Torsion Sheaves & Cohen-Lenstra Heuristics for Ш(E/k)

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(joint w/ K. Sugahara)

I. Torsion Coherent Sheaves & Zeta Functions

I.1. Torsion Coherent Sheaves. Local One

Def. $X/\mathbb{F}_q$: algebraic variety

(1) $T = (T_Q)_{Q\in X}$: (pure) torsion (coherent) sheaf if

(i) $\exists$ injective morphism of locally free sheaves $\phi: V_1 \to V_2$ of same rank s.t. $T = \mathrm{Coker}(\phi)$.

(ii) $\forall'$ (= for all but finitely many) Q, $T_Q = 0$.

(2) $\mathrm{Aut}(T) := \mathrm{Aut}_{\mathcal{O}_X}(T) \cong \prod_{Q\in X} \mathrm{Aut}_{X,Q}(T_Q)$

$N(T) := \prod_Q |T_Q|$, Norm of T w/ $T_Q = q^{\deg T_Q}$, $q_Q := q^{\deg(Q)}$

Indeed $T_Q = \mathcal{O}_{X,Q}/m_{X,Q}^{n_Q} \Rightarrow \{\deg T_Q = n_Q\cdot \deg Q$.

(3) Zeta function of X for torsion sheaves:

$$\zeta_X^T(s) := \sum_{T/X:\ \text{pure torsion coherent sheaves}} \frac{1}{\# \mathrm{Aut}(T)} \cdot N_T^{-s} \qquad \mathrm{Re}(s) > 1.$$

Thm A. $\zeta^T_X(s) = \prod_{k\geq 1} \zeta_X(s+k)$.

w/ $\zeta_X(s) = \exp\left(\sum_m N_m \frac{t^m}{m}\right)$, $t := q^{-s}$

Proof: Using Euler product $\zeta_X(s) = \prod_{Q\in X} \frac{1}{1-N_Q^{-s}} = \sum_m \#(\mathrm{Sym}^{(m)} X(\mathbb{F}_q))\cdot q^{-sm}$.

Remark: $\exists$ Motivic version for X/k ~ any base field

Motivic Euler product $\Rightarrow$ Thm A': $\zeta^T_{X^{(m)}}(s) = \prod_{k\geq 1} \zeta_X(s+k)$ $\forall X/k$