Day 3. Riemann Hypothesis for Elliptic Curves $/\mathbb{F}_q$ & Distribution of Zeros

jointly w/ Zagier!

L. Weng. Nov 17, 2015
MSC, Tsinghua
Beijing.

§1 Non-Abelian Zeta Function for $X/\mathbb{F}_q$

$X/\mathbb{F}_q$: curve of genus g. $n\in\mathbb{Z}_{>0}$ $\Rightarrow$ Non-Abelian Zeta Function

$$\hat{\zeta}_{X,n}(s) = \sum_{m\geq 0}\ \sum_{[V]\in \mathcal{M}_{X,n}(mn)} \frac{\#(H^0(X,V)\setminus\{0\})}{\#\mathrm{Aut}\,V}\cdot (q^{-s})^{\chi(X,V)}.$$

Set
$$\beta_{X,n}(d) = \sum_{[V]\in\mathcal{M}_{X,n}(d)} \frac{1}{\#\mathrm{Aut}\,V},\qquad \alpha_{X,n}(d) = \sum_{V\in\mathcal{M}_{X,n}(d)} \frac{q^{h^0(X,V)}-1}{\#\mathrm{Aut}(V)}$$
Arithmetic of Brill-Noether Loci

$q^{-s}=T$, $Q=q^n$
$$= \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn)\left(\left(\frac{1}{T}\right)^{g-1-m} + (QT)^{g-1-m}\right) + \alpha_{X,n}((g-1)n) + \beta_{X,n}(0)\frac{QT^g}{(1-T)(1-QT)}.$$

In particular, if $X=E$ elliptic curve $/\mathbb{F}_q$.

$$\hat{\zeta}_{E,n}(s) = \alpha_{E,n}(0) + \beta_{E,n}(0)\frac{QT}{(1-T)(1-QT)}$$
$$= \alpha_{E,n}(0)\frac{(1-\omega_{n,1}T)(1-\omega_{n,2}T)}{(1-T)(1-QT)}$$

Thm (W-Z) $|\omega_{n,i}| = \sqrt{Q} = Q^{\frac{1}{2}}$, $i=1,2$. Riemann-Hypothesis.

§2. $\boxed{\text{Counting Miracle}}$ & Main Structures.

Thm (W-Z). $\alpha_{E,n+1}(0) = \beta_{E,n}(0)$. Thm (S) $\alpha_{X,n+1}(0) = q^{n(g-1)}\beta_{X,n}(0)$.

generalized by Sugahara.

Thm (~~HN~~, R-D, Z)
$$\frac{\beta_{X,n}(0)}{q^{\frac{n(n+1)}{2}(g-1)}} = \sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\dots+n_k=n}\frac{\prod_{i=1}^{k}\hat{v}_{X,n_i}}{\prod_{j=1}^{k-1}(q^{n_j+n_{j+1}}-1)}$$
$\hat{v}_{X,n} := \prod_{i=1}^{n}\hat{\zeta}_X(i)$ w/ $\hat{\zeta}_X(1) = \frac{\#\mathrm{Jac}(\mathbb{F}_q)}{q-1}$.

Weng's formulation!!
Parabolic Reduction, Stability & the Masses.

$\exists$ Number Field Analogue (W)