Def A (C-L). f: $\mathbb{C}$-valued function / iso. classes of finite A-modules.

(i) $W_k(f;\mathfrak{a}) := \hat{\sum}_{G(\mathfrak{a})} w_k(G)\cdot f(G)$, $\quad \zeta_k(f,s) := \sum_{\mathfrak{a}} W_k(f;\mathfrak{a})\cdot N\mathfrak{a}^{-s}$

(ii) (k,r)-Average of f:

$$M_{k,r}(f) := \lim_{x\to\infty} \frac{\hat{\sum}_{N\mathfrak{a}\le x} \frac{1}{(N\mathfrak{a})^r}\cdot \hat{\sum}_{G(\mathfrak{a}),\varphi_r} f(G/\mathrm{Im}\,\varphi)\cdot w_k(G)}{\sum_{N\mathfrak{a}\le x} \frac{1}{(N\mathfrak{a})^r}\cdot \hat{\sum}_{G(\mathfrak{a}),\varphi_r} w_k(G)}$$

$= \sum_{N\mathfrak{a}\le x} W_k(\mathfrak{a})$

Prop (C-L) Set $\zeta_k(f;s+r)\cdot\frac{\zeta_k(s)}{\zeta_k(s+r)} =: \hat{\sum}_{\mathfrak{a}} a_{k,r}(f;\mathfrak{a})\cdot N\mathfrak{a}^{-s}$. Then

(i) $$M_{k,r}(f) = \lim_{x\to\infty}\frac{\sum_{N\mathfrak{a}\le x} a_{k,r}(f;\mathfrak{a})}{C_k\cdot\log a}$$

(ii) Assume $\zeta_k(f;s)-\frac{\exists c}{s}$ can be analytically continued to $\mathrm{Re}(s)\ge 0$, then

$$M_{k,r}(f) = \begin{cases} \lim_{s\to 0} \zeta_k(f;s)/\zeta_k(s) = c/C_k & \text{if } r=0 \\ \zeta_k(f;r)\cdot\frac{C_r}{C_{r+k}} = \frac{\zeta_k(f;u)}{\zeta_k(r)} & \text{if } r\ne 0 \end{cases}$$

Def B# (C-L). $F=\mathbb{Q}(\sqrt{d}) \Rightarrow r_F = \mathrm{rank}(\mathcal{O}_F^\times) = \begin{cases}1 & \text{real} \\ 0 & \text{imaginary}\end{cases} =: r$.

$$M^\#(f) := \lim_{x\to\infty}\frac{\sum_{|D_F|\le x} f(Cl^{(2)}_{F/\mathbb{Q}})}{\sum_{|D_F|\le x} 1}$$

Cohen-Lenstra Heuristic Assumption#: $\boxed{M_r(f) = M^\#(f)}$ holds for imaginary / real separately

Example: For imaginary quadratic fields

Probability of $Cl^{(2)}_F$ to be cyclic $= \frac{\zeta(2)\cdot\zeta(3)}{3\cdot\zeta(6)}\cdot\frac{1}{C_\infty\cdot\eta_\infty(2)} \approx 97.7575\%$.