II.2. Heuristics for Elliptic Curves

A) Number Fields versus Elliptic Curves

Thm (Dedekind) F: # field, $\zeta_F(s)$: Dedekind zeta function

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \#Cl(F)\cdot\frac{2^{r_1}(2\pi)^{r_2}\cdot R}{w\cdot\sqrt{|D_F|}}$$

R ← regulator

$Cl(F)$: Class group

w: # of units $\cap \mu_\infty$.

D_F: Discriminant

r_1: # of $F \hookrightarrow \mathbb{R}$

r_2: # of $F \hookrightarrow \mathbb{C}/\sim$.

BSD Conjecture: $E/\mathbb{Q}$: elliptic curve, $\zeta_E(s)$: Hass-Weil zeta function for $E/\mathbb{Q}$

$$\lim_{s\to 1}\frac{1}{(s-1)^r}\zeta_E(s) = \#Ш(E/\mathbb{Q})\cdot\frac{R_E\cdot Tam(E)\cdot\Omega_E^+}{\#E(\mathbb{Q})_{tor}}$$

$Ш(E/\mathbb{Q})$: Shafarevich-Tate group

R_E: Regulator

$Tam(E)$: Tamagawa #

Ω_E^+ ← period.

$E(\mathbb{Q})_{tor}$: torsion of $E(\mathbb{Q})$

So it appears.

# field	$\leftrightarrow$	Elliptic curve
$r_F = \mathrm{rank}\,\mathcal{O}_F^\times$	"	$r_E = \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q})$
$Cl(F)$	"	$Ш(E/\mathbb{Q})$.
Finite Abelian Groups = torsion sheaves$/\mathrm{Spec}\,\mathcal{O}_F$	$\leftrightarrow$	torsion coherent sheaves$/\{E_v\}_v$

B) Relative Shafarevich-Tate Groups

$E/\mathbb{Q}$: elliptic curve $L/\mathbb{Q}$: quadratic field. $E := E(\overline{\mathbb{Q}}) \circlearrowleft G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Def. Relative Shafarevich-Tate Groups.

$$Ш(E/L) := \mathrm{Ker}\Big(H^1(G_L, E) \to \prod_v \prod_{w|v} H^1(G_w, E_v)\Big), \quad G_w = G_{L_w}$$

① Inflation-Restriction Exact Sequences

- $0 \to H^1(G_{L/K}, E(L)) \to H^1(G_K, E) \to H^1(G_L, E)$
- $0 \to H^1(G_{L_w/K_v}, E(L_w)) \to H^1(G_v, E_v) \to H^1(G_w, E_v)$

Even assuming $Ш(E/\mathbb{Q})$, Not easy to see.