II. Cohen-Lenstra Heuristics for Ш ⊂ E(L)

II.1. Original Cohen-Lenstra Heuristics for $Cl_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}$.

• A: Dedekind ring $\mathfrak{a} \triangleleft A$. $f \in \mathrm{Hom}_A(A^k, G)$

G: finite A-module = torsion sheaf / Spec A. $= \bigoplus A/\mathfrak{a}_i \Rightarrow \chi_A(G) := \prod_i \mathfrak{a}_i$

$\Downarrow$

$|G| = N(\chi_A(G))$

$$W_k(G) = \frac{\#\{A^k \twoheadrightarrow G\}}{\# G}\cdot\frac{1}{\#\mathrm{Aut}\,G}$$

$$W_\infty(G) = W(G) := \frac{1}{\#\mathrm{Aut}\, G}.$$

• $\forall\, \mathfrak p \triangleleft A$ prime. $\eta_k(\mathfrak p) = \prod_{i=1}^k (1-N\mathfrak p^{-i})$, $\eta_\infty(\mathfrak p) := \prod_{i=1}^\infty(1-N\mathfrak p^{-i})$

$\zeta_A^*(1) := \mathrm{Res}_{s=1}\zeta_A(s)$, $C_k := \prod_{i=1}^k \zeta_A(i)$, $C_\infty = \prod_{i=1}^\infty \zeta_A(i)$.

$r_\mathfrak p(G) = \mathfrak p$-rank of $G = \dim_{A/\mathfrak p} G/\mathfrak p G$ e.g. $r_p(\mathbb Z/p^{e_1}\oplus\cdots\oplus \mathbb Z/p^{e_r}) = r$. $(e_i \neq 0)$.

__Thm__ (C-L). (i) $\sum_G W_k(G)\cdot \#\{G_1\le G: G_1\simeq S,\ G/G_1\simeq Q\} = W_k(S)\cdot W_k(Q)$ for fixed S & Q.

(ii) $W_k(G) = \left(\prod_{\mathfrak p|\mathfrak a} \eta_k(\mathfrak p)/\eta_{k-r_\mathfrak p(G)}(\mathfrak p)\right)\cdot \frac{1}{\#\mathrm{Aut}\, G}$

(iii) $\lim_{k\to\infty} W_k(G) = W_\infty(G)$

(iv) $W_k(\mathfrak a) := \sum_{G(\mathfrak a)} W_k(G) = \frac{1}{N\mathfrak a}\prod_{\mathfrak p^\alpha\|\mathfrak a} \begin{bmatrix} \alpha+k-1\\ \alpha\end{bmatrix}_\mathfrak p$ $=$ $\eta_{\alpha}(\mathfrak p)/\eta_{\alpha+k}(\mathfrak p)\cdot\eta_{k}(\mathfrak p)$ cro. $k\neq\infty$.

$W(\mathfrak a) = \frac{1}{N\mathfrak a}\prod_{\mathfrak p^\alpha\|\mathfrak a}\eta_\alpha(\mathfrak p)^{-1}$.

$W(p^\alpha) = \left(p^\alpha(1-\frac1p)(1-\frac1{p^2})\cdots(1-\frac1{p^\alpha})\right)^{-1}$.

(v) $\sum W_k(\mathfrak a)\cdot N\mathfrak a^{-s} = \prod_{j=1}^k \zeta_A(s+j)$

__Remarks.__ 1. Counting should be compared w/ those for Atiyah bundles

(2) (i) $\longleftrightarrow$ analogue in the study of Hall algebras