I.2. Strange Duality

Thm B. $X = E$: elliptic curve

$$\beta^{T}_{E,0}(n) := \sum_{\substack{T/E:\ \text{pure torsion} \\ \deg T = n \\ \mathrm{rk}(T)=0}} \frac{1}{\#\mathrm{Aut}\,T} = \sum_{\substack{V/E:\ \text{s-stable} \\ \mathrm{rk}(V)=n \\ \deg(V)=0}} \frac{1}{\#\mathrm{Aut}(V)} =: \beta^{(V)}_{E,n}(0).$$

Proof. By the main thm of W-Zagier on RH of elliptic curve $/\mathbb{F}_q$.

$$\sum_{n\geq 0} \beta_{X,n}(0)\, q^{-ns} = \prod_{k\geq 1} \zeta_E(s+k)$$
$$= \sum_{n\geq 0} \beta^{T}_{X,0}(n)\, q^{-ns} \quad \text{by Thm A.}$$
$\#$.

We expect: Analogue holds for Abelian varieties $/\mathbb{F}_q$

& probably, also holds for certain Calabi-Yau.

I.3 Global One: Torsion Coherent Sheaves.

X/F: ~~elliptic curve~~ arith variety / # field $\quad \mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_F$ integral model

$\mathcal{X}_v/\mathbb{F}_{q_v}$: $\mathfrak{p}_v$-reduction

Def. $\mathcal{T} = (\mathcal{T}_{v,Q})_{v\in \mathrm{Spec}\,\mathcal{O}_F,\ Q\in \mathcal{X}/v}$: pure torsion coherent sheaf if

(i) $\exists$ injective morphism of locally free sheaves $\phi: V_1 \to V_2$ of same rank s.t. $\mathcal{T} = \mathrm{Coker}(\phi)$

(ii) $\forall'\ (v,Q)$, $\mathcal{T}_{v,Q} = 0$.

This then would lead to a discussion on curves & their points over p-adic field. Very interesting & deep. But for our limited purpose, we take a semi-global one.