n}} \left(e^{h^0(F,\Lambda)} - 1\right) \cdot (e^{-s})^{\deg(\Lambda)} \, d\mu(\Lambda). \qquad \mathrm{Re}(s) > 1.$$

w/ $h^0(F,\Lambda) :=$ 0-th arithmetic cohomology of Λ.

$$= \log\left(\sum_{x \in \Lambda} e\left(-\pi \sum_{\sigma : \mathbb{R}} \|x\|^2_{\rho_\sigma} - 2\pi \sum_{\tau : \mathbb{C}} \|x\|^2_{\rho_\tau} \right) \right)$$

Zeta Facts. (0) $\hat{\zeta}_{F,1}(s) \doteq \hat{\zeta}_F(s)$, complete Dedekind zeta function for F.

(1) Well-defined & holomorphic for $\mathrm{Re}(s) > 1$

& $\exists$ meromorphic continuation to the whole s-plane

(2) (Functional Equation) $\hat{\zeta}_{F,n}(1-s) = \hat{\zeta}_{F,n}(s)$

(3) (Singularities) Only two of them, at $s=0,1$ all simple poles

$$\mathrm{Res}_{s=1} \hat{\zeta}_{F,n}(s) = \mathrm{Vol}\left(\mathcal{M}_{F,n}[1]\right)$$

Proof. Refined Arithmetic Riemann-Roch + Vanishing theorem for $h^1(F,\Lambda)$

$$\Longrightarrow \hat{\zeta}_{F,n}(s) = \int_{[\Lambda] \in \mathcal{M}_{F,n}[\geq 1]} \left(e^{h^0(F,\Lambda)} - 1\right) \cdot \mathrm{Vol}(\Lambda)^s \cdot d\mu(\Lambda)$$

$$+ \int_{[\Lambda] \in \mathcal{M}_{F,n}[\geq 1]} \left(e^{h^0(F,\Lambda)} - 1\right) \mathrm{Vol}(\Lambda)^{1-s} \, d\mu(\Lambda)$$

$$+ \mathrm{Vol}\left(\mathcal{M}_{F,n}[1]\right) \cdot \left(\frac{1}{s-1} - \frac{1}{s}\right)$$

w/ $\mathcal{M}_{F,n}[\geq T]$ = moduli space of s.stable $\mathcal{O}_F$-lattices of rank n & volumes $\geq T$.