II SL_n-Zeta Functions $/X$. (2-4)

$G=SL_n \longleftrightarrow$ Root system $\langle V, \langle , \rangle, \Phi=\Phi^+\cup\Phi^-, \Delta=\{\alpha_1,\dots,\alpha_{n-1}\}, \varpi:=\{\varpi_1,\dots,\varpi_{n-1}\}, W\}$

simple roots; fundamental wt

$V = \operatorname{span}\Delta = \{\sum x_i e_i : \sum x_i = 0\}$.

$\mathbb{R}^{n+1} = \mathbb{R}e_1+\mathbb{R}e_2+\cdots+\mathbb{R}e_n$ standard Euclidean space

$\Phi=\{e_i-e_j\} \supset \Phi^+=\{\alpha_{ij}, i<j\}$, $\alpha_{ij}=e_i-e_j$, $\alpha_i=\alpha_{i\,i+1}=e_i-e_{i+1}$

$$\varpi_i := \frac{n-i}{n}(e_1+\cdots+e_i) - \frac{i}{n}(e_{i+1}+\cdots+e_n).$$

$\langle \alpha_i, \varpi_j\rangle = \delta_{ij}$

In particular $\varpi_{n-1} = \frac{1}{n}(e_1+\cdots+e_{n-1}-(n-1)e_n)$

$\alpha^\vee = \frac{2}{\langle\alpha,\alpha\rangle}\cdot\alpha = \alpha \quad \forall \alpha\in\Phi$. coroots.

$W=S_n \ni \sigma: \sum a_i e_i \mapsto \sum a_i e_{\sigma(i)}$.

$$\rho = \frac12\sum_{\alpha>0}\alpha = \frac12((n-1)e_1+(n-3)e_2+\cdots-(n-1)e_n).$$

$P=P_{n-1,1} \ni \begin{pmatrix} \boxed{}_{n-1} & * \\ 0 & * \end{pmatrix} \Longleftrightarrow \Delta_P=\Delta\setminus\{\alpha_p\}=\{\beta_{P,1},\dots,\beta_{P,n-2}\}$, $\alpha_p=\alpha_{n-1}$.

$\Phi_P = \Phi\cap\{\varpi_p\}^\perp$. $\Phi_P^+=\Phi_P\cap\Phi^+$.

$c_P := 2\langle\varpi_p-\rho_P, \alpha_p^\vee\rangle = n$.

$$\rho_P = \frac12\sum_{\alpha\in\Phi_P^+}\alpha = \frac12((n-2)e_1+(n-4)e_2+\cdots-(n-2)e_{n-1})$$

$\rho-\rho_P = \frac{n}{2}\varpi_{n-1}$.

$$\lambda := \sum_{j=1}^{n-1}(1+s_j)\varpi_j = \rho+\sum_{j=1}^{n-1}s_j\varpi_j.$$

Def. (i) Period of G for X

$$\omega_X^G(\lambda) := \sum_{w\in W}\frac{1}{\prod_{\alpha\in\Delta}\left(1-\zeta^{-\langle w\lambda-\rho,\alpha^\vee\rangle}\right)}\prod_{\substack{\alpha>0\\ w\alpha<0}}\frac{\hat\zeta_X(\langle\lambda,\alpha^\vee\rangle)}{\hat\zeta_X(\langle\lambda,\alpha^\vee\rangle+1)}$$

(ii) Period of (G,P) for X, $s=s_p=s_{n-1}$

$$\omega_X^{(G,P)}(s) := \operatorname{Res}_{\langle\lambda-\rho,\beta_{P,j}^\vee\rangle=0 \quad j=1,\dots,n-2}\ \omega_X^G(\lambda)$$

$$= \operatorname{Res}_{s_{n-1}=0,\dots,\widehat{s_p=0},\dots,s_1=0}\ \omega_X^G(\lambda).$$

Def. $\hat\zeta_X^{G/P}(s) := \operatorname{Norm}(\omega_X^{(G,P)}(s))$ ~ clean out ζ's in the deminator.

Thm. $\hat\zeta_X^{G/P}(-c_P-s) = \hat\zeta_X^{G/P}(s)$.