Prop (Special Counting Miracle)

$$\alpha^{At}_{E,n+1}(0) = \beta^{At}_{E,n}(0) = \sum_{\substack{V:\text{Atiyah} \\ \mathrm{rk}\,V = n}} \frac{1}{\#\mathrm{Aut}\,V}.$$

Proof: Generating Function $\begin{cases} \text{Don: Easy} \\ \text{Me: Slightly Difficult.} \end{cases}$

Prop. Special CM $\Rightarrow$ CM

$\because$) Atiyah's classification $/\mathbb{F}_q$ Jordan-Hölder Graded bdl

J-H filtration (not !) But ! $\oplus G_i(V) = G(V) = L_1 \oplus \cdots \oplus L_n$

$V \supset V_1 \supset \cdots \supset V_k$ s.t. $G_i(V) = V_i/V_{i+1}$ $\begin{cases} \text{stable} \\ \deg 0. \end{cases}$ Difficulty: $L_i/\overline{\mathbb{F}}_q$, not $/\mathbb{F}_q$

$\because$) ℓ-torsion of Jacobian $\mathrm{Jac}(E)/\mathbb{F}_{q^k}$ plays key roles

Intrinsic Arithmetic

$$\alpha_{E,n}(0) = \sum \frac{q^{h^0(X,V)} - 1}{\#\mathrm{Aut}(V)} \quad \text{However} \Longleftarrow$$

$\boxed{\text{(a) Only } h^0(X,V) \neq 0 \text{ has non trivial contributions!}}$

(b) $\exists$ decomposition $/\mathbb{F}_q$ $V = U \oplus W$ as s.stable bdls

s.t. $G(U) \cong \mathcal{O}_E^{\oplus i}$, $G(W) \cong L_{i+1} \oplus \cdots \oplus L_n$ ($\Rightarrow \deg L_i = 0$ but $L_i \not\cong \mathcal{O}_E$)

$h^0(E,V) = h^0(E,U)$ & $\mathrm{Aut}(V) \simeq \mathrm{Aut}(U) \times \mathrm{Aut}(W)$.

$$\Rightarrow \alpha_{E,n}(0) = \sum_{i\geq 1}^{n} \alpha^*_{E,i} \cdot \beta^*_{E,n-i} \overset{SCM}{=\!=} \beta_{E,n-1}(0).$$

Atiyah bdl $\rightarrow$

$$\alpha^*_{E,i} = \sum_{\substack{U:\text{s.stable} \\ G(U) \cong \mathcal{O}_E^{\oplus i}}} \frac{q^{h^0(U)} - 1}{|\mathrm{Aut}\,U|}, \qquad \beta^*_{E,n-i} = \sum_{\substack{W:\text{s.stable} \\ G(W) = L_1 \oplus \cdots \oplus L_{n-i} \\ L_i \neq \mathcal{O}_E}} \frac{1}{|\mathrm{Aut}\,W|}$$

∎

4. Multiplicative structure of β-invariants

Main Thm $\sum \beta_{E,n}(0)(q^{-s})^n = \prod_{k\geq 1} \zeta_E(s+k)$

Proof: $\begin{cases} \text{Don: } \begin{cases} \text{Heavy} \\ \text{Very Difficult} \end{cases} \\ \text{Me: in heaven!} \end{cases}$

$\Rightarrow (q^n - 1)\beta_n = (q^n + q^{n-1} - a)\beta_{n-1} - (q^{n-1} - q)\beta_{n-2}$ w/ $\beta_n = \beta_{E,n}(0)$, $a = q + 1 - \#E(\mathbb{F}_q)$