## C. Averages

**Def** (i) Elliptic Average (for a fixed r)

bad prime

$$\mathcal{M}_{E,r}(f) := \lim_{x\to\infty} \frac{\sum_{\substack{L: |D_L| \le x \\ r(E(L)) = r}} f(Ш^{(N)}(E/L))}{\sum_{\substack{L: |D_L| \le x \\ r(E(L)) = r}} 1}$$

(ii) Torsion Average: F: $\mathbb{C}$-valued / iso. classes of pure torsion coherent sheaves.

$$\mathcal{M}_r(F) := \lim_{x\to\infty} \frac{\sum_{N(\mathfrak{a}) \le x} \frac{1}{N\mathfrak{a}^r} \sum_{T(\mathfrak{a}), \varphi_r} F(T/\mathrm{Im}\,\varphi) \cdot \frac{1}{\# \mathrm{Aut}\, T}}{\sum_{N(\mathfrak{a}) \le x} \frac{1}{N\mathfrak{a}^r} \cdot \sum_{T(\mathfrak{a}), \varphi_r} \frac{1}{\# \mathrm{Aut}(T)}}$$

w/ $\sum_{T(\mathfrak{a}), \varphi_r} = \sum_{\substack{T: \chi(T) = \mathfrak{a} \\ \varphi \in \mathrm{Hom}(\mathcal{O}^r, T)}}$ w/ $\chi(T) := \prod_v \prod_Q \prod_i p_Q^{n_{v,Q}^{(i)}}$

if $T = (T_{v,Q})_{v,Q}$ w/ $T_{v,Q} = \bigoplus_{i=1}^{\ell_Q} \mathcal{O}_{E_v} / \mathfrak{m}_{E_v,Q}^{n_{v,Q}^{(i)}}$

## D. Elliptic Curve Heuristics

$\exists$ natural 1-1 correspondence $f \longmapsto F$ (for restricted $f$)

s.t. $\forall$ $E/\mathbb{Q}$ elliptic curve, we have

$$\mathcal{M}_{E,r}(f) = \mathcal{M}_r(F)$$

**Ex**
- G-cyclic $\Leftrightarrow$ $G_p$: cyclic $\simeq \mathbb{Z}/p^i\mathbb{Z}$
- $f_0 = \mathbb{1}_{G^{odd} = \text{cyclic}}$
- $F_0 = \mathbb{1}_{T = \text{simple}}$ i.e., $T_{v,Q} = \mathfrak{m}_{v,Q}/\mathfrak{m}_{v,Q}^{n_{v,Q}}$ $\forall v, Q \in E_v$.

**Cor.** Assume Elliptic Curve Heuristics,

The Probability of $Ш^{odd}(E/L)$ being cyclic among all quadratic $L/\mathbb{Q}$ s.t. $\mathrm{rk}\, E(L) = 0$ is given by

$$\frac{\zeta_{E_{p=2}}(6)}{\zeta_{E_{p=2}}(2)\,\zeta_{E_{p=2}}(3)} \cdot \frac{\zeta_E(2) \cdot \zeta_E(3)}{\zeta_E(6)} \cdot C_\infty(E/\mathbb{Q})$$

Local Adjustment; Hasse-Weil zeta; $\mathrm{Res}_{s=1} \prod \zeta_E(s+k)$