Day 5. Periods, Cohomology & Non-Abelian Zetas for Number Fields

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I. Non-Abelian Zeta Functions for Number Fields

~~Ex. Def.~~

F: number fields, $\mathcal{O}_F$ = ring of integers. P: rank n projective $\mathcal{O}_F$-module.

$\Rightarrow$ $P \simeq \mathcal{O}_F^{(n-1)} \oplus \mathfrak{a}$ w/ $\mathfrak{a}$: fractional ideal of $\mathcal{O}_F$. $\because$ $\mathcal{O}_F$ = Dedekind.

$$\downarrow$$

$$(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^n \simeq (\mathbb{R}^n)^{r_1} \times (\mathbb{C}^n)^{r_2} =: V_n(F)$$ of Minkowski embedding $F \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$

w/ $\sigma_i: F \hookrightarrow \mathbb{R}$, $i = 1, \dots, r_1$

$\tau_j: F \hookrightarrow \mathbb{C}/\sim$ $j = 1, \dots, r_2$.

$g = (g_\sigma; g_\tau)$ Hermetian metric on $V_n(F)$.

Def. (i) $\mathcal{O}_F$-lattice $\Lambda := (P; g)$ of rank n.

$\Rightarrow$ (co)volume of $\Lambda := \mathrm{Vol}(V_n F/P, g)$.

(ii) Λ: semi-stable $\overset{\text{def}}{\Longleftrightarrow}$ $\forall \Lambda_1 \subseteq \Lambda$ sublattice (i.e. Λ/Λ_1 also $\mathcal{O}_F$-lattice)

$$\mu(\Lambda_1) \leq \mu(\Lambda) = \deg(\Lambda)/\mathrm{rank}(\Lambda)$$

$$\Longleftrightarrow \quad \mathrm{Vol}(\Lambda_1)^{\mathrm{rank}\,\Lambda} \geq \mathrm{Vol}(\Lambda)^{\mathrm{rank}(\Lambda_1)} \quad \forall \Lambda_1 \subseteq \Lambda$$

$\because$) Arakelov - Riemann - Roch thm:

$$-\log \mathrm{Vol}(\Lambda) =: \chi(F, \Lambda) = \deg(\Lambda) - \frac{\mathrm{rk}\,\Lambda}{2} \cdot \log |\Delta_F|$$

Δ_F: discriminant of F

$\mathcal{M}_{F,n} :=$ moduli space of s.stable $\mathcal{O}_F$-lattices of rank n

$\mathcal{M}_{F,n}[T] :=$ Moduli space of s.stable $\mathcal{O}_F$-lattice of rank n & volume T.

Facts (i) $\mathcal{M}_{F,n} = \dot{\bigcup}_{T \in \mathbb{R}_{>0}} \mathcal{M}_{F,n}[T]$

(ii) $\mathcal{M}_{F,n}[T]$ compact

(iii) $\exists$ natural measure $d\mu$ & $d_T\mu$ on $\mathcal{M}_{F,n}$ & $\mathcal{M}_{F,n}[T]$

s.t. $$d\mu = \frac{dT}{T} \times d_T\mu.$$