Difficult to use. In fact for E, Much refined structures!!

**Thm** (W-Z) ← Main Thm

Let $\zeta_{E/\mathbb{F}_q}(s) := \sum_{n=0}^{\infty} \beta_{E,n}(0) \cdot q^{-ns} = \sum_{V} \frac{1}{\#\mathrm{Aut}\, V} \cdot q^{nk(V)-s}$

(V: s.s. stable of deg 0) — Informed by Deninger

Then $\zeta_{E/\mathbb{F}_q}(s) = \prod_{k=1}^{\infty} \zeta_{E/\mathbb{F}_q}(s+k)$ ← Similar structure as in Cohen-Lenstra

⇓

**Thm** (W-Z) For $n \geq 2$

$$1 < \frac{\beta_{E,n}(0)}{\beta_{E,n-1}(0)} < \frac{Q^{\frac{1}{2}}+1}{Q^{\frac{1}{2}}-1}.$$

**Cor** (RH) $|\omega_{E,i}| = Q^{\frac{1}{2}}$. $i=1,2$.

## 3. Counting Miracle. (CM for short)

### 3.1. Atiyah Bundle

Inductively define $I_n$ as. $I_1 = \mathcal{O}_E$, $\underset{n\geq 2}{I_n}$: ! non-trivial extension of $I_{n-1}$ by $I_1$

$$0 \to \mathcal{O}_E \to I_n \to I_{n-1} \to 0$$

**Atiyah Bundle**: $V = \bigoplus_{k\geq 1} I_k^{\oplus m_k}$.

!, ∵) $\mathrm{Ext}^1(I_n, \mathcal{O}_E) \cong H^1(E, I_n^\vee) \simeq H^0(E, I_n) = \mathbb{F}_q$.

**Prop.** (i) $h^0(V) = \sum_{k\geq 1} m_k$ ← (Atiyah)

(ii) $|\mathrm{Aut}(V)| = q^{N(m_1, m_2, \cdots)} \prod_{k\geq 1} q^{-m_k^2} \times |GL_{m_k}(\mathbb{F}_q)|$ ← $GL_{m_k}(\mathbb{F}_q)$.

w/ $N(m_1, m_2, \cdots) := \sum_{k,l\geq 1} m_k m_l \min(k,l)$.

∵) Key ⓐ $\mathrm{Aut}\, I_k \leftrightarrow \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ & \ddots & & \vdots \\ 0 & & x_1 & x_2 \\ & & & x_1 \end{pmatrix} \Rightarrow \#\mathrm{Aut}\, I_k = q^{k-1}(q-1)$

ⓑ $\#\mathrm{Aut}(I_k^{\oplus m}) = q^{(k-1)m^2} \cdot |GL_m(\mathbb{F}_q)|$

ⓒ $|\mathrm{Aut}(V)| = \prod_{k\neq l} |\mathrm{Hom}(I_k, I_l)|^{m_k m_l} \cdot \prod_k |\mathrm{Aut}(I_k^{\oplus m_k})|$ #.